NAME __________________________
Signature ________________________

Math 111 - Section 001

Final Exam

March 25, 2006

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*Show all work for full credit.*

*Problems 3 and 5 are worth 15 points each.*

*All other problems are worth 10 points each.*
1. Suppose you have 5 coins, one of each is counterfeit (either heavier or lighter than the other four). You use a pan balance scale to find the bad coin and determine whether it is heavier or lighter.

(a) Prove that 2 weighings are not enough to guarantee that you find the bad coin and determine whether it is heavier or lighter.

Two weighings yield a 3-ary tree of height 2, which has at most 9 leaves, but 5 coins require a tree with 10 leaves.

(b) Draw a decision tree for weighing the coins to determine the bad coin (and whether it is heavier or lighter) in the minimum number of weighings.

Use the weighing 1 and 2 against 3 and 4 as the root. If the four coins have the same weight, weigh 1 against 5 to determine whether 5 is heavy or light. If 1 and 2 are lighter or heavier than 3 and 4, weigh 1 against 2. If 1 and 2 balance, weigh 3 against 4 to find out which of these coins is heavier or lighter; if 1 and 2 do not balance, then immediate information is obtained regarding coins 1 and 2.
2. Form a binary search tree from the words of the sentence:

This is my discrete mathematics final which was not so difficult but long

using alphabetical order, inserting words in the order they appear in the sentence.
3. The expression: / $\uparrow$ $-$ $a \cdot 7 \cdot c \cdot 4 \cdot 3 \ b$ is written in prefix notation.

a) Rewrite the expression in usual algebraic form.

b) What is the value of this expression if $a = 9, b = 2, c = 1$?

b) Represent the expression you obtained in part a) using a binary tree.

c) Write this expression in postfix notation.
5. (a) A full 3-ary tree has 100 internal vertices. How many leaves does it have?
(b) A full 4-ary tree has 100 leaves. How many internal vertices does it have?

a) A full 3-ary tree with 100 internal vertices has:

\[ l = (3 - 1) \cdot 100 + 1 = 201 \] leaves

b) A full 4-ary tree with 100 leaves has:

\[ i = \frac{100 - 1}{4 - 1} = 33 \] internal vertices
6. Find the solution of the recurrence relation \( a_n = 4a_{n-1} - 3a_{n-2} + n2^n \) with \( a_0 = 1 \) and \( a_1 = 4 \).

The associated homogeneous recurrence relation.

\[ a_n = 4a_{n-1} - 3a_{n-2} \]

The characteristic equation is \( r^2 - 4r + 3 = 0 \) with roots \( r = 1 \) and \( r = 3 \). Thus the general solution to the associated homogeneous equation is:

\[ a_n^{(h)} = \alpha \cdot 1^n + \beta \cdot 3^n = \alpha + \beta \cdot 3^n \]

Next, we find a particular solution to the given recurrence relation. We look for a solution of the form \( a_n^{(p)} = (pn + q) \cdot 2^n \). We determine the constants \( q, p \):

\[
(pn + q) \cdot 2^n = 4(pn - p + q) \cdot 2^{n-1} - 3(pn - 2p + q) \cdot 2^{n-2} + n2^n \\
(pn + q) \cdot 2^n = 2(pn - p + q) \cdot 2^n - \frac{3}{4}(pn - 2p + q) \cdot 2^n + n2^n
\]

After cancelling the term \( 2^n \) we get:

\[
pn + q = 2pn - 2p + 2q - \frac{3}{4}pn + \frac{1}{2}p - \frac{3}{4}q + n \\
- \frac{1}{4}pn - n = -\frac{1}{2}p + \frac{1}{4}q
\]

Therefore we must have \( p = -4 \) and \( q = -8 \). Thus we get:

\[ a_n^{(p)} = -4(n + 2) \cdot 2^n = -(n + 2) \cdot 2^{n+2} \]

The solution is:

\[ a_n = \alpha + \beta \cdot 3^n - (n + 2) \cdot 2^{n+2} \]

Find the particular solution to the given recurrence relation when \( a_0 = 1 \) and \( a_1 = 4 \).

We determine \( \alpha \) and \( \beta \) from the given initial conditions, as follows:

\[
a_0 = 1 = \alpha + \beta - 8 \\
a_1 = 4 = \alpha + 3\beta - 24
\]

We obtain: \( \alpha = -1/2 \) and \( \beta = 19/2 \) and the solution is in this case:

\[ a_n = -\frac{1}{2} + \frac{19}{2} \cdot 3^n - (n + 2) \cdot 2^{n+2} \]
7. What is the generating function for \( \{a_k\} \) where \( a_k \) is the number of solutions of 
\[
x_1 + x_2 + x_3 + x_4 = k
\]
when \( x_1, x_2, x_3 \) and \( x_4 \) are integers with \( x_1 \geq 3, 1 \leq x_2 \leq 5, 0 \leq x_3 \leq 4 \) and \( x_4 \geq 1 \)? Write your answer in closed form.

The restriction of \( x_1 \) gives the factor \( x^3 + x^4 + x^5 + \ldots \).
The restriction on \( x_2 \) gives the factor \( x + x^2 + x^3 + x^4 + x^5 \).
The restriction on \( x_3 \) gives the factor \( 1 + x + x^2 + x^3 + x^4 \).
The restriction on \( x_4 \) gives the factor \( x + x^2 + x^3 + \ldots \).

Thus the answer is the product of these:
\[
G(x) = (x^3 + x^4 + x^5 + \ldots)(x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \ldots)
\]

We use algebra to rewrite this in closed form:
\[
G(x) = x^5 \left(1 + x + x^2 + x^3 + x^4\right)^2 \cdot \frac{1}{(1-x)^2} = x^5 \frac{(1-x^5)^2}{(1-x)^4}
\]
8. Use your answer to problem 7 to find the coefficient $a_7$.

We want the coefficient of $x^7$ in the series:

$$G(x) = x^5(1 - 2x^5 + x^{10}) \cdot \frac{1}{(1 - x)^4}$$

which is the same as the coefficient of $x^2$ in the series for

$$\frac{1}{(1 - x)^4}$$

The coefficient of $x^2$ in $1/(1 - x)^4$ is $C(4 + 2 - 1, 2)$.

Thus the answer is:

$$C(4 + 2 - 1, 2) = \binom{5}{2} = 10$$
9. Find the number of positive integers \( \leq 1000 \) that are multiples of at least one of 3, 5, 11.

There are 333 integers divisible by 3.
There are 200 integers divisible by 5.
There are 90 integers divisible by 11.
There are 66 integers divisible by \( 3 \cdot 5 = 15 \).
There are 30 integers divisible by \( 3 \cdot 11 = 33 \).
There are 18 integers divisible by \( 5 \cdot 11 = 55 \).
There are 6 integers divisible by \( 3 \cdot 5 \cdot 11 = 165 \).

According to the inclusion-exclusion principle we get:

\[
    n = 333 + 200 + 90 - 66 - 30 - 18 + 6 = 515
\]
10. a) Does the graph $K_{2,5}$ have an Euler circuit? If not, does it have an Euler path?
b) Does the graph $K_{2,5}$ have a Hamilton path? Justify all your answers.

a) The graph $K_{2,5}$ has two vertices of degree 5 and five vertices of degree 2. Hence it has an Euler path and no Euler circuit.

b) There is no Hamilton path in this graph since any path containing all five vertices of degree 2 must visit some of the vertices of degree 5 more than once.
11. How many nonisomorphic simple graphs are there with six vertices and four edges?

There are 9 such graphs: one with a 4-cycle, two with a triangle, three with no cycles and all edges in the same connected component; three possibilities when there are two connected components (two paths of length two, a path of length three plus a single edge, a star with three edges plus a single edge).
12. Determine whether this graph is planar. Justify your answer.

The graph is not planar. The graph contains a subgraph isomorphic to $K_{3,3}$, using
{$1, 3, 5$} and {$2, 4, 6$} as the two vertex sets.
13. Give a recurrence relation for $e_n$ = the number of edges of the graph $K_n$.

$$e_n = e_{n-1} + n - 1$$
14. List all positive integers such that $C_n$ is bipartite. What is the chromatic number of the graph $C_n$? For full credit you have to justify all your answers.

$n$ must be even.

Chromatic number: if $n$ is even then $\chi(G) = 2$; when $n$ is odd $\chi(G) = 3$. 