Math 125A - Section 001
**Final Exam**
March 19, 2004

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*Show all work for full credit. Each problem is worth 20 points.*
1. Nine chairs in a row are to be occupied by six students and professors Alpha, Beta and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professors will be between two students. In how many ways can professors Alpha, Beta and Gamma choose their chairs? (Hint: the two end chairs must be occupied by students.)

**First solution:**
The two end chairs must be occupied by students, so the professors have seven middle chairs from which to choose, with no two adjacent. If these chairs are numbered from 2 to 8, the three chairs can be:

\[(2,4,6) \quad (2,4,7) \quad (2,4,8) \quad (2,5,7)\]
\[(2,5,8) \quad (2,6,8) \quad (3,5,7) \quad (3,5,8) \quad (3,6,8) \quad (4,6,8)\]

Within each triple, the professors can arrange themselves in 3! ways, so the total number is \(10 \times 6 = 60\).

**Second solution:**
Imagine the six students standing in a row before they are seated. There are 5 spaces between them, each of which may be occupied by at most one of the 3 professors. Therefore, there are \(P(5, 3) = 5 \times 4 \times 3 = 60\) ways the professors can select their places.
2. Two of the squares of a $4 \times 4$ checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

Let us assume that we labeled the board as in the picture:

```
1  2  3  4
5  6  7  8
9 10 11 12
13 14 15 16
```

You will receive credit if you count the possible colorings equivalent under rotations or under rotations and reflections.

If you consider both rotations and reflections you get the following:

For the first yellow square it will suffice to consider the positions from the upper left $2 \times 2$ square, that is $(1,2,5,6)$ only.
If the first yellow square is in position 1 then the inequivalent colorings are given by choosing for the second yellow square one of the following: $5, 6, 9, 10, 11, 13, 14, 15, 16$. So we have 9 inequivalent colorings in this case.

Let us assume that we place the first yellow square in position 5. We count all the possible positions of the second yellow square again, under the diagonal or on the diagonal, but excluding the corner squares (since $(5, 13)$ is equivalent to $(1, 9)$; $(5, 16)$ is equivalent to $(1, 15)$). We get: $6, 9, 10, 11, 14, 15$. There are 6 inequivalent colorings in this case.

If the first yellow square is in position 2, then the second can be in one of the positions $(5, 14, 15)$, all other colorings being equivalent with colorings previously counted. So there are 3 inequivalent colorings in this case.

Finally, let us assume that the first yellow square is in position 6. Then, for the second square we can choose one of the following positions $10, 11, 14$; all the other choices being counted already. Notice that: $(6, 13)$ is equivalent to $(1, 10)$; $(6, 16)$ is equivalent to $(1, 11)$; $(6, 9)$ is equivalent to $(5, 10)$; $(6, 15)$ is equivalent to $(5, 11)$.

Finally, we get $9 + 6 + 3 + 3 = 21$ inequivalent colorings.
3. What is the probability that a five-card poker hand contains cards of five different kinds? (Hint: there are 13 kinds and 4 suits in a 52 deck of cards.)

There are
\[ C(52, 5) = \frac{52!}{5! \cdot 47!} \]
different hands, and we assume by symmetry that they are all equally likely.

We need to count the number of hands that have 5 different kinds (ranks). There are \( C(13, 5) \) ways to choose the kinds. For each card, there are then 4 ways to choose the suit. Therefore there are

\[ C(13, 5) \cdot 4^5 = \frac{13!}{5! \cdot 8!} \cdot 4^5 \]
ways to choose the hand. Thus the probability is:

\[ P = \frac{C(13, 5) \cdot 4^5}{C(52, 5)} = \frac{\frac{13!}{5! \cdot 8!} \cdot 4^5}{\frac{52!}{5! \cdot 47!}} = \frac{13! \cdot 47! \cdot 4^5}{8! \cdot 52!} = \frac{3 \cdot 11 \cdot 4^3}{5 \cdot 17 \cdot 49} \approx 0.51 \]
4. How many ways are there to travel in $xyzw$ space from the origin $(0, 0, 0, 0)$ to the point $(4, 3, 5, 3)$ by taking steps one unit in positive $x$, positive $y$, positive $z$ or positive $w$ direction?

We can describe such a travel in a unique way by a sequence of 4 $x$’s, 3 $y$’s, 5 $z$’s and 3 $w$’s. Therefore there are

\[
\frac{(4 + 3 + 5 + 3)!}{4! \cdot 3! \cdot 5! \cdot 3!} = \frac{15!}{4! \cdot 3! \cdot 5! \cdot 3!} = 5 \cdot 7 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15
\]

such sequences.
5. Let $n$ be a positive integer. Show that:

\[
\binom{2n}{n+1} + \binom{2n}{n} = \frac{1}{2} \binom{2n+2}{n+1}
\]

The left hand side of the above gives:

\[
\binom{2n}{n+1} + \binom{2n}{n} = \frac{(2n)!}{(n+1)! \cdot (2n-n-1)!} + \frac{(2n)!}{n! \cdot (2n-n)!} = \\
= \frac{(2n)!}{(n+1)! \cdot (n-1)!} + \frac{(2n)!}{n! \cdot n!} = \\
= \frac{(2n)!}{(n+1) \cdot n! \cdot (n-1)!} + \frac{(2n)!}{n! \cdot n \cdot (n-1)!} = \\
= \frac{(2n)!}{n! \cdot (n-1)!} \left( \frac{1}{n+1} + \frac{1}{n} \right) = \\
= \frac{(2n)!}{n! \cdot (n-1)!} \frac{n + n + 1}{n(n+1)} = \frac{(2n+1)!}{n! \cdot (n+1)!}
\]

The right hand side of the equality we have to prove is:

\[
\frac{1}{2} \left( \binom{2n+2}{n+1} \right) = \frac{1}{2} \frac{(2n+2)!}{(n+1)! \cdot (2n+2-n-1)!} = \\
= \frac{1}{2} \frac{(2n+2)!}{(n+1)! \cdot (n+1)!} = \\
= \frac{1}{2} \frac{(2n+1)! \cdot (2n+2)}{(n+1) \cdot n! \cdot (n+1)!} = \\
= \frac{(2n+1)!}{n! \cdot (n+1)!}
\]

Since the two expressions are equal we have proved the equality and we are done.
6. Suppose that there are three possible final exam times: Monday, Wednesday and Friday mornings. Suppose that there are six courses to be scheduled: Anthropology, Calculus, Economics, English, History and Physics. Also suppose that the following table summarizes which courses have common students:

<table>
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<tr>
<th></th>
<th>Anthropology</th>
<th>Calculus</th>
<th>Economics</th>
<th>English</th>
<th>History</th>
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<tr>
<td>Physics</td>
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The $i,j$ entry is 1 if courses $i$ and $j$ have common students and 0 otherwise. Is it possible to schedule the final exams in such a way that no student has a schedule conflict?

Start by drawing a graph whose vertices are the 6 courses, two vertices are adjacent if the corresponding courses have students in common. A final exam scheduling will then correspond to a three coloring of the graph (that is, a coloring with three colors).

There is more than one answer to this problem. For example one choice would be:

- Monday: Anthropology, Economics
- Wednesday: Calculus, English
- Friday: History, Physics
7. Suppose that \( A(x) \) is the ordinary generating function for the sequence \((1, 3, 9, 27, 81, \ldots)\) and \( B(x) \) is the ordinary generating function for the sequence \((b_k)\). Find \((b_k)\) if:

\[
B(x) = \frac{dA(x)}{dx} + x + 1
\]

Let \((a_k)\) be the sequence whose ordinary generating function is \(A(x)\). We see that \(a_k = 3^k\) and thus:

\[
A(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 3^k x^k
\]

Therefore:

\[
\frac{dA(x)}{dx} = \sum_{k=0}^{\infty} k 3^k x^{k-1} = \sum_{k=1}^{\infty} k 3^k x^{k-1}
\]

After shifting the summation index we get:

\[
\frac{dA(x)}{dx} = \sum_{m=0}^{\infty} (m + 1) 3^{m+1} x^m = \sum_{k=0}^{\infty} (k + 1) 3^{k+1} x^k
\]

Thus the sequence generated by \(dA(x)/dx\) is \((c_k) = (3, 2 \cdot 9, 3 \cdot 27, 4 \cdot 81, \ldots)\).

The sequence generated by the function \(x + 1\) is \((d_k) = (1, 1, 0, \ldots)\).

Finally, we get:

\[
(b_k) = (c_k + d_k) = (4, 19, 3 \cdot 27, 4 \cdot 81, \ldots)
\]
8. Suppose that there are \( p \) kinds of objects. Assume that the supply of objects of each kind is \( M \) objects. Let \( b_k \) be the number of distinguishable ways of choosing \( k \) objects, if only an odd number of objects of each kind can be taken. Set up a generating function for \( (b_k) \).

There are two cases to consider: \( M \) is an odd number and \( M \) is an even number.

\( i. \) Let us assume first that the supply of each objects is \( M \), an odd number. Then, from each kind of objects we can pick 1, 3, 5 \ldots \( M \) objects. Hence we consider the expression:

\[
\left[(a_1 x)^1 + (a_1 x)^3 + \ldots (a_1 x)^M\right] \cdot \left[(a_2 x)^1 + (a_2 x)^3 + \ldots (a_2 x)^M\right] \ldots \left[(a_p x)^1 + (a_p x)^3 + \ldots (a_p x)^M\right]
\]

This becomes:

\[
\left(a_1 x + a_1^3 x^3 + \ldots a_1^M x^M\right) \cdot \left(a_2 x + a_2^3 x^3 + \ldots a_2^M x^M\right) \ldots \left(a_p x + a_p^3 x^3 + \ldots a_p^M x^M\right)
\]

Setting \( a_1 = a_2 = \ldots = a_p = 1 \) we obtain that the generating function for \( b_k \) is:

\[
G(x) = \left(x + x^3 + \ldots x^M\right)^p
\]

\( ii. \) Let us assume now that the supply of each objects is \( M \), an even number. Then, from each kind of objects we can pick 1, 3, 5 \ldots \( M - 1 \) objects. Repeating the steps from part \( i. \), with \( M - 1 \) replacing \( M \), we obtain for the generating function in this case:

\[
G(x) = \left(x + x^3 + \ldots x^{M-1}\right)^p
\]
9. Find the chromatic polynomial of the following graph:

![Graph Image]

Then find the number of ways to color the graph in at most 3 colors.

The chromatic polynomial of the graph is:

\[
P(G, x) = x^2(x - 1)^2(x - 2)^2 - 2x(x - 1)^4 + 4x(x - 1)^3 - 2x(x - 1)(x - 2) + x^2(x - 1)(x - 2) - 2x(x - 1)^2
\]

The number of ways to color the graph in at most three colors is obtained by evaluating the chromatic polynomial at \( x = 3 \):

\[
P(G, 3) = 36 - 96 + 96 - 12 + 18 - 24 = 18 \text{ ways}
\]
10. Find the coefficient of $x^5$ in the expansion of $(1 - 2x)^{-4} x^2$.

The Binomial Theorem gives:

$$(1 + y)^u = 1 + uy + \frac{u(u - 1)}{2!} y^2 + \frac{u(u - 1)(u - 2)}{3!} y^3 + \ldots$$

Set $y = -2x$ and $u = 4$ to get:

$$(1 - 2x)^{-4} = 1 + (-4)(-2x) + \frac{(-4)(-4 - 1)}{2!} (-2x)^2 + \frac{(-4)(-4 - 1)(-4 - 2)}{3!} (-2x)^3 + \ldots$$

Then, the coefficient of $x^5$ in the expansion of $(1 - 2x)^{-4} x^2$ is:

$$\frac{(-4)(-4 - 1)(-4 - 2)}{3!} \cdot (-2)^3 = \frac{4 \cdot 5 \cdot 6 \cdot 8}{2 \cdot 3} = 160$$