Extreme Value Theorem, Absolute Extrema, and Optimization

By: Grace Clarke, Grace Kendall, and Ryan Barock
Extreme Value Theorem

If a function is **continuous** on a **closed interval** $[a,b]$, then $f$ has both an absolute maximum value and an absolute minimum value on our interval $[a,b]$.

- If it’s not continuous, then the Extreme Value Theorem does not work
- If it’s on an open interval, then the Extreme Value Theorem does not work
Using the Extreme Value Theorem

1. Establish that the function is continuous on the closed interval
2. Determine all critical points in the given interval and evaluate the function at these critical points and at the endpoints of the interval
3. The largest and smallest values from step two will be the maximum and minimum values, respectively
Extreme Value Theorem examples

Find the maximum and minimum values of \( f(x) = x^4 - 3x^3 - 1 \) on \([-2, 2]\).

1. The function is continuous on \([-2, 2]\) since the function is a polynomial.

2. \( f'(x) = 4x^3 - 9x^2 = 0 \)
   
   \[ X^2 (4x-9) = 0 \]
   
   \[ X = 0 \ x = 9/4 \]
   
   9/4 is not on the interval.

3. Maximum at \( x = -2 \) Minimum at \( x = 2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>39</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>-9</td>
</tr>
</tbody>
</table>
Absolute Extrema

**Absolute Maximum**: Let $c$ be a number in domain of $f$. Then $f(c)$ is the absolute maximum value of $f$ if $f(c) \geq f(x)$ for all $x$ in the domain.

- Simply put, this is the highest point on the domain of the function.

**Absolute Minimum**: Let $c$ be a number in the domain of $f$. If $f(c) \leq f(x)$ for all $x$ in the domain of $f$.

- Simply put, this is the lowest point on the domain of the function.
Steps to Find Absolute Extrema

1. Verify the function is continuous on a closed interval.
2. Find all critical points of the function that are on the closed interval.
3. Evaluate the function at the critical points (find y-values).
4. Evaluate the function at the end points (find y-values).
5. Identify the largest and smallest y-values.

Example:

Absolute Maximum: \((e, f(e))\)
Absolute Minimum: \((a, f(a))\)
Absolute Extrema Word Problem

The altitude (in feet) attained by a bird when searching for food at \( t \) seconds is given by the function \( h(t) = -\frac{1}{3}t^3 + 4t^2 + 20t + 2 \) \( (t \geq 0) \) Find the absolute maximum.

\[ h'(t) = -t^2 + 8t + 20 \]

\[ 0 = -t^2 + 8t + 20 \]

\[ t = 10, t = -2 \] \( \leftrightarrow \) does not work with the domain

\[ h(0) = 2 \]

\[ h(10) = \frac{806}{3} \]

Absolute max
Optimization

Optimization in calculus involves finding the optimal value of a quantity. In optimization problems, you are sometimes given the function, and sometimes you must find the appropriate function to optimize. Optimization often has constraints that must be considered, such as the length or height of something. The constraints are often helpful when solving optimization problems. Optimal values are often either the maximum or minimum values of a function.

There are many real-world uses for optimization, such as:

- a company finding out maximum profit, or minimizing average cost
- an engineer finding a container with the optimum shape and volume
- a doctor figuring out maximum concentration of a drug for a patient.
Optimization Strategy

1. Assign a letter to each variable mentioned in the problem. Draw a picture if possible.
2. Find an expression for the quantity to be optimized.
3. Use the given conditions to write Step 2 as a function of one variable. Note any physical restrictions.
4. Use methods to optimize.
Optimization Word Problem (given function)

(p. 301) Acrosonic’s total profit (in dollars) from manufacturing and selling x units of their model F loudspeaker systems is given by:

\[ P(x) = -0.02x^2 + 300x - 200,000 \quad (0 \leq x \leq 20,000) \]

How many units of the loudspeaker system must Acrosonic produce to maximize its profits?

1. \[ P'(x) = -0.04x + 300 \] so \( x = 7500 \)
2. \[ P(0) = -200,000 \quad P(7500) = 925,000 \quad P(20,000) = -2,200,000 \]
3. So, we see that the absolute maximum value of the function \( P \) is 925,000. By producing 7500 units, Acrosonic will achieve a maximum profit of $925,000.
Optimization problem (not given function)

(p. 313) A man wishes to have a rectangular-shaped garden in his backyard. He has 50 feet of fencing with which to enclose his garden. Find the dimensions for the largest garden he can have if he uses all of the fencing.

1. Make x and y the dimensions representing adjacent sides of the garden. Let A denote the Area.
2. A=xy is the quantity to be maximized
3. The perimeter (2x+2y) must equal 50 feet. So y=25-x
4. Which means A=x(25-x), and A’=-2x+25. So, x=12.5
5. f(0)=0, f(12.5)=156.25, f(25)=0
6. The absolute maximum value of the function f is 156.25, where x and y both equal 12.5. Thus, the garden of maximum area is a square with sides of length 12.5 feet.