Problem Set 1

(1) This problem is another area computation using an "in-and-out" argument similar to the one found in the *Nine Chapters* for the gou-gu theorem. Consider the parallelogram in the plane with vertices as pictured in Figure 1 at the points $(0,0)$, (a,b), (c,d), and $(a+c, b+d)$ (the drawing implicitly assumes that $\frac{b}{a} < \frac{d}{c}$–why?–and we will assume this). Label in terms of a, b, c, d all the side-lengths marked with a red line (these are all parallel to the x or y-axes). Then show that the area of the parallelogram is $ad - bc$ by computing the area of the large rectangle and then subtracting the areas of the four triangles and the two small rectangles.

(2) In class we used a "secant method" to classify rational solutions to $x^2 + y^2 = 1$ (recall: we drew lines with rational slope through $(0,1)$, a fixed rational point on the circle, and determined where else they intersected with the circle), and thus to classify integral solutions to $a^2 + b^2 = c^2$. Mimic the method, but now for an ellipse rather than a circle, to (a) classify (in terms of a single "slope" parameter as we did in class) rational solutions to $2x^2 + y^2 = 1$; and (b) produce an integral solution to $2a^2 + b^2 = c^2$ with a and b both at least 10 and not sharing any common factor.

(3) Compute using Euclid's algorithm the greatest common divisors of the following pairs of integers: $(101, 47)$; $(2524, 196)$.

(4) For what value of x is a rectangle with sides x and 1 *similar* to the rectangle (shaded red in the picture) obtained by cutting out a square of side 1 from the x-by-1 rectangle? (See picture.) Now perform the analogous operation on the smaller rectangle: cut out the square with side length equal to $x - 1$, and show the remaining rectangle (shaded red again) is again similar to the previous two rectangles. Repeat this procedure–if we had constructed the side labeled x by the geometric procedures of *Elements* II, what results would be an example of anthyphairesis applied to geometric magnitudes–and explain why this procedure must continue indefinitely (i.e., none of the smaller rectangles produced is ever a square), and why this implies x must be irrational (i.e., the sides of our original rectangle are incommensurable).

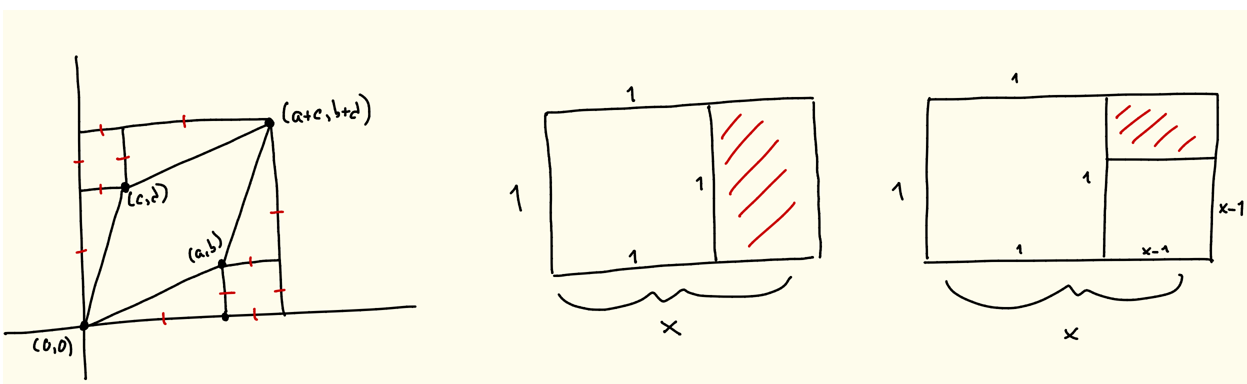


FIGURE 1. The left diagram is for problem 1. The two diagrams to the right are for problem 4.

Problem Set 2

(1) Find an integer solution to the equation $118x - 27y = 1$ using the kuṭṭaka.

(2) Find an integer x whose remainder when divided by 71 is 3 and whose remainder when divided by 45 is 2. Use Euclid's algorithm and the counting-board method encapsulated in our "magic table."

(3) Compute the first few steps in the continued fraction expansion of $\sqrt{2}$ (i.e. execute anthyphairesis on the pair $(\sqrt{2}, 1)$), and conjecture what form the full continued fraction takes. Set up a quadratic equation (as we did in class for $\frac{1+\sqrt{5}}{2}$) that allows you to show that this continued fraction expansion is correct.

(4) One solution to the equation $x^2 - 2y^2 = -1$ is given by $(x, y) = (1, 1)$.
   (a) Use the "composition" idea that underlies the chakravala to produce a solution to $x^2 - 2y^2 = +1$ with x and y both integers greater than 10.
   (b) Use the "magic table" generated from the continued fraction to find one more solution (not found in the previous part) to $x^2 - 2y^2 = 1$ with x and y integers greater than 10.

Writing Assignment 1

Respond to *one* of the following topics. In all cases, your responses should be clearly organized and should make specific reference (with citation) to the primary texts. You should address along the way each of the questions contained in your chosen topic, but your essay should not just be a bullet-point response, but rather a reflection on the topic that synthesizes your thoughts on the various individual questions. You may want to (and in some cases will have to) read beyond our assigned passages. Do not hesitate to take a stance, nor to raise a question that you find puzzling and might want to think about more. 600-750 words is an appropriate length.

(1) Try reading Book I Proposition 2 of the *Elements* without reference to the diagram. In what sense are the points G and L (all references here are to the Joyce webpage) "underspecified" when they are first introduced? At what point in the argument, if any, is the ambiguity resolved? Similarly analyze Book 1 Proposition 3. Reread (and possibly read more of) other geometrical arguments in the *Elements* and discuss the relationship between the lettered diagrams and the verbal arguments. Identify and compare cases where the verbal argument seems to be totally sufficient for the proof, and where the diagram seems to be essential to guide the reader through the argument. Speculate on what significance your observations might have for how we think about Euclid or other Greek geometers.

(2) Study the Definitions in Book I of the *Elements*. Considering their use in specific propositions of Book I, what role do the definitions seem to be playing? Do your preconceptions about the role of the definitions change if you know–as is in fact the case–that in manuscripts of Euclid the definitions are not separately numbered at the start of the book, but rather are written one after the other in an introductory prose paragraph? Read sections 1-2 (pages 2-3 of the edition posted on Canvas) of Hilbert's *Foundations of Geometry*, and compare and contrast in Hilbert and Euclid the role of the definitions and their relationship to the axioms and propositions.

(3) Read Āryabhaṭa's explanation of the "pulverizer" (kuṭṭaka) in the *Sourcebook*, along with Bhāskara's commentary (paragraphs 2.32-2.33 & commentary on page 415-16; remember the *Sourcebook* is on reserve in the library, and there are scanners right by the reserve shelf). Give a phrase-by-phrase explanation of Āryabhaṭa's description of the method; what is included, and what is omitted or only implicit? What does Bhāskara's commentary add? Work out another example of the kuṭṭaka, explaining which steps in Āryabhaṭa's explanation corresponds to your arithmetic steps. Once you have understood the method carefully, can you speculate on how it might have been discovered?

(4) Return to the three versions of (in our terms) "solving a quadratic equation" that we encountered in the first week of class: the Babylonian tablet YBC 6967, Proposition II.11 of Euclid's *Elements*, and the passage from Chapter V of al-Khwārizmī's *Al-jabr wa'l-muqābala*. As we have mentioned, what is or is not classified as "algebra" has been a source of considerable controversy in recent history-writing; but as with many phenomena, the label we choose for it is a poor substitute for thinking carefully about it. Go through these three texts carefully, and describe similarities and differences between them. Formulate a (provisional! you need not swear by this at any later point) definition of "algebra" or "algebraic reasoning." Explain what motivates your definition, and describe to what extent each of these texts conforms to your definition.

Problem Set 3

(1) Use your favorite integration method to show that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

(2) Use your favorite integration method to show that the surface area of a sphere with radius r is $4\pi r^2$.

(3) In this problem,[1] you will use your knowledge of calculus to give another proof of Archimedes' theorem on the quadrature of parabolas. Consider the parabola $y = ax^2$ for some $a > 0$ (the quadrature of any parabola in the plane can be reduced to that of one of this form by rotating and shifting the parabola). Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be any two points on the parabola. Let R be the (generalized) vertex of the parabola given by the intersection of the parabola with the (generalized) axis through the midpoint of PQ.

   (a) What is the equation of the line through P and Q? What is the midpoint of the segment PQ? What are the coordinates of the point R?

   (b) Compute the area of the triangle PQR.

   (c) Compute using integration the area of the parabolic segment PQR.

   (d) Confirm the result of Archimedes' theorem: the parabolic segment has area equal to $4/3$ that of the triangle.

   Note: if you are having trouble with the algebra in this problem, you may carry out the argument in the special case $a = 1$, $P = (1, 1)$, and $Q = (-2, 4)$, and you will receive partial credit.

[1] This problem will count as two problems when graded.

Writing Assignment 2

Respond to *one* of the following topics. In all cases, your responses should be clearly organized and should make specific reference (with citation) to the primary texts. You should address along the way each of the questions contained in your chosen topic, but your essay should not just be a bullet-point response, but rather a reflection on the topic that synthesizes your thoughts on the various individual questions. You may want to (and in some cases will have to) read beyond our assigned passages. Do not hesitate to take a stance, nor to raise a question that you find puzzling and might want to think about more. 600-750 words is an appropriate length.

(1) Familiarize yourself with some of the basic methods of the counting-board (you might wish to play with toothpicks or some such), and then study the counting board implementation of the root extraction methods in *The Nine Chapters*. (You will have to look up the Shen-Crossley-Lun book in the library: please just scan or photocopy pages you wish to read, so that the book remains available for other students.) Come up with your own examples of quadratic equations, and explain their solutions on the counting-board—your explanation should include diagrams representing the procedure, which you may draw by hand. Also give in parallel a symbolic "algebraic" explanation of the procedure. Compare these two approaches at a practical and/or conceptual level.

(2) In *Quadrature of Parabolas*, Archimedes begins with yet another proof (that we did not discuss), of the quadrature of parabolas, culminating in Proposition 17 of the book. Like the argument in *The Method*, this proof uses physical principles like the law of the lever. Study this argument, and compare and contrast it with the argument in *The Method*. Discuss carefully how each argument works with—or works around—concepts of the infinite that arise when studying non-rectilinear areas.

(3) Read through the definitions and postulates in *Sphere and Cylinder I*. Carefully go over all of the ingredients in Archimedes' proof of the calculation of the surface area of a sphere, and illustrate the use of the various postulates. Some of these postulates are particularly interesting, because they are from a modern perspective interesting mathematical results capable of proof via analytic methods. Read the section of the commentary of Eutocius (also translated in Netz's edition of *Sphere and Cylinder I*, to which we have unlimited ebook access via the university library) on Archimedes' definition and postulates in *Sphere and Cylinder I*: how does Eutocius understand the role of the postulates? In your view, how does Archimedes understand the role of the postulates?

(4) Come up with your own topic. Whatever it is, you should state it clearly, and your essay should involve detailed reading of at least one primary text.

Problem Set 4

(1) Al-Kuhi studied the centers of gravity of the following series of shapes. Fix a semicircle ABG with diameter AG, center (of the corresponding circle) D along AG, and BD perpendicular to AG (see figure below). Then form the triangle ABG and the (segment of the) parabola ABG, as pictured. Al-Kuhi considered both these plane figures and the solids obtained by rotating them about the line BD (sweeping out a hemisphere, a cone, and a paraboloid). He announced in a letter to al-Sabi that the centers of gravity of the various planar and solid figures lie along BD at points P dividing BD in the following ratios:

(a) Triangle: $DP/DB = 1/3$.
(b) Cone: $DP/DB = 1/4$.
(c) Parabola: $DP/DB = 2/5$.
(d) Paraboloid: $DP/DB = 2/6$.
(e) Semicircle: $DP/DB = 3/7$.
(f) Hemisphere: $DP/DB = 3/8$.

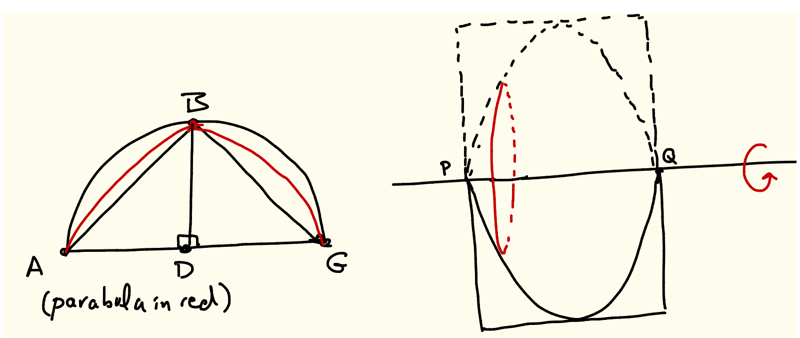
Such calculations are great feats prior to the introduction of calculus, but they are not all correct. Use calculus to derive the correct results for the planar figures (triangle, parabola, semicircle). Here is an outline of one way to proceed: we may assume the circle has radius 1, and that the points A and D are $(0,0)$ and $(1,0)$. For any of the figures considered, let $f(x)$ denote the height at x, and let A denote the total area, so that $A = \int_0^2 f(x)dx$. Calculus tells us that the point P is then $(1, p)$ where

$$p = \frac{\int_0^2 \frac{f(x)^2}{2} dx}{A}.$$

Use this formula to compute p for each of the three shapes. For the semicircle, you will find that your answer does not agree exactly with that of al-Kuhi: what value of π would be implied by the semicircle answer? (Our reading from the *Sourcebook*, 572-573, is in fact al-Kuhi's response to a contemporary who questioned the result of his calculation, since it seemed at variance with Archimedes' approximations to π. See 568-572 of the *Sourcebook* for more of this exchange.)

(2) This week we will study Ibn al-Haytham's work on the volume of a paraboloid. Consider a parabola and a line perpendicular to its axis that intersects the parabola in points P and Q and thereby determines a parabolic segment. Consider the solid obtained by rotating the parabolic segment about the line. Also consider the cylinder obtained by rotating the rectangle whose top is PQ and whose bottom is obtained by translating PQ so that it is tangent to the parabola at the vertex. (See picture.) Show that the volume of the paraboloid is $\frac{8}{15}$ of the volume of the cylinder in the following steps:

(a) Write down an equation of the parabola and an equation of the line through P and Q.
(b) Compute the volume of the cylinder (in terms of the parameters from the previous part).
(c) Set up and compute an integral that computes the volume of the paraboloid, and compare it to your cylinder calculation.



Problem Set 5

(1) Following al-Khayyam (and "translating" his work into analytic geometry), find the equations of two conics in the plane that intersect at a point (x, y) such that $x^3 - x + 2 = 0$.

(2) Continue with the cubic equation $x^3 - x + 2 = 0$ from the previous problem. How many real solutions does it have? Adapt the cubic formula from the case we discussed in class to find all real solutions. Be careful about the signs, since in class we gave Tartaglia's formula for solving $x^3 + cx = d$ when c and d were both positive.

(3) Consider a right spherical triangle ABC with right angle at A. Label as usual the sides opposite the angles A, B, and C as a, b, and c. Suppose that $b = \frac{\pi}{6}$, $c = \frac{\pi}{3}$.
   (a) Draw a picture of such a configuration.
   (b) Use the spherical law of cosines discussed in class to determine $\cos(a)$, and consequently $\sin(a)$.
   (c) Then use the spherical law of sines to determine A, B, and C.
   (d) Combine your knowledge of A, B, and C with Girard's theorem to compute (a decimal approximation is fine) the area of the spherical triangle ABC.

(4) The classical definition of the conic sections as the intersections of a cone (extending infinitely in both directions) with a plane allows for the following reinterpretation of the conics from the point of view of projective geometry: *any conic section is the central (one-point) projection of a circle onto a plane*. Indeed, a cone is simply the configuration of lines obtained by fixing a circle and a point, and drawing all lines (infinitely extended) connecting the fixed point to each point on the circle. Thus, the intersection of a plane and a cone is precisely the projection of a circle from the cone-point onto the plane. **Your task:** for each of the conic sections (parabola, ellipse, hyperbola), show that you have understood this argument by drawing an example of a central projection that maps a circle onto this conic. For the hyperbola case, how does it happen that the circle, which is only one connected piece (you can draw it without picking up your pencil), gets mapped to a hyperbola that is two separate pieces (you have to lift up your pencil to draw the two separate components)? (Bonus: is it possible to choose the plane so that the circle does not project to either a parabola, an ellipse, or a hyperbola?)

Problem Set 6

(1) Read the statements of results X.5 and X.11 from the translation (see next page) of Isaac Barrow's *Geometrical Lectures*, and then do the following:
  (a) Translate X.5 into modern terms as follows: let AP denote the x-axis, and let AEG be the graph of a function $y = f(x)$. Describe the curve AFI in these terms, i.e., if $AD = x$, what is DF in terms of x? Then state in terms of these coordinates the conclusion of X.5 ("RF touches the curve AFI"), and verify this using your knowledge of calculus.
  (b) Translate X.11 into modern terms, just as you did with X.5. Let ZGE be the graph of $y = g(x)$, explain how the curve AIF is defined, and state in familiar terms, and verify using your knowledge of calculus, the conclusion of X.11.

(2) Recall Newton's general form of the binomial theorem: for all real numbers α,

$$(1+t)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} t^i,$$

where $\binom{\alpha}{i} = \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}$.
  (a) Check that Newton's formula agrees with the Taylor series (centered at $t = 0$) of $(1+t)^\alpha$.
  (b) Check in this generality the familiar binomial identity $\binom{\alpha-1}{i-1} + \binom{\alpha-1}{i} = \binom{\alpha}{i}$.
  (c) In class, we showed how Newton used this formula with $\alpha = \frac{1}{2}$ to deduce a power series expression $\sin^{-1}(y) = y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \ldots$. We then carried out the first steps of Newton's procedure for inverting (in the sense of function composition) this expression to determine the power series for $\sin(x)$. Carry out this calculation carefully, justifying the expansion $\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \ldots$ as far as the x^7 term.

Problem Set 7

(1) Newton used the following claims about ellipses in Proposition 11 (Book 1, Section 3) of the *Principia*. For the letter labeling, refer to the diagram accompanying the proposition in the text.

 (∗) The angles $\angle SPR$ and $\angle HPZ$ are equal, i.e. in an ellipse with foci S and H containing a point P with tangent $\overline{RPZ}$ at P, the angles $\angle SPR$ and $\angle HPZ$ are equal. (Apollonius, *Conics*, Proposition III-48.)

 (∗∗) The product $\overline{AC} \cdot \overline{BC}$ equals the product $\overline{PF} \cdot \overline{CD}$ ("the parallelograms spanned by the semi-axes of any two conjugate diameters all have equal areas"). (Apollonius, *Conics*, Proposition VII-31.)

Complete the attached (see next page) exercises 4.23(a)-(e) from Katz's *A History of Mathematics* text to establish the claim (∗∗) about areas spanned by conjugate diameters. You do not have to explain the first claim (∗) about the angles, but see the later exercises 29-32 in Katz Chapter 4 for an overview of Apollonius's development of this material.

Problem Set 8

(1) Use Hilbert's "axioms of connection" (*Foundations of Geometry* section 2) to give a careful proof of Theorem 1 (page 3). Be sure you note explicitly just which axioms you are using and when.

(2) For each of the following sets, identify whether it has (a) smaller cardinality than $\mathbb{N}$ (i.e., is finite); (b) the same cardinality as $\mathbb{N}$; (c) the same cardinality as $\mathbb{R}$; or (d) larger cardinality than $\mathbb{R}$. Explain your reasoning carefully (you may use results proven in class).
   (a) The Cartesian plane $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.
   (b) The set of prime numbers.
   (c) The set of functions $f : \mathbb{R} \to \{0, 1\}$.
   (d) The set of solutions (in the complex numbers) to *all* equations of the form $x^2 + ax + b = 0$, where a and b are *integers*.
   (e) Bonus: The domain Ω defined in section 9 of Hilbert's *Foundations of Geometry*.

Exam Information

The exam is Friday December 13, 10:30am-12:30pm in our usual classroom, LCB 215. The exam will consist of three parts:

(1) **Math problems:** There will be three questions, based on the following:
   (a) One problem very closely related to your assigned primary readings from Newton's calculus.
   (b) One problem comparing and/or contrasting aspects of Euclidean and non-Euclidean geometry. For the latter, keep in mind our in-class discussion, reading materials, and homeworks on projective geometry and spherical geometry from earlier in the course, and the Hilbert and Poincaré readings from more recently.
   (c) One problem on cardinality, and specifically on the Cantor diagonalization argument (see Stillwell reading, class notes, and problem set 8).

(2) **Passage identification:** There will be a *matching* section where you will have to match excerpts from our primary readings to a list of authors and/or works. These will be drawn from the entire course.

(3) **Essay:** There will be one essay on the history of conic sections. It will test your understanding of the many ways conic sections have appeared throughout our course, and the different theoretical perspectives mathematicians have brought to the study of conic sections. You should be prepared to reference in detail primary readings from throughout the class, and to discuss broader trends in the history of mathematics with reference to the history of conics.

Math 3010 Final Exam

December 13, 2019

Complete parts 1 and 2 (math problems and passage identification) on the exam itself. Complete part 3 (essay) on a separate sheet of paper. In part 1, the number in parentheses after each sub-question indicates how many points it is worth. (The exam total is 66.7 points.) Good luck!

Your Name (1 point): ______________________________

1. Problems (9 points each)

(1) The following question is a simpler version of Problem III Example 1 from Newton's 1671 *Treatise on fluxions*: Find the greatest and the least values of x satisfying the equation

$$x^2 + x - xy + y^2 = 0.$$

(For 1 bonus point: When you consider the equation $x^2 + x - axy + y^2 = 0$ for varying values of a, what kinds of curves arise when $a = 1$, $a = 2$, $a = 3$?) (9)

(2) The following question is about Euclidean and non-Euclidean geometry.

(a) How are "points" and "lines" defined in spherical geometry? (The question about points is not a trick! For 1 bonus point, how does elliptic geometry define points differently from spherical geometry, and what Euclidean axiom does this restore?) (2)

(b) Contrast the possible values for the *sum of the angles in a triangle* in spherical, Euclidean, and hyperbolic geometry. For each of these geometries, carefully draw a picture of a triangle that exemplifies the contrast (for hyperbolic geometry, draw your triangle in the Poincaré disk model, which accurately reflects angle measurements). (4)

(c) On a sphere of radius 1, what is the area of a spherical triangle with three angles equal to $\frac{\pi}{2}$? (3)

(3) The following question is about the formal understanding of the infinite that developed in the 19^{th} century.

(a) Define what it means for two sets S and T to have the same cardinality ("size"). (2)

(b) Recall that for any set S, the power set $\mathcal{P}(S)$ is the *set* whose *elements* are the *subsets* of S. (Remember that the "empty set" and S itself are both considered to be subsets of S.) How many elements does $\mathcal{P}(S)$ have when S has 3 elements? (2)

(c) Show that $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ do not have the same cardinality. (For 1 extra point you may if you wish instead prove the more general statement that for any set S, S and $\mathcal{P}(S)$ do not have the same cardinality.) (5)

2. Passage Identification (22 points)

2.1. **Matching (20 points):** Match ten of the members of the following list of works to the ten texts given on the next page. You will not use any work from this list more than once, and you will not use seven of the listed works. There are no tricks on this list: every text listed is one from which you have read selections, so you need not worry that, for instance, the Wallis text you read was some *other* text by Wallis. **Write the number labeling the author/work next to the appropriate passage in the margin on the next page.** (Feel free to separate the pages of the exam if this is more convenient.)

I have preserved the text as in your actual reading assignments, with the following exceptions. In passage (A) I have replaced a proper name with "[he]." I have also replaced some older fonts (e.g., a 17$^{\text{th}}$ century "long s") with more familiar fonts. Note that because most texts are in translation, some of the notation or symbolism in the translation (which is what you read) is not as in the original work; however, as a hint, passage (G) *does* represent the author's notation.

(1) Anonymous, *Zhou bi suan jing*
(2) Archimedes, *Quadrature of Parabolas*
(3) Archimedes, *Sphere and Cylinder*
(4) Descartes, *Geometry*
(5) Descartes, *Discourse on the Method*
(6) Euclid, *Elements*
(7) Fermat, *Various mathematical works*
(8) Hilbert, *Foundations of Geometry*
(9) Al-Kāshī, *The Reckoner's Key*
(10) Al-Khayyām, *Algebra*
(11) Al-Khwārizmī, *Treatise on Hindu Reckoning*
(12) Liu Hui, Li Chunfeng, et al., *Nine Chapters on the Mathematical Art*
(13) The Banū Mūsā, *Apollonius's Conic Sections*
(14) Newton, *Mathematical Principles of Natural Philosophy* (which we usually called the *Principia*)
(15) Newton, *Treatise on Fluxions*
(16) Plato, *Republic*
(17) Wallis, *On Conic Sections*

2.2. **For free (2 points):**

(1) What text (out of all that we read) did you most enjoy reading this term? ______________________

(2) What text did you have the hardest time reading this term? ______________________

(**Write your answers in the left-hand margin.**)

(A) "Then [he] through his prowess and superiority in the science of geometry, succeeded in the theory of the science of the section of the cylinder…And he discovered the proof of the fact that to every elliptical section of this type there corresponds some cylinder which contains the equivalent section. So [he] at that point composed a treatise on what he had discovered of that science and died."

(B) "But when X was put in the place of one and was made in the second place, and its form was the form of one, they needed a form for the tens because of the fact that it was similar to the form of one, so that they might know by means of it that it was X. So they put one space in front of it and put in it a little circle like the letter o…"

(C) "Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal."

(D) "Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height."

(E) "Furthermore, suppose that the volume of the cylinder means square rate 12 and that of its inscribed sphere means circle rate 9, so that the volume of the sphere is 3/4 that of the circumscribed cylinder …But the supposition is incorrect. How can this be verified? Take eight cubic blocks with 1-*cun* sides to form a cube with a 2-*cun* side. Cut the cube horizontally by two identical cylindrical surfaces perpendicular to each other, 2 *cun* both in diameter and in height, then their common part looks like the surface of two four-ribbed umbrellas put together." [*cun* is a unit of measurement]

(F) "If, then, we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction—to those that are unknown as well as to those that are known. Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between these lines, until we find it possible to express a single quantity in two ways. This will constitute an equation…"

(G) "I suppose at the start (according to the *Geometria indivisibilium* of Bonaventura Cavalieri) that any plane whatever consists, as it were, of an infinite number of parallel lines. Or rather (which I prefer) of an infinite number of parallelograms of equal altitude; of which indeed the altitude of a single one is $\frac{1}{\infty}$ of the whole altitude, or an infinitely small divisor; (for let ∞ denote an infinite number); and therefore the altitude of all of them at once is equal to the altitude of the figure."

(H) "Now since the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little accessions of the flowing Quantities x and y, by which those Quantities are increased through the several indefinitely little intervals of Time; it follows, that those Quantities x and y, after any indefinitely small interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$."

(I) "Having shown that the axioms of the above system are not contradictory to one another, it is of interest to investigate the question of their mutual independence. In fact, it may be shown that none of them can be deduced from the remaining ones by any logical process of reasoning."

(J) "But, as for the proof concerning these kinds [of equations], if the subject of the question is simply a number, neither we nor any of the algebraists have been able to do it except in the three first degrees: number, thing, and *māl*. But perhaps someone else, who will come after us, will know [how to do] it."

3. Essay (16 points)

On separate paper, discuss the following in a *well-structured essay that makes detailed reference to our course materials*. Weave in as many of our texts as you can; 4-6 is a reasonable number, depending on how much detail you go into about each.

> The theory of conic sections has been a steady companion for us throughout our course. **What about this science made it a stimulus to mathematical inquiry for two millenia?** As long as your essay is structured and closely references our course materials, you may respond to this question in any way you like; if you wish, you may take the following more detailed prompt as your starting-point:
>
> From the point of view of modern analytic (algebraic) geometry, the science of conic sections is the theory of plane curves of degree two, which is notably simpler than the theory of higher-degree equations. Historically, the conic sections both (i) serve as a testing-ground for an era's mathematical technology, as it tries to prove itself on the simplest non-linear problems; and (ii) model more complex phenomena than linear equations, and so serve as a tool for studying more complicated problems than would be accessible otherwise. Elaborate on this dual role of conic sections as both a tool and a proving-ground for other mathematical theories.

4. Freebie (.67 points)

Pick a quote, any quote (circle one for .67 points):

(1) "The essence of mathematics lies precisely in its freedom."–Georg Cantor
(2) "Mathematics is the science of the infinite."–Hermann Weyl
(3) "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."–Eugene Wigner
(4) "Mathematics is the art of giving the same name to different things."–Henri Poincaré