EFFECTIVE BOUNDS FOR ADJOINT BLOCH–KATO SELMER GROUPS OF HILBERT MODULAR FORMS

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Abstract. Let F be a totally real field and let Π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Under the usual Taylor–Wiles residual irreducibility hypothesis, we prove the vanishing of the adjoint Bloch–Kato Selmer group with characteristic-zero coefficients and give an explicit upper bound for a corresponding Selmer group with divisible coefficients. The argument revisits Wiles's use of level raising at Taylor–Wiles primes. In place of freeness or almost freeness of automorphic cohomology over a Hecke algebra, we compare congruence modules at the original and auxiliary levels and control the part of the Hecke cotangent space detected by the automorphic module. This produces Galois cohomology classes with prescribed ramification at the auxiliary primes. The resulting bounds depend explicitly on congruence data at the original automorphic level and on torsion in cotangent spaces of local deformation rings.

1. Introduction

Let Π be a regular C-algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ over a totally real field F, and let $\rho = \rho_{\Pi,\iota}\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representation associated to Π (for a choice of isomorphism $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$). For some finite extension $E/\mathbb{Q}_p$, with ring of integers $\mathcal{O}$, we can assume (replacing ρ by a conjugate) ρ factors through $\mathrm{GL}_2(\mathcal{O})$. We can then consider different versions of the Bloch-Kato Selmer groups for the trace-zero adjoint representation. Let $W = \mathrm{ad}^0(\rho)$, $W_E = \mathrm{ad}^0(\rho) \otimes E$, and $W_{E/\mathcal{O}} = \mathrm{ad}^0(\rho) \otimes_{\mathcal{O}} E/\mathcal{O}$. Then if S is a finite set of places such that Π is unramified outside S, and we let $\Gamma_{F,S} = \mathrm{Gal}(F(S)/F)$ denote the Galois group of the maximal unramified outside S extension of F, we first have the Selmer groups $H^1_g(\Gamma_{F,S}, W_E)$ and $H^1_f(F, W_E)$: the former classifies lifts of $\rho \otimes E$ to $\mathrm{GL}_2(E[\epsilon])$, with fixed determinant $\det(\rho)$, that are unramified outside S and de Rham at all places $v|p$; the latter consists of cohomology classes that are unramified away from p and at $v|p$ lie in the local Bloch-Kato group $H^1_f(\Gamma_{F_v}, W_E)$, which in fact in this case coincides with the previous local condition $H^1_g(\Gamma_{F_v}, W_E)$ that the lift be de Rham (see [All16, Remark 1.2.9]). Then the Bloch-Kato conjecture (building on conjectures of Deligne, Beilinson, and Fontaine-Mazur) in this setting first asserts that the groups $H^1_f(F, W_E)$ and $H^1_g(\Gamma_{F,S}, W_E)$ vanish, a prediction whose automorphic interpretation is the familiar fact that $L(\mathrm{ad}^0(\Pi), 1)$ is non-zero.

Bloch and Kato's innovation was moreover, at least when one has a motivic interpretation of ρ, to predict the special value of the L-function in terms of a transcendental period, local Tamagawa factors (arising from places in S where ρ is ramified), and the torsion variant of the above Selmer group, $H^1_f(F, W_{E/\mathcal{O}})$, where the local conditions are by definition the images of the local conditions for W_E; this can be translated to predictions about the size of the more relaxed Selmer groups $H^1_g(\Gamma_{F,S}, W_{E/\mathcal{O}})$ via global duality (as in [DFG04, Lemma 2.1]). Galois deformation theory relates

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both the rational and the torsion Selmer groups to tangent spaces of suitable Galois deformation rings R for $\bar{\rho}$ with local conditions at the primes in S. Set up properly, if $\phi\colon R \to \mathcal{O}$ is the augmentation corresponding to the deformation ρ, with $\mathfrak{p} = \ker(\phi)$, the tangent space $\mathrm{Hom}(\mathfrak{p}/\mathfrak{p}^2, E)$ will be isomorphic to $H^1_g(\Gamma_{F,S}, W_E)$, and its torsion variant $\mathrm{Hom}(\mathfrak{p}/\mathfrak{p}^2, E/\mathcal{O})$ will be some quantifiable variant on $H^1_g(\Gamma_{F,S}, W_{E/\mathcal{O}})$.

Thus we are faced with a hierarchy of possible results toward the Bloch–Kato conjecture in this or related settings:

(1) Prove the vanishing of $H^1_g(\Gamma_{F,S}, W_E)$, or equivalently the finiteness of $H^1_g(\Gamma_{F,S}, W_{E/\mathcal{O}})$.
(2) Prove $H^1_g(\Gamma_{F,S}, W_{E/\mathcal{O}})$ or $\mathrm{Hom}(\mathfrak{p}/\mathfrak{p}^2, E/\mathcal{O})$ is finite, with an explicit bound or even exact formula for the order.
(3) Prove the full Bloch-Kato conjecture for $H^1_f(F, W_{E/\mathcal{O}})$.

In his work on Fermat's Last Theorem, Wiles proved many cases of (1), along with (in combination with the main theorem of [TW95]) an exact formula for corresponding cases of (2) (see [Wil95, Theorem 3.3]). Wiles's work depends on a number of simplifying hypotheses: the field F is $\mathbb{Q}$, the residual representation $\bar{\rho}$ has "large" image as required by the traditional Taylor–Wiles method, and the local representations $\rho|_{\Gamma_{F_v}}$ satisfy simplifying hypotheses, particularly at $v|p$ (e.g., Π should be weight 2 and either unramified at p or ordinary at p). The structure of Wiles's formula depends on proving an $R = \mathbb{T}$ theorem identifying R with an appropriate reduced Hecke algebra $\mathbb{T}$, and on the fact that these rings are complete intersections. This allows, via Wiles's numerical criterion, the size of $\mathfrak{p}/\mathfrak{p}^2$ (now known to be finite, since $\mathbb{T}$ is reduced) to be reinterpreted as the size of $\mathcal{O}/\eta_{\mathbb{T}}$, where $\eta_{\mathbb{T}}$ is the congruence ideal.

The mechanism of Wiles's proof is to introduce auxiliary (Taylor-Wiles) primes Q of ramification in the deformation problem and show using automorphic congruences that the growth of the Selmer group with relaxed ramification at Q is maximal. In [Wil95, Theorem 3.1(i)], this method is used to prove the upper bound in (2); the full force of an integral $R = \mathbb{T}$ theorem, relying on the Taylor-Wiles patching method, can then be used to deduce that the upper bound is exact ([Wil95, Theorem 3.3]). While Wiles does not exactly establish (3), Diamond–Flach–Guo ([DFG04]) were able to prove (3) for $F = \mathbb{Q}$, under the hypotheses $p > k$ and Π_p unramified, by analyzing an explicit motivic setup in combination with the Taylor-Wiles method and an idea of Hida that interprets the special L-value in terms of a congruence ideal.

We describe some of the subsequent work on questions (1)-(3). Kisin ([Kis04]), still when $F = \mathbb{Q}$, took up Wiles's method and proved the vanishing of $H^1_g(\Gamma_{\mathbb{Q},S}, \mathrm{ad}(\rho))$ for Π of arbitrary weight and power of p in the level. His proof requires a difficult analysis of the integral p-adic Hodge theory of modular curves but is very close in spirit to Wiles's method. Kisin's method does not yield an explicit bound as in (2), but it follows more closely the argument of [Wil95, Chapter 3] and in particular does not rely on an $R = T$ theorem of any kind. Later works have relied heavily on Taylor-Wiles patching and have proven only (1). We highlight the work of Allen ([All16]), which proves vanishing of $H^1_g(\Gamma_{F,S}, W_E)$ in the much more general setting where $W = \mathrm{ad}(\rho_{\Pi,\iota})$ for Π a regular algebraic polarizable (in the sense of [BLGGT14]) cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ where K is a CM field with totally real subfield F;[1] Allen's work allows for arbitrary local behavior of Π, but it requires $p > 2$ and that $\bar{\rho}(\Gamma_{K(\zeta_p)})$ have large (adequate) image to allow the usual higher-rank Taylor-Wiles method (developed in [CHT08], [Tho12]) to be applied. In a major innovation, Newton and Thorne ([NT23]) and Thorne ([Tho22]) have proven (1) for all

[1] Note that $\rho_{\Pi,\iota}$ is defined on Γ_K, but there is an outer action of Γ_F involved in the definition of the Selmer group, which only classifies suitably polarized deformation.

regular algebraic polarizable cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_K)$, subject only to the very weak requirement that $\rho_{\Pi,\iota}|_{\Gamma_{K(\zeta_{p^\infty})}}$ be irreducible. Results along the lines of (2) (let alone (3)) seem to be inaccessible via these methods. In the direction of (3), work in progress of Diamond-Iyengar-Khare-Manning seeks exact Bloch-Kato formulas beyond the setting of [DFG04], but still restricted to $F = \mathbb{Q}$, using integral $R = \mathbb{T}$ theorems and the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Our work, which in the present paper is restricted to GL_2 over our totally real field F, revisits the arguments of [Wil95, Chapter 3] and gives a framework where we can prove explicit upper bounds for $H^1_g(\Gamma_{F,S}, W_{E/\mathcal{O}})$ as in (2) for regular algebraic cuspidal automorphic representations Π of $GL_2(\mathbb{A}_F)$. Our method, like the argument of Wiles in [Wil95, Chapter 3], is not able to give formulas for the exact order of this Selmer group. We also rely on the Taylor–Wiles hypothesis that $\bar{\rho}|_{\Gamma_{F(\zeta_p)}}$ is absolutely irreducible, but are otherwise able to prove meaningful upper bounds with no restriction on the local behavior of Π. See §4 for possible generalizations: we are optimistic that via these methods we can establish explicit upper bounds as in (2) but in the generality of Allen's work, and we have interesting lines of inquiry that may eventually make possible a version of (2) in the Newton–Thorne setting.

More precisely, we work on a quaternion algebra D that is ramified precisely at infinity, and with automorphic representations of the associated algebraic group G^D over F (see §2); in general we reduce to this case by base-change, which is harmless for establishing (1) but leads to some inefficiency in (2). We refer to Theorem 3.15 for the result in the Hilbert modular case, and for the rest of the introduction we assume that Π arises from an automorphic representation π of $G^D(\mathbb{A}_F)$ via the Jacquet-Langlands correspondence. Letting $U \subset G^D(\mathbb{A}_F^\infty) \cong \mathrm{GL}_2(\mathbb{A}_F^\infty)$ be the conductor subgroup, we have a space of p-adically integral automorphic forms $S(U, \mathcal{O})$ on G^D to which $(\pi^\infty)^U$ contributes, and we have the reduced Hecke algebra $\mathbb{T}(U)$ generated by good Hecke operators T_v, $v \notin S$ acting on $S(U, \mathcal{O})$. Since $(\pi^\infty)^U$ is 1-dimensional, we obtain an eigencharacter $\lambda\colon \mathbb{T}(U) \to \mathcal{O}$ giving the Hecke eigenvalues. Let $I = \ker(\lambda)$, and let $\mathfrak{m} = \ker(\lambda \pmod{\varpi})$. Our results will be stated in terms of $M_\emptyset = S(U, \mathcal{O})_\mathfrak{m}$ and $I \subset \mathbb{T}_\emptyset = \mathbb{T}(U)_\mathfrak{m}$. In §3 we will define Selmer conditions $\mathcal{D}_S$ at all torsion levels and the resulting torsion Selmer group $H^1_{\mathcal{D}_S}(F, W_{E/\mathcal{O}})$ (see the discussion following Definition 3.5). For a finite $\mathcal{O}$-module A, write $\lg(A) = \mathrm{length}_\mathcal{O}(A)$; when a cohomology group is finite, write $h^i_\mathcal{L}(F, V) = \lg(H^i_\mathcal{L}(F, V))$. Then we show:

Theorem 1.1. *Assume that $[F(\zeta_p) : F] > 2$, that $\bar{\rho}|_{\Gamma_{F(\zeta_p)}}$ is absolutely irreducible, and if $p = 5$ and $\bar{\rho}$ has projective image $\mathrm{PGL}_2(\mathbb{F}_5)$, then $F(\zeta_5)$ is not contained in $\overline{F}^{\ker(\mathrm{proj}(\bar{\rho}))}$.*[2] *Then $H^1_g(\Gamma_{F,S}, W_E) = 0$, and $H^1_{\mathcal{D}_S}(\Gamma_{F,S}, W_{E/\mathcal{O}})$ is finite with length (as $\mathcal{O}$-module) at most $c_S + d_r$ where:*

- *$c_S = \sum_{v\in S} c_v$ is a sum of local error terms arising from singularities of local deformation rings: $c_v = \lg(\Omega_{\overline{R}^\square_v/\mathcal{O}} \otimes_{\overline{R}^\square_v,\rho} \mathcal{O})_{\mathrm{tor}}$ where for $v \nmid p$, $\overline{R}^\square_v$ is the local lifting ring with fixed determinant $\det(\rho)$ for $\bar{\rho}|_{\Gamma_{F_v}}$; and for $v|p$ it is a certain potentially semistable deformation ring specified in Lemma 3.11.*
- *$r = \dim_k H^1_{\mathcal{D}^\perp_S}(\Gamma_{F,S}, \mathrm{ad}^0(\bar{\rho})^*) + \sum_{v\in S} t_v$; here t_v (defined in Equation (7)) is the number of generators of $(\Omega_{\overline{R}^\square_v/\mathcal{O}} \otimes_{\overline{R}^\square_v,\rho} \mathcal{O})_{\mathrm{tor}}$.*
- *$d_r = r(j_\emptyset + a_\emptyset) + b_\emptyset - \lg(\Psi_\lambda(M_\emptyset))$, where*
  - *$a_\emptyset = \lg((M_\emptyset/IM_\emptyset)_{\mathrm{tor}})$.*

[2] We can allow $p = 3$ if U is assumed sufficiently small; the method would let us take $p = 3$ with no restriction on U, but the final bound would be different, and the technical complications obfuscating for our present purposes.

– $b_\emptyset = \lg(IM_\emptyset/I^2M_\emptyset)$, *and*

$$\varpi^{j_\emptyset}\mathcal{O} = \mathrm{Ann}(IM_\emptyset/I^2M_\emptyset)\,\mathrm{Ann}((M_\emptyset/IM_\emptyset)_{\mathrm{tor}})^2.$$

– $\Psi_\lambda(M_\emptyset)$ *is the congruence module defined in* (2).

Remark 1.2. Let us demystify this formula in the simple case where $M_\emptyset$ is free of rank 1 over $\mathbb{T}_\emptyset$ (analogous to Wiles's situation where his Hecke algebra is Gorenstein, and the localized cohomology of the modular curve is free rank 2 over the Hecke algebra; in our definite quaternion algebra setting, the generic ranks will be 1 rather than 2). Then $a_\emptyset = 0$, $\Psi_\lambda(M_\emptyset) \simeq \mathcal{O}/\eta_\lambda(\mathbb{T}_\emptyset)$, and $b_\emptyset = \lg(I/I^2)$. Theorem 1.1, would then bound the length of $H^1_{\mathcal{D}_S}(\Gamma_{F,S}, W_{E/\mathcal{O}})$ by

$$\dim_k H^1_{\mathcal{D}_S^\perp}(\Gamma_{F,S}, \mathrm{ad}^0(\bar\rho)^*)j_\emptyset + \lg(I/I^2) - \lg(\mathcal{O}/\eta_\lambda(\mathbb{T}_\emptyset)) + c_S.$$

If we further simplify to the case where the c_v (and thus t_v) are all zero, then the Greenberg-Wiles formula transforms this bound into

$$\dim_k H^1_{\mathcal{D}_S}(\Gamma_{F,S}, \mathrm{ad}^0(\bar\rho)) \cdot j_\emptyset + \lg(I/I^2) - \lg(\mathcal{O}/\eta_\lambda(\mathbb{T}_\emptyset)).$$

This mod p Selmer dimension gives the maximum number of generators needed for the deformation ring and thus bounds from above the number of $\mathcal{O}$-module generators of I/I^2, so this $\dim_k H^1_{\mathcal{D}_S}(\Gamma_{F,S}, \mathrm{ad}^0(\bar\rho)) \cdot j_\emptyset$ term is a weakened version (expected number of generators times exponent being an upper bound on length) of $\lg(I/I^2)$. Wiles's sharp formula in [Wil95, Theorem 3.1] is $2\lg(I/I^2) - \lg(\mathcal{O}/\eta_\lambda(\mathbb{T}_\emptyset))$. Thus our general formula is not quite as strong as Wiles's, but it is conceptually closely related. It would be interesting to expand on this connection and try to refine our arguments to yield a sharper final result in certain simplified settings.

Our method, like Wiles's, produces, for each $n \geq 1$, Taylor-Wiles sets Q_n of level n and enough ramified Galois cohomology classes in $H^1_{\mathcal{D}_{S\cup Q_n}}(\Gamma_{F,S\cup Q_n}, \mathrm{ad}^0(\rho) \otimes \mathcal{O}/p^n)$ (the local conditions as before in S but now allowing all local deformations at Q_n) to (provided Q_n is chosen to annihilate the dual Selmer group mod p^n) bound $H^1_{\mathcal{D}_S}(\Gamma_{F,S}, \mathrm{ad}^0(\rho) \otimes \mathcal{O}/p^n)$ using the Greenberg-Wiles formula (see Lemma 3.11 and Proposition 3.12). At each Taylor-Wiles level U_{Q_n}, we have a reduced Hecke algebra with diamond operators $\mathcal{O}[\Delta_{Q_n}] \to \mathbb{T}_{Q_n}$ acting on a space M_{Q_n} of automorphic forms, and we have a choice of stabilization $\lambda_{Q_n}\colon \mathbb{T}_{Q_n} \to \mathcal{O}$ of λ. The linchpin of the method is Proposition 3.6 and Proposition 3.9, which show in a rather direct manner that $\mathfrak{a}_{Q_n}/\mathfrak{a}^2_{Q_n} \to I_{Q_n}/I^2_{Q_n}$ ($I_{Q_n} = \ker(\lambda_{Q_n})$) has large image by considering their respective images in $\mathrm{Hom}(M_{Q_n}/I_{Q_n}M_{Q_n}, I_{Q_n}M_{Q_n}/I^2_{Q_n}M_{Q_n}) \subset \mathrm{End}(M_{Q_n}/I^2_{Q_n}M_{Q_n})$. The method depends in an essential way on $M_{Q_n}/I_{Q_n}M_{Q_n}$ having generic rank 1 and on the length of its torsion submodule being bounded independently of Q_n; in the Taylor-Wiles setting used here $M_{Q_n}/I_{Q_n}M_{Q_n}$ is in fact independent of Q_n.

We normalize class field theory to send uniformizers to geometric Frobenius elements. We normalize Hodge-Tate weights so that the cyclotomic character κ has Hodge-Tate weight 1.

2. Setup and preliminary reductions

Let F_0 be a totally real field, and let Π_0 be a regular C-algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{F_0})$. Fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Via ι, each embedding $\tau\colon F_0 \to \overline{\mathbb{Q}}_p$ induces $\iota\tau$ and thus a unique $v|\infty$; since F_0 is totally real, there is no ambiguity in writing $\tau_v\colon F_0 \to \mathbb{R}$ for $\iota\tau$. The Harish–Chandra parameter of $\Pi_{0,v}$ has the form $\{(-m_{\tau_v} + \frac{1}{2}, -n_{\tau_v} - m_{\tau_v} - \frac{1}{2})\}_{v|\infty}$ for some

integers $n_{\tau_v} \geq 0$ and m_{τ_v} (this normalization will be convenient below). We write $n_\tau = n_{\tau_v}$ and $m_\tau = m_{\tau_v}$, where τ and τ_v are associated as above.

By the work of many people, notably Carayol, Taylor, and Tsuji, there is a continuous representation $\rho_{\Pi_0,\iota}\colon \Gamma_{F_0} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that

(1) $\rho_{\Pi_0,\iota}$ is odd.
(2) For all finite places $v \nmid p$, $\mathrm{WD}(\rho_{\Pi_0,\iota}|_{\Gamma_{F_{0,v}}})^{F\text{-ss}} \cong \iota(\mathrm{rec}_v(\Pi_{0,v}|\det|^{-1/2}))$.
(3) For $v|p$, $\rho_{\Pi_0,\iota}|_{\Gamma_{F_{0,v}}}$ is de Rham, and the τ-labeled Hodge-Tate weights are $\{-m_\tau, -n_\tau - m_\tau - 1\}$. Moreover, this representation is semistable (respectively, crystalline) when $\Pi_{0,v}$ is an unramified twist of the Steinberg representation (respectively, an unramified representation).

We fix an integral model, still denoted $\rho_{\Pi_0,\iota}\colon \Gamma_{F_0} \to \mathrm{GL}_2(\mathcal{O})$, defined over the ring of integers $\mathcal{O}$ in a finite extension $E/\mathbb{Q}_p$, which we assume large enough to contain the images of all embeddings $F_0 \hookrightarrow \overline{\mathbb{Q}}_p$. After choosing the auxiliary field F below, we enlarge E further if necessary. At one point later in the argument (part (5) of Proposition 3.6), if p is less than the difference of the Hodge-Tate weights at some embedding, we will need to replace $\mathcal{O}$ by its quadratic extension $\mathcal{O}[\sqrt{\varpi}]$; we do this from now, so that we don't have to change $\mathcal{O}$ later on. We continue to denote the resulting coefficient field and its ring of integers by E and $\mathcal{O}$, and let e be the ramification index of this coefficient field over $\mathbb{Q}_p$.

Let S_0 be the finite set of finite places of F_0 consisting of the places above p and the places at which Π_0 is ramified. For any extension F/F_0 considered below, let S denote the set of places of F above S_0.

Let F/F_0 be any totally real finite solvable extension such that $[F : \mathbb{Q}]$ is even (later on we will make more requirements on F). Let $\Pi = \mathrm{BC}_{F/F_0}(\Pi_0)$. It similarly has an associated Galois representation $\rho_{\Pi,\iota}$, and we will mainly analyze this restriction. Let D be the quaternion algebra over F whose set of ramified places is exactly $S_\infty = \{v|\infty\}$. Let $G = G^D$ be the reductive group over F associated to D, so that $G^D(R) = (D \otimes_F R)^\times$ for every F-algebra R. Fix a maximal order $\mathcal{O}_D$ of D along with isomorphisms $\mathcal{O}_{D,v} \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$ for all finite places v; from now on we make this identification implicitly and use it to identify $G(\mathbb{A}_F^\infty) \cong \mathrm{GL}_2(\mathbb{A}_F^\infty)$. Then Π lies in the image of the Jacquet–Langlands correspondence from infinite-dimensional cuspidal automorphic representations of $G(\mathbb{A}_F)$ to cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$, and we let π be the cuspidal automorphic representation of $G(\mathbb{A}_F)$ such that $\mathrm{JL}(\pi) = \Pi$. Let χ be the common central character of π and Π. Let $U = \prod_v U_v \subset G(\mathbb{A}_F^\infty)$ be the (compact open) conductor subgroup of π, so $(\pi^\infty)^U$ is one-dimensional.

By the explicit form of the Jacquet-Langlands correspondence, for each $v|\infty$, $\pi_v \cong \mathrm{Sym}^{n_{\tau_v}}(\mathbb{C}^2) \otimes \det^{-n_{\tau_v}-m_{\tau_v}}$. As notational shorthand, we will write $W_\iota = \otimes_{v|\infty} \mathrm{Sym}^{n_{\tau_v}}(\mathbb{C}^2) \otimes \det^{m_{\tau_v}}$, isomorphic to $\pi_\infty^\vee$. Necessarily $w = n_\tau + 2m_\tau + 1$ is independent of τ, and note then that $\chi|_{F_\infty^\times} = N_{F_\infty/\mathbb{R}}^{1-w}$. Associated with χ we have the p-adic character $\chi_\iota\colon \mathbb{A}_F^\times/\overline{F^\times F_\infty^\times} \to \overline{\mathbb{Q}}_p^\times$ given by

$$\chi_\iota(x) = \prod_{\tau\colon F \to \overline{\mathbb{Q}}_p} \tau(x_{v_\tau})^{1-w} \cdot \iota^{-1}\left(N_{F_\infty/\mathbb{R}}(x_\infty)^{w-1} \cdot \chi(x)\right),$$

and by the global Artin map (normalized to send geometric Frobenii to uniformizers) we may regard χ_ι as a Galois character $\Gamma_F \to \overline{\mathbb{Q}}_p^\times$. We then see that $\det(\rho_{\Pi,\iota}) = \kappa^{-1}\chi_\iota$, where κ denotes the p-adic cyclotomic character.

We can now introduce the spaces of (p-adic) integral automorphic forms. A slight technicality intervenes, since we will want to have $\mathcal{O}$-linear self-dualities on these spaces, even when some n_τ

is bigger than $p-1$. Thus following [Kis09, §3.1.9] and [CDT99, Lemma 3.3.1], we note that for any $n \geq 0$ there is an $\mathcal{O}$-lattice $L_n \subset \mathrm{Sym}^n(E^2)$ that is stable under $M_2(\mathcal{O})$ and on which the usual E-linear duality $\mathrm{Sym}^n E^2 \times \mathrm{Sym}^n E^2 \to \det^n$ restricts to an $\mathcal{O}$-linear perfect pairing. For any $\mathcal{O}$-algebra A, let $L_{n,A} = L_n \otimes_{\mathcal{O}} A$, and consider the A-module $S(A) = S_{\underline{n},\chi_\iota}(A)$ of functions

$$\varphi\colon D^\times\backslash(D\otimes\mathbb{A}_F^\infty)^\times \to \otimes_{\tau\colon F\to\overline{\mathbb{Q}}_p} L_{n_\tau,A}\otimes\det{}^{m_\tau}(A^2)$$

such that

- $\varphi(gz) = \chi_\iota(z)\varphi(g)$ for all $g \in (D\otimes\mathbb{A}_F^\infty)^\times$ and $z \in (\mathbb{A}_F^\infty)^\times$; and
- there exists a compact open subgroup $V \subset G(\mathbb{A}_F^\infty)$ such that $\varphi(gv) = v_p^{-1}\varphi(g)$ for all $v \in V$, $g \in G(\mathbb{A}_F^\infty)$.

When $A = \mathcal{O}$, we let $\langle\cdot,\cdot\rangle$ be the perfect pairing induced on $\otimes_{\tau\colon F\to\overline{\mathbb{Q}}_p} L_{n_\tau}\otimes\det{}^{m_\tau}$ by the above pairings; we will only need this in Proposition 3.6. We define an action of $G(\mathbb{A}_F^{\infty,p})\times(\mathrm{GL}_2(F_p)\cap M_2(\mathcal{O}_{F,p}))$ on $S(A)$ by $(u\cdot\varphi)(g) = u_p\varphi(gu)$, where u_p denotes projection to the p-component. When A is an E-vector space, this extends to an action of $G(\mathbb{A}_F^\infty)$. For a particular compact open subgroup V, we write $S(V,A)$ for the V-invariants with respect to this action. Then we have the Hecke operators $[VgV] \in \mathrm{End}_A(S(V,A))$, given by $[VgV]\varphi = \sum x_i\cdot\varphi$ where $VgV = \bigsqcup_i x_iV$, which make sense in general for $g \in G(\mathbb{A}_F^{\infty,p})\times(\mathrm{GL}_2(F_p)\cap M_2(\mathcal{O}_{F,p}))$, and when A is an E-vector space make sense for any $g \in G(\mathbb{A}_F^\infty)$.

There is a $G(\mathbb{A}_F^\infty)$-equivariant isomorphism

$$j\colon S(\mathcal{O})\otimes_{\mathcal{O},\iota}\mathbb{C} \xrightarrow{\sim} \mathrm{Hom}_{G(F_\infty)}(W_\iota^\vee, \mathcal{A}_{G,\chi}) \xrightarrow{\sim} \bigoplus_\sigma \sigma^\infty, \tag{1}$$

where $\mathcal{A}_{G,\chi}$ denotes the space of (complex) automorphic forms on $G(\mathbb{A}_F)$ with central character χ, and where the direct sum is taken over automorphic representations σ of $G(\mathbb{A}_F)$ with central character χ and such that σ_∞ is isomorphic to π_∞.[3] Taking U-invariants, we obtain an analogous isomorphism, now equivariant for the Hecke operators. Let $\mathbb{T}(U)$ be the reduced Hecke algebra at level U, defined as the $\mathcal{O}$-subalgebra of $\mathrm{End}(S(U,\mathcal{O}))$ generated by all T_v, $v \notin S$.

Now, by the choice of U, $(\pi^\infty)^U$ is 1-dimensional, so we obtain the eigencharacter $\lambda\colon \mathbb{T}(U)\to\mathcal{O}$ such that $\lambda(T_v) = N(v)^{1/2}(\chi_{1,v}(\varpi_v)+\chi_{2,v}(\varpi_v))$ when $\pi_v \cong \mathrm{ind}_B^{G(F_v)}(\chi_{1,v}\times\chi_{2,v})$ (normalized induction). Reducing mod ϖ, we obtain a maximal ideal $\mathfrak{m} \subset \mathbb{T}(U)$, and we let $\mathbb{T} = \mathbb{T}(U)_{\mathfrak{m}}$ be the localization, with $\lambda\colon \mathbb{T}\to\mathcal{O}$ the homomorphism induced by λ, and with $I = \ker(\lambda)$.

We will also need these spaces with level structure added at auxiliary (Taylor-Wiles) primes. First let Q be any set of primes of F disjoint from S. We first introduce the Taylor-Wiles level subgroups

$$U_{Q,v} = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in U_0(v) : \frac{a}{d} \mapsto 1 \in \Delta_v\right\},$$

where Δ_v is the p-Sylow subgroup (equivalently, quotient) of the cyclic group $\kappa(v)^\times$, and set $U_Q = \prod_{v\notin Q} U_v \times \prod_{v\in Q} U_{Q,v}$. $S(U_Q,\mathcal{O})$ has commuting actions of $\Delta_Q = \prod_{v\in Q}\Delta_v$ and $\mathbb{T}_Q(U_Q)$, the commutative $\mathcal{O}$-subalgebra of $\mathrm{End}(S(U_Q,\mathcal{O}))$ generated by all T_v, $v \notin S\cup Q$ and by U_{ϖ_v}, $v \in Q$, for some fixed choice of uniformizers ϖ_v of $\mathcal{O}_{F_v}$.

Provided $U_0(Q)$ is sufficiently small, $S(U_Q,\mathcal{O})$ is a finite free $\mathcal{O}[\Delta_Q]$-module by [Gee22, Prop. 5.4], and $S(U_Q,\mathcal{O})/\Delta_Q \xrightarrow{\sim} S(U_0(Q),\mathcal{O})$ (with the map induced by summing over the Δ_Q-action). In §3, we will work under an Assumption (Assumption 3.1) that guarantees this always holds.

[3] Define $j(\varphi)(w^\vee)(g) = w^\vee(g_\infty^{-1}g_p\varphi(g^\infty))$.

Choose a root α_v of $X^2 - \lambda(T_v)X + N(v)\chi(\varpi_v)$. For any $v \in Q$, $\pi_v^{U_{Q,v}}$ is two-dimensional, and the α_v-eigenspace of U_{ϖ_v} is one-dimensional (whether or not the roots of the Hecke polynomial are distinct). We obtain a character $\lambda_Q \colon \mathbb{T}_Q(U_Q) \to \mathcal{O}$ corresponding to the choices $(\alpha_v)_{v\in Q}$ (and the initial choice of λ). Let $I_Q = \ker(\lambda_Q)$, and let $\mathfrak{m}_Q = \ker(\lambda_Q \pmod{\varpi})$.

Define $M_Q = S(U_Q, \mathcal{O})_{\mathfrak{m}_Q}$; it is a finite $\mathbb{T}_Q := \mathbb{T}_Q(U_Q)_{\mathfrak{m}_Q}$-module, finite free as an $\mathcal{O}$-module.

Lemma 2.1. *$(M_Q/I_QM_Q) \otimes_{\mathcal{O}} E$ is one-dimensional over E.*

Proof. The decomposition of Equation (1) shows that $S(U_Q, E) \otimes_\iota \mathbb{C}$ is $\mathbb{T}_Q(U_Q)$-equivariantly isomorphic to $\bigoplus_\sigma \sigma^{U_Q}$, where the sum is indexed over isomorphism classes of irreducible admissible representations σ of $G(\mathbb{A}_F^\infty)$ such that $\sigma \otimes \pi_\infty$ is isomorphic to a cuspidal automorphic representation of $G(\mathbb{A}_F)$: note that the multiplicities are all 1, which is part of the Jacquet-Langlands theorem, and that when all $n_\tau = 0$, we must include 1-dimensional automorphic representations that factor through the reduced norm. Since the 1-dimensional constituents do not share Hecke eigenvalues with the infinite-dimensional constituents (by purity), the Jacquet–Langlands correspondence and strong multiplicity one theorem further imply that when we pass to $(M_Q/I_QM_Q) \otimes_\iota \mathbb{C}$, the only summand that survives is

$$\otimes_{v\notin S\cup Q} \pi_v^{U_v}/(T_v - \lambda_Q(T_v)) \otimes_{v\in S} \pi_v^{U_v} \otimes_{v\in Q} \pi_v^{U_{Q,v}}/(U_{\varpi_v} - \lambda_Q(U_{\varpi_v})),$$

which we must check is one-dimensional. For $v \notin S \cup Q$, $\pi_v^{U_v}$ is 1-dimensional with T_v acting by $\lambda_Q(T_v) = \lambda(T_v)$. For $v \in S$, $\pi_v^{U_v}$ is one-dimensional by the choice of U, while for $v \in Q$, $\pi_v^{U_{Q_v}}/(U_{\varpi_v} - \alpha_v)$ is one-dimensional since the α_v-eigenspace in $\pi_v^{U_{Q_v}}$ is one-dimensional (whether or not U_{ϖ_v} has distinct eigenvalues). □

We end this section by summarizing the commutative algebra of congruence modules that we will require. We define the congruence module as in [DDT94, §4.4, p. 125, Lemma 4.17], with some broader context taken from [IKM24], and the congruence ideal as in [IKM24]. Let $\mu \colon T \to \mathcal{O}$ be an augmentation of a complete Noetherian local $\mathcal{O}$-algebra, and put $J = \ker(\mu)$. Assume that the localization T_J is regular of dimension 0, i.e., $(T, \mu) \in \mathcal{C}_{\mathcal{O}}(0)$ in the sense of [IKM24]. Equivalently, $T_J \simeq E$. Let L be a finitely generated T-module of positive depth such that $L_J \neq 0$. In our work, L is finite flat over $\mathcal{O}$, which implies the positive depth and finite generation hypothesis.

The congruence module of L at μ is

$$\Psi_\mu(L) := \frac{L}{L[J] + L[\mathrm{Ann}_T(J)]}. \tag{2}$$

In the self-dual finite-flat setting, this is the module appearing in [DDT94, § 4.4, Lemma 4.17, p. 125]; by [IKM24, Proposition 2.10], it is the codimension zero case of the more general construction in *loc. cit.*.

Regarding $\mathcal{O}$ as a T-module through μ, define the congruence ideal of L by

$$\eta_\mu(L) := \mathrm{im}\left(\mathrm{Hom}_T(\mathcal{O}, L) \otimes_{\mathcal{O}} \mathrm{Hom}_T(L, \mathcal{O}) \longrightarrow \mathcal{O}\right), \tag{3}$$

where the map is given by composition. This is [IKM25, § 2, equation (2.4)], specialized to codimension zero. For $L = T$, it recovers the usual congruence ideal

$$\eta_\mu(T) = \mu(\mathrm{Ann}_T(J)).$$

More precisely, write

$$\Psi_\mu(L) \simeq \bigoplus_{i=1}^{d} \mathcal{O}/\varpi^{e_i}\mathcal{O}, \qquad 0 \le e_1 \le \cdots \le e_d,$$

where $d = \dim_E L_J$. Then

$$\eta_\mu(L) = \varpi^{e_1}\mathcal{O}, \qquad \mathrm{Fitt}_{\mathcal{O}}(\Psi_\mu(L)) = \varpi^{e_1+\cdots+e_d}\mathcal{O};$$

see [IKM25, Lemma 2.19]. Thus the congruence ideal records the least elementary divisor of the congruence module. In particular, when $d = 1$,

$$\Psi_\mu(L) \simeq \mathcal{O}/\eta_\mu(L). \tag{4}$$

We will only need the following consequence: if L carries a perfect $\mathcal{O}$-linear pairing $\langle\cdot,\cdot\rangle$ for which T acts self-adjointly, and φ spans $L[J]$, then (3) is the image of the restriction of this pairing to the μ-eigenspace, hence $\Psi_\mu(L) \cong \mathcal{O}/(\langle\varphi,\varphi\rangle)$.

3. Taylor-Wiles primes

We now make the more precise hypotheses on Π_0 that we will use for our main analysis. See §4 for discussion of how to relax these hypotheses.

Assumption 3.1. Assume that Π_0, F_0, and p satisfy the following:

- $[F_0(\zeta_p) : F_0] > 2$ (in particular, p is odd).
- $\bar\rho_{\Pi_0,\iota}|_{\Gamma_{F_0(\zeta_p)}}$ is absolutely irreducible.
- If $p = 5$ and $\bar\rho_{\Pi_0,\iota}$ has projective image $\mathrm{PGL}_2(\mathbb{F}_5)$, then $F_0(\zeta_5)$ is not contained in $\overline{F}_0^{\ker(\mathrm{proj}(\bar\rho_{\Pi_0,\iota}))}$.

Our analysis will require working with automorphic forms on definite quaternion algebras. To descend general Π_0 via the Jacquet-Langlands correspondence to a definite quaternion algebra, we have to make a base-change first.

Lemma 3.2. *There exists a quadratic Galois extension F/F_0 such that:*

- *F is totally real.*
- *Every place in S_0 splits in F.*
- *$[F(\zeta_p) : F] > 2$.*
- *$\bar\rho_{\Pi_0,\iota}|_{\Gamma_{F(\zeta_p)}}$ remains absolutely irreducible, and, when $p = 5$, still satisfies the last item in Assumption 3.1.*

Proof. This follows from the Grunwald–Wang theorem. In addition to imposing the stated local splitting conditions, we choose F linearly disjoint from the compositum of $F_0(\zeta_p)$ and the field cut out by $\bar\rho_{\Pi_0,\iota}$, using a local condition at an auxiliary prime outside S_0. This preserves the condition $[F(\zeta_p) : F] > 2$, the absolute irreducibility of the residual representation, and the exceptional condition when $p = 5$. □

For the rest of the paper, if $[F_0 : \mathbb{Q}]$ is even we take $F = F_0$, and otherwise we now fix an F as in Lemma 3.2. We then make the constructions of §2 and adopt its notation. (For instance, we let $\Pi = \mathrm{BC}_{F/F_0}(\Pi_0)$, and we descend Π to a cuspidal automorphic representation π of $G(\mathbb{A}_F)$, where $G = G^D$ is associated to the quaternion algebra D over F ramified exactly at $\{v|\infty\}$.) For shorthand we set $\rho = \rho_{\Pi,\iota} = \rho_{\Pi_0,\iota}|_{\Gamma_F}$, valued in $\mathrm{GL}_2(\mathcal{O})$, and we let $\bar\rho = \bar\rho_{\Pi,\iota}$ be its reduction.

Lemma 3.3. *For a fixed conductor f_v, there are finitely many inertial types arising from Weil-Deligne representations of fixed dimension of conductor at most f_v. In particular, for each $v|p$, there is a finite extension F'_v/F_v such that all automorphic representations σ of $G(\mathbb{A}_F)$ contributing to M_Q for any auxiliary set Q have associated Galois representations $\rho_{\sigma,\iota}$ that become semistable over F'_v.*

Proof. The first claim follows directly in dimension two. Since $p \neq 2$, the Weil–Deligne representations in question are sums of characters, twists of the Steinberg parameter, or inductions of characters from quadratic extensions of F_v; in each case local class field theory gives only finitely many inertial types of bounded conductor. For the second claim, for each $v|p$, the conductor subgroup U_v of Π_v bounds the conductor of all Weil–Deligne representations at v that can arise from automorphic representations σ with level U_Q (a group with local component U_v at v). There are finitely many inertial types with this bounded conductor, and they all factor through some finite extensions F'_v/F_v. It follows from local–global compatibility at $v|p$ that $\rho_{\sigma,\iota}|_{\Gamma_{F_v}}$ becomes semistable over F'_v. □

Remark 3.4. This lemma will be needed to define the maps $R^{\mathrm{pst}}_{\bar\rho,S\cup Q_n} \to \mathbb{T}_{Q_n}$ below. As an alternative to the following discussion (and to the above lemma), we could have chosen the extension F/F_0 to make Π_v either unramified or an unramified twist of the Steinberg representation at all $v|p$. We would then have $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ or $U_0(v)$ for all $v|p$, and the potentially semistable (becoming semistable over F'_v) deformation ring used below could simply be replaced with a semistable deformation ring. This would work just as well for proving the eventual vanishing of $H^1_g(\Gamma_{F_0,S}, \mathrm{ad}^0(\rho)\otimes E)$, but for the quantitative bound on the torsion analogue we think this setup is more transparent: for instance, when $[F_0:\mathbb{Q}]$ is even, it allows us always to take $F = F_0$.

We next fix integers $a < b$ such that all (labeled) Hodge-Tate weights of $\rho_{\Gamma_{F_v}}$ (for all $v|p$) lie in $[a,b]$. We can then consider the category $\mathcal{E}^{\mathrm{pst}}_{F,S}$ of finite $\mathbb{Z}_p[\Gamma_{F,S}]$-modules whose restrictions to Γ_{F_v} for all $v|p$ are isomorphic to quotients of lattices in potentially semistable representations, becoming semistable over F'_v, with Hodge-Tate weights in $[a,b]$. Note that this full subcategory of all finite $\mathbb{Z}_p[\Gamma_{F,S}]$-modules is stable under isomorphisms, subquotients, and finite direct sums. The closure properties just noted, together with [Ram93, Proposition 1.1], show that the following definition makes sense.

Definition 3.5. Let $R^{\mathrm{pst}}_{\bar\rho,S}$ be the quotient of the universal deformation ring of

$$\bar\rho\colon \Gamma_{F,S} \longrightarrow \mathrm{GL}_2(k)$$

characterized as follows. For $A \in \mathcal{C}_{\mathcal{O}}$, an A-valued deformation τ factors through $R^{\mathrm{pst}}_{\bar\rho,S}$ if and only if

- $\det(\tau) = \kappa^{-1}\chi_\iota$;
- for every $j \geq 1$, the underlying finite $\mathbf{Z}_p[\Gamma_{F,S}]$-module of $\tau \bmod \mathfrak{m}_A^j$ belongs to $\mathcal{E}^{\mathrm{pst}}_{F,S}$.

We caution the reader that $R^{\mathrm{pst}}_{\bar\rho,S}$, like $\mathcal{E}^{\mathrm{pst}}_{F,S}$, depends on $[a,b]$ and on $(F'_v)_{v|p}$, although these have been omitted from the notation.

For notational simplicity, in what follows we let $W = \mathrm{ad}^0(\rho_{\Pi,\iota})$, $W_E = W\otimes_{\mathcal{O}} E$, and $W_{E/\mathcal{O}} = W\otimes_{\mathcal{O}} E/\mathcal{O}$. We also write W_m for $\mathrm{ad}^0(\rho_{\Pi,\iota})\otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$, and set $W_m^* = \mathrm{Hom}_{\mathcal{O}}(W_m, E/\mathcal{O}(1))$. For each integer $m \geq 1$ and finite set $T \supset S$, we consider the following Selmer condition $\mathcal{D}_{T,m}$:

- For all $v \in T \setminus \{v|p\}$, $\mathcal{D}_{T,m,v} = H^1(\Gamma_{F_v}, W_m)$.
- For $v \mid p$, $\mathcal{D}_{T,m,v}$ consists of the classes $c \in H^1(\Gamma_{F_v}, W_m)$ for which the associated first-order deformation
$$\rho_{m,c}\colon \Gamma_{F_v} \longrightarrow \mathrm{GL}_2((\mathcal{O}/\varpi^m)[\epsilon]/(\epsilon^2))$$
has determinant $\det(\rho \bmod \varpi^m)$ and whose underlying finite $\mathbf{Z}_p[\Gamma_{F_v}]$-module is a quotient of a Γ_{F_v}-stable lattice in a potentially semistable representation that becomes semistable over F'_v and has Hodge–Tate weights in $[a,b]$.
- For all $v \notin T$, $\mathcal{D}_{T,m,v} = H^1_{\mathrm{unr}}(\Gamma_{F_v}, W_m)$, the unramified classes.

Thus the Selmer group $H^1_{\mathcal{D}_{T,m}}(F, W_m)$ classifies self-extensions U of $\rho \otimes \mathcal{O}/\varpi^m$, as $\mathcal{O}/\varpi^m[\Gamma_{F,T}]$-module, that lie in $\mathcal{E}^{\mathrm{pst}}_{F,T}$ and have the fixed determinant condition. Note that for all $m \le m'$, under the reduction modulo ϖ^m map $W_{m'} \to W_m$, $\mathcal{D}_{T,m',v}$ maps to $\mathcal{D}_{T,m,v}$ in all cases. Similarly, the maps $W_m \xrightarrow{\varpi^{m'-m}} W_{m'}$ induce maps $\mathcal{D}_{T,m,v} \to \mathcal{D}_{T,m',v}$, and the preimage of $\mathcal{D}_{T,m',v}$ under the induced map $H^1(\Gamma_{F_v}, W_m) \to H^1(\Gamma_{F_v}, W_{m'})$ is $\mathcal{D}_{T,m,v}$. Similar remarks apply to the dual Selmer conditions $\mathcal{D}^{\perp}_{T,m,v}$ (the annihilators of the $\mathcal{D}_{T,m,v}$ under local duality).

We are concerned with controlling the Selmer group (compare [DDT94, Lemma 2.40])

$$H^1_{\mathcal{D}_S}(F, W_{E/\mathcal{O}}) = \varinjlim_m H^1_{\mathcal{D}_{S,m}}(F, W_m) \xleftarrow{\sim} \varinjlim_m \mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{O}/\varpi^m).$$

Analogous isomorphisms hold with $\mathcal{D}_{S\cup Q}$ in place of $\mathcal{D}_S$, and $\mathfrak{p}_Q$ in place of $\mathfrak{p}$, for any Taylor-Wiles set Q. To simplify our Selmer group notation, we will often write $H^1_{\mathcal{D}_S}(F, W_m)$ for $H^1_{\mathcal{D}_{S,m}}(F, W_m)$.

For each n, let Q_n be a finite set of primes such that for all $v \in Q_n$, $N(v) \equiv 1 \pmod{p^n}$, and $\bar\rho(\mathrm{Frob}_v)$ has distinct eigenvalues; we refer to any such set as a set of Taylor–Wiles primes of level n. As in §2 we choose a root α_v of $X^2 - \lambda(T_v)X + N(v)\chi(\varpi_v)$, letting β_v be the other root, and we let $\mathfrak{m}_{Q_n} = \ker(\lambda_{Q_n} \bmod \varpi)$. Thus $\mathfrak{m}_{Q_n}$ is the maximal ideal of $\mathbb{T}_{Q_n}(U_{Q_n})$ generated by ϖ, $(T_v - \mathrm{tr}(\rho_{\Pi,\iota})(\mathrm{Frob}_v))_{v\notin S\cup Q_n}$, and $(U_{\varpi_v} - \alpha_v)_{v\in Q_n}$.

The irreducibility of $\bar\rho$ implies that the universal modular pseudodeformation gives a universal deformation

$$\rho_{\mathfrak{m}_{Q_n}} : \Gamma_{F,S\cup Q_n} \to \mathrm{GL}_2(\mathbb{T}_{Q_n})$$

and thus by Lemma 3.3 a homomorphism $R^{\mathrm{pst}}_{\bar\rho,S\cup Q_n} \to \mathbb{T}_{Q_n}$, which by construction surjects onto the reduced subalgebra $\mathbb{T}^{S\cup Q_n}_{Q_n}$ generated by the good Hecke operators T_v, $v \notin S \cup Q_n$. It follows however from the assumption $\alpha_v \not\equiv \beta_v \pmod{\varpi}$ for $v \in Q_n$ that U_{ϖ_v} is in the image of $R^{\mathrm{pst}}_{\bar\rho,S\cup Q_n}$ for all $v \in Q_n$ (see [Tay06, Corollary 1.8]). Thus $R^{\mathrm{pst}}_{\bar\rho,S\cup Q_n} \to \mathbb{T}_{Q_n}$ is surjective. Furthermore, restricting the universal deformation $\rho^{\mathrm{univ}}_{S\cup Q_n}$ to Γ_{F_v}, $v \in Q_n$,

$$\rho^{\mathrm{univ}}_{S\cup Q_n}|_{\Gamma_{F_v}} \sim \begin{pmatrix} \psi_{\alpha_v} & 0 \\ 0 & \psi_{\beta_v} \end{pmatrix}$$

for a unique choice of characters $\psi_{\alpha_v}, \psi_{\beta_v}$ with $\psi_{\alpha_v}(\mathrm{Frob}_v) \equiv \alpha_v \pmod{\varpi}$. In the usual manner we obtain from local class field theory a homomorphism

$$\Delta_v \xrightarrow{\psi_{\alpha_v}} (R^{\mathrm{pst}}_{\bar\rho,S\cup Q_n})^\times \to \mathbb{T}^\times_{Q_n},$$

and the fundamental consequence of local-global compatibility at primes in Q_n is that composed with the action of $\mathbb{T}_{Q_n}$ on M_{Q_n}, this agrees with the diamond operator action of Δ_v on M_{Q_n} ([Tay06, Corollary 1.8]).

We summarize our modular input for Bloch-Kato vanishing:

Proposition 3.6. *For any set Q_n as above:*

(1) The $\mathcal{O}$-module $I_{Q_n}/I^2_{Q_n}$ is finitely-generated with an upper bound g on the number of generators depending only on $|Q_n|$.

(2) M_{Q_n} is a free $\mathcal{O}[\Delta_{Q_n}]$-module of rank equal to the $\mathcal{O}$-rank of $S(U,\mathcal{O})_{\mathfrak{m}}$, and $M_{Q_n}/\mathfrak{a}_{Q_n}M_{Q_n} \xrightarrow{\sim} S(U,\mathcal{O})_{\mathfrak{m}}$.

(3) The augmentation ideal $\mathfrak{a}_{Q_n}$ is contained in I_{Q_n}. The $\mathcal{O}$-module $M_{Q_n}/I_{Q_n}M_{Q_n}$ is independent of Q_n, isomorphic to $M_\emptyset/IM_\emptyset = S(U,\mathcal{O})_{\mathfrak{m}}/IS(U,\mathcal{O})_{\mathfrak{m}}$. Thus it is isomorphic to $\mathcal{O} \oplus A_\emptyset$ for some finite torsion $\mathcal{O}$-module $A_\emptyset$, and we set $a_\emptyset := \lg(A_\emptyset) = \lg((S(U,\mathcal{O})_{\mathfrak{m}}/IS(U,\mathcal{O})_{\mathfrak{m}})_{\mathrm{tor}})$.

(4) The induced map

$$\mathfrak{a}_{Q_n} M_{Q_n}/\mathfrak{a}_{Q_n}^2 M_{Q_n} \to I_{Q_n} M_{Q_n}/I_{Q_n}^2 M_{Q_n}$$

has cokernel with length $b_\emptyset = \lg(IM_\emptyset/I^2M_\emptyset)$ independent of Q_n.

(5) The congruence module $\Psi_{\lambda_{Q_n}}(M_{Q_n})$ of (2) is cyclic and satisfies

$$\lg(\Psi_{\lambda_{Q_n}}(M_{Q_n})) = \lg(\Psi_\lambda(M_\emptyset)) + \lg(\mathcal{O}/|\Delta_{Q_n}|).$$

In particular,

$$\lg(\Psi_{\lambda_{Q_n}}(M_{Q_n})) \geq \lg(\mathcal{O}/|\Delta_{Q_n}|) = e\, v_p(|\Delta_{Q_n}|) \geq ne|Q_n|.$$

Proof. (1): Let $\mathfrak{p}_{Q_n}$ be the kernel of the composite $R^{\mathrm{pst}}_{\bar\rho,S\cup Q_n} \to \mathbb{T}_{Q_n} \xrightarrow{\lambda_{Q_n}} \mathcal{O}$. There exists $g \in \mathbb{Z}$ depending only on $\bar\rho$ and $|Q_n|$ such that there is a surjection $\mathcal{O}[[X_1,\dots,X_g]] \twoheadrightarrow R^{\mathrm{pst}}_{\bar\rho,S\cup Q_n}$: indeed, we can take

$$g = \dim_k H^1_{\mathcal{D}_{S\cup Q_n}}(F, \mathrm{ad}^0(\bar\rho)) \leq \dim_k H^1_{\mathcal{D}_S}(F, \mathrm{ad}^0(\bar\rho)) + |Q_n|,$$

since Q_n being a set of Taylor-Wiles primes ensures $\dim H^1(\Gamma_{F_v}, \mathrm{ad}^0(\bar\rho))/H^1(\Gamma_{F_v}/I_{F_v}, \mathrm{ad}^0(\bar\rho)) = 1$ for all $v \in Q_n$. The $\mathcal{O}$-module surjection $\mathfrak{p}_{Q_n}/\mathfrak{p}_{Q_n}^2 \twoheadrightarrow I_{Q_n}/I_{Q_n}^2$ then implies $I_{Q_n}/I_{Q_n}^2$ is generated by at most g elements.

(2): By our assumption $[F(\zeta_p) : F] > 2$, the level subgroup U is sufficiently small in the sense that p does not divide the order of $(U(\mathbb{A}_F^\infty)^\times \cap g^{-1}D^\times g)/F^\times$ for any $g \in (D\otimes\mathbb{A}_F^\infty)^\times$: see the proof of [Tay06, Lemma 1.1]. Part (2) is then a standard consequence of the Q_n being sets of Taylor-Wiles primes: see [Tay06, Lemma 2.3].

(3): We have noted above that the diamond operators $\mathcal{O}[\Delta_v]$ are in $\mathbb{T}_{Q_n}$, and $\delta \in \Delta_v$ acts trivially on the U_{ϖ_v}-eigenform (with eigenvalue α_v) inducing λ_{Q_n} (see the calculation leading to [Gee22, Fact 5.7]). Thus $\mathfrak{a}_{Q_n}$ is contained in I_{Q_n}. Moreover,

$$M_{Q_n}/I_{Q_n}M_{Q_n} \xrightarrow{\sim} (M_{Q_n}/\mathfrak{a}_{Q_n}M_{Q_n})/(I_{Q_n}/\mathfrak{a}_{Q_n})(M_{Q_n}/\mathfrak{a}_{Q_n}M_{Q_n}) \xrightarrow{\sim} S(U,\mathcal{O})_{\mathfrak{m}}/IS(U,\mathcal{O})_{\mathfrak{m}},$$

where $I = \ker(\lambda)$. In particular, this last quotient is a finite $\mathcal{O}$-module of generic rank 1 (Lemma 2.1) that is independent of Q_n.

(4): The cokernel in question is isomorphic to $I_{Q_n}(M_{Q_n}/\mathfrak{a}_{Q_n})/I_{Q_n}^2(M_{Q_n}/\mathfrak{a}_{Q_n})$; since Q_n is a Taylor-Wiles set of primes, as noted above $\mathbb{T}_{Q_n}$ equals the reduced subalgebra $\mathbb{T}_{Q_n}^{S\cup Q_n}$, and, by part (2), the cokernel is isomorphic, as a $\mathbb{T}_{Q_n}^{S\cup Q_n} \twoheadrightarrow \mathbb{T}_\emptyset$-module, to $IM_\emptyset/I^2M_\emptyset$. As such, it is independent of Q_n, and it is finite since it is a finitely generated module over $\mathbb{T}_\emptyset/(I + \mathrm{Ann}_{\mathbb{T}_\emptyset}(I))$, which has finite cardinality since $\mathbb{T}_\emptyset$ is reduced.

(5): We use the congruence formalism of (2)–(4). The three modules M_Q, $M_Q^{\mathrm{St}} := S(U_0(Q),\mathcal{O})_{\mathfrak{m}_Q}$, and $M_\emptyset$ carry perfect $\mathcal{O}$-linear pairings $\langle\cdot,\cdot\rangle_V$, for $V = U_Q, U_0(Q), U$, for which the Hecke algebras $\mathbb{T}_Q$, $\mathbb{T}_Q^{\mathrm{St}}$, and $\mathbb{T}$ consist of self-adjoint operators (see [Tay06, p. 742, 752]). More precisely, recall from §2 the perfect pairing $\langle\cdot,\cdot\rangle$ on the codomain of our integral automorphic forms. We obtain perfect pairings $\langle\cdot,\cdot\rangle_{0,V}$ on each $S(V,\mathcal{O})$ using the formula from [Tay06, p.742], and *loc. cit.* also observes that the adjoint of the Hecke operator $[VgV]$ with respect to this pairing is $\chi_\iota(g)[Vg^{-1}V]$. It follows easily that for $v \notin S\cup Q$, T_v is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{0,V}$. Since we also want this self-adjointness for the operators U_{ϖ_v}, $v \in Q$, we instead use the pairing

$$\langle\varphi_1,\varphi_2\rangle_V = \prod_{v\in Q}\chi_\iota(\varpi_v)^{-1}\langle\varphi_1, w_Q\varphi_2\rangle_{0,V},$$

where $w_Q \in \mathrm{GL}_2(\mathbb{A}_F^\infty)$ has component $\begin{pmatrix} 0 & 1 \\ \varpi_v & 0 \end{pmatrix}$ for each $v \in Q$, and component 1 elsewhere. For this modified pairing, the full Hecke algebras $\mathbb{T}_Q$ and $\mathbb{T}_Q^{\mathrm{St}}$ are self-adjoint, and in particular we get induced perfect pairings on M_Q, M_Q^{St}, and $M_\emptyset$ to which we can apply the formalism of (2)–(4).

We have by part (2) (as explained in more detail in [Tay06, Lemmas 2.2–2.3 and Corollary 2.4]), $\mathbb{T}_Q \twoheadrightarrow \mathbb{T}_Q^{\mathrm{St}} \twoheadrightarrow \mathbb{T}$-equivariant maps $\beta = \beta_2 \circ \beta_1$

$$\beta\colon M_Q \xrightarrow{\beta_1} M_Q^{\mathrm{St}} \xrightarrow[\sim]{\beta_2} M_\emptyset$$

where β_1 is the trace, and $\beta_2^{-1}(f) = \prod_{v\in Q}\left(A_v - \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix}\right) f$, where $A_v \in \mathbb{T}$ is the unique root of $X^2 - T_v X + N(v)\chi_\iota(\varpi_v)$ lifting α_v. For any of these maps $\beta_?$, denote by $\beta_?^\dagger$ the adjoint with respect to the given pairings. Let φ span $M_\emptyset[I]$, and let φ_Q span $M_Q[I_Q]$. Since β is surjective, $\mathrm{coker}(\beta^\dagger)$ is torsion-free, and thus $\beta^\dagger(\varphi)$ equals φ_Q up to a unit $u \in \mathcal{O}^\times$. We now compute

$$\langle \varphi_Q, \varphi_Q\rangle_{U_Q} = u^{-2}\langle \beta^\dagger\varphi, \beta^\dagger\varphi\rangle_{U_Q} = u^{-2}\langle \varphi, \beta\circ\beta^\dagger\varphi\rangle_U.$$

We can compute $\beta\beta^\dagger = \beta_2\beta_1\beta_1^\dagger\beta_2^\dagger = |\Delta_Q|\beta_2\beta_2^\dagger$, since the definition of the pairings implies $\beta_1^\dagger$ is just the inclusion, and each diamond operator acts trivially on $\beta^\dagger(\varphi)$. Finally, β_2 is an isomorphism, so $\beta_2^\dagger$ is as well, and we conclude that $\lg(\Psi_{\lambda_Q}(M_Q)) = \lg(\Psi_\lambda(M_\emptyset)) + \lg(\mathcal{O}/|\Delta_Q|)$, which implies (5). □

Remark 3.7. In the toy case where $M_\emptyset$ is free of rank 1 over $\mathbb{T}_\emptyset$, $a_\emptyset = 0$, and $b_\emptyset = \lg(I/I^2)$ is the cotangent space in Wiles's numerical criterion.

Here and below, $\mathfrak{a}M/I^2M$ denotes the image of $\mathfrak{a}M$ in M/I^2M. We use the following simple lemma in the proof of Proposition 3.9.

Lemma 3.8. *Let $\lambda\colon T \to \mathcal{O}$ be an augmentation of a complete Noetherian local $\mathcal{O}$-algebra, let $I = \ker(\lambda)$, and let $\mathfrak{a} \subseteq I$ be an ideal of T. Let M be a finite T-module, and suppose that*

$$M/IM \simeq \mathcal{O}x \oplus A, \qquad A = (M/IM)_{\mathrm{tor}}.$$

Then the cokernel of the multiplication map

$$\mathfrak{a}/\mathfrak{a}^2 \longrightarrow \mathrm{Hom}_{\mathcal{O}}(M/IM, \mathfrak{a}M/I^2M)$$

is annihilated by $\mathrm{Ann}_{\mathcal{O}}(A)^2$.

Proof. Let t generate $\mathrm{Ann}_{\mathcal{O}}(A)$, and let

$$f \in \mathrm{Hom}_{\mathcal{O}}(M/IM, \mathfrak{a}M/I^2M).$$

Then tf vanishes on A. Write

$$tf(x) = \sum_i b_i m_i \pmod{I^2M}, \qquad b_i \in \mathfrak{a},$$

and write the image of m_i in M/IM as

$$m_i = c_i x + a_i, \qquad c_i \in \mathcal{O}, \quad a_i \in A.$$

Since $tA = 0$, we have

$$t^2 f(x) = \left(\sum_i t c_i b_i\right) x \pmod{I^2M}.$$

Multiplication by $\sum_i tc_ib_i \in t\mathfrak{a}$ kills $A \subset M/IM$ as a map to $\mathfrak{a}M/I^2M$. Hence $t^2 f$ is induced by multiplication by $\sum_i tc_ib_i$, and therefore lies in the image of $\mathfrak{a}/\mathfrak{a}^2 \longrightarrow \mathrm{Hom}_{\mathcal{O}}(M/IM, \mathfrak{a}M/I^2M)$. □

Proposition 3.9. *Fix an integer $r \geq 0$. There is a constant d'_r such that for all $n \geq 1$, and all Taylor–Wiles sets Q_n of level n and size $|Q_n| = r$, the image of $\mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2 \to I_{Q_n}/I_{Q_n}^2$ has length at least $ner - d'_r$. In terms of the constants of Proposition 3.6, we can take $d'_r = b_\emptyset + ra_\emptyset$.*

There is also a constant d_r such that the image of

$$\mathrm{Hom}_{\mathcal{O}}(I_{Q_n}/I_{Q_n}^2, \mathcal{O}/\varpi^{ne}) \to \mathrm{Hom}_{\mathcal{O}}(\mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2, \mathcal{O}/\varpi^{ne})$$

has length at least $\lg(\mathrm{Hom}_{\mathcal{O}}(\mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2, \mathcal{O}/\varpi^{ne})) - d_r = ner - d_r$.

Proof. Using Proposition 3.6, we write $M_{Q_n}/I_{Q_n}M_{Q_n} = \mathcal{O}x \oplus A_{Q_n}$, where x is not torsion, and $A_{Q_n} \cong A_\emptyset$ is up to isomorphism independent of Q_n. We control the image of $\mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2 \to I_{Q_n}/I_{Q_n}^2$ by considering their respective images in $\mathrm{End}(M_{Q_n}/I_{Q_n}^2M_{Q_n})$, via the commutative diagram

$$\begin{array}{ccc} \mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2 & \xrightarrow{\beta} & \mathrm{Hom}(M_{Q_n}/I_{Q_n}M_{Q_n}, \mathfrak{a}_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ I_{Q_n}/I_{Q_n}^2 & \xrightarrow{\delta} & \mathrm{Hom}(M_{Q_n}/I_{Q_n}M_{Q_n}, I_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n}), \end{array} \tag{5}$$

where the maps β and δ are induced by the action of $\mathbb{T}_{Q_n}$ on M_{Q_n}.

The image of β has length at least that of its quotient (by restricting the endomorphism) $\mathfrak{a}_{Q_n} \cdot x \subset \mathfrak{a}_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n}$, and again using that $M_{Q_n}/I_{Q_n}M_{Q_n} = \mathcal{O}x \oplus A_{Q_n}$ we find that

$$\lg(\mathrm{im}(\beta)) \geq \lg(\mathfrak{a}_{Q_n} \cdot x) \geq \lg(\mathfrak{a}_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n}) - c_{|Q_n|},$$

where $c_{|Q_n|} \leq |Q_n| \lg(A_{Q_n}) = |Q_n|a_\emptyset$ is bounded by the number of $\mathcal{O}$-module generators of $\mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2$ (namely $|Q_n|$) times the length of each such generator's image on A_{Q_n}.

Since γ is injective, and the cokernel of $\mathfrak{a}_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n} \to I_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n}$ has length bounded by $b_\emptyset$ (by part (4) of Proposition 3.6), we conclude that $\lg(\mathrm{im}(\gamma\circ\beta)) \geq \lg(I_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n}) - b_\emptyset - c_{|Q_n|}$. We can bound this in terms of the congruence ideal using basic properties of Fitting ideals:

$$\begin{aligned} \mathrm{Fitt}_{\mathcal{O}}(I_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n}) &= \mathrm{Fitt}_{\mathbb{T}_{Q_n}/I_{Q_n}}(I_{Q_n}M_{Q_n} \otimes_{\mathbb{T}_{Q_n}} \mathbb{T}_{Q_n}/I_{Q_n}) = \mathrm{Fitt}_{\mathbb{T}_{Q_n}}(I_{Q_n}M_{Q_n}) \otimes_{\mathbb{T}_{Q_n}} \mathbb{T}_{Q_n}/I_{Q_n} \\ &\subset \mathrm{Ann}_{\mathbb{T}_{Q_n}}(I_{Q_n}M_{Q_n}) \otimes_{\mathbb{T}_{Q_n}} \mathbb{T}_{Q_n}/I_{Q_n} = \lambda_{Q_n}(\mathrm{Ann}_{\mathbb{T}_{Q_n}}(I_{Q_n}M_{Q_n})) \\ &\subseteq \mathrm{Ann}_{\mathcal{O}}(\Psi_{\lambda_{Q_n}}(M_{Q_n})) = \eta_{\lambda_{Q_n}}(M_{Q_n}). \end{aligned}$$

Thus by part (5) of Proposition 3.6, we conclude (recall from part (5) of Proposition 3.6 that $\Psi_{\lambda_{Q_n}}(M_{Q_n})$ is cyclic) that

$$\lg(\mathrm{im}(\gamma \circ \beta)) \geq \lg(\Psi_\lambda(M_\emptyset)) + \lg(\mathcal{O}/|\Delta_{Q_n}|) - b_\emptyset - |Q_n|a_\emptyset,$$

and since $\gamma \circ \beta$ factors through α we obtain the same lower bound for $\lg(\mathrm{im}(\alpha))$. This proves the claim about d'_r. Since the source of $\gamma \circ \beta$ has length $\lg(\mathcal{O}/|\Delta_{Q_n}|)$, we also find that $\lg \ker(\gamma \circ \beta) \leq b_\emptyset + |Q_n|a_\emptyset - \lg(\Psi_\lambda(M_\emptyset))$.

For the second claim, we take the Pontryagin dual (denoted $(?)^*$) of the diagram (5). By exactness of the Pontryagin dual, $\lg(\mathrm{im}(\beta^* \circ \gamma^*)) = \lg(\mathrm{im}(\gamma \circ \beta))$. Moreover, $\mathrm{coker}(\gamma)$ is annihilated by $\mathrm{Ann}(IM_\emptyset/I^2M_\emptyset)$ by part (4) of Proposition 3.6, while Lemma 3.8 shows that $\mathrm{coker}(\beta)$ is annihilated by $\mathrm{Ann}(A_\emptyset)^2$. Thus $\mathrm{coker}(\gamma \circ \beta)$ is killed by

$$\varpi^{j_\emptyset}\mathcal{O} = \mathrm{Ann}(IM_\emptyset/I^2M_\emptyset)\ \mathrm{Ann}(A_\emptyset)^2.$$

We thus have an exact sequence

$$0 \to K_{Q_n} \to \mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2 \xrightarrow{\gamma\circ\beta} \mathrm{Hom}(M_{Q_n}/I_{Q_n}M_{Q_n}, I_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n}) \to L_{Q_n} \to 0$$

where $\lg(K_{Q_n})$ is bounded in a way depending only on $|Q_n|$, and the exponent of L_{Q_n} is bounded independently of Q_n (by $j_\emptyset$). If $ne \le j_\emptyset$, the asserted lower bound is automatic. We may therefore assume that $ne > j_\emptyset$. Now, any element of

$$\ker((\mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2)^* \to K_{Q_n}^*)[\varpi^{ne-j_\emptyset}]$$

is in the image of $(\gamma\circ\beta)^*[\varpi^{ne}]$, the restriction of $(\gamma\circ\beta)^*$ to the ϖ^{ne}-torsion. Hence

$$\lg(\mathrm{im}((\gamma\circ\beta)^*[\varpi^{ne}])) \ge \lg\Big(\mathrm{Hom}(\mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2, \mathcal{O}/\varpi^{ne-j_\emptyset})\Big) - \lg(K_{Q_n}).$$

In particular, since $\gamma\circ\beta$ factors through α, the image of

$$\mathrm{Hom}(I_{Q_n}/I_{Q_n}^2, \mathcal{O}/\varpi^{ne}) \to \mathrm{Hom}(\mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2, \mathcal{O}/\varpi^{ne})$$

has length at least

$$(ne - j_\emptyset)|Q_n| - b_\emptyset - a_\emptyset|Q_n| + \lg(\Psi_\lambda(M_\emptyset)) = ner - d_r,$$

where $r = |Q_n|$ and

$$d_r = r(j_\emptyset + a_\emptyset) + b_\emptyset - \lg(\Psi_\lambda(M_\emptyset)).$$

□

We now use a variant of Wiles's argument from [Wil95, Theorem 3.1(i)] to convert Proposition 3.9 for Taylor-Wiles sets into a bound on the Selmer group $H^1_{\mathcal{D}_S}(F, W_{E/\mathcal{O}})$. The following central proposition translates Proposition 3.9 into the growth of the Selmer groups $H^1_{\mathcal{D}_{S\cup Q_n}}(F, W_{ne})$.

Corollary 3.10. *For any $r \ge 0$ and any Taylor–Wiles set Q_n of level n and size $|Q_n| = r$,*

$$h^1_{\mathcal{D}_{S\cup Q_n}}(\Gamma_{F,S\cup Q_n}, W_{ne}) - h^1_{\mathcal{D}_S}(\Gamma_{F,S}, W_{ne}) \ge ner - d_r,$$

where d_r is the constant in Proposition 3.9.

Proof. The $\mathcal{O}$-algebra homomorphisms $\mathcal{O}[\Delta_{Q_n}] \xrightarrow{\theta} R^{\mathrm{pst}}_{\bar\rho,S\cup Q_n} \to \mathbb{T}_{Q_n}$ give rise to a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{O}/\varpi^{ne}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}_{Q_n}/\mathfrak{p}_{Q_n}^2, \mathcal{O}/\varpi^{ne}) & \xrightarrow{\theta^*} & \mathrm{Hom}_{\mathcal{O}}(\mathfrak{a}_{Q_n}/\mathfrak{a}_{Q_n}^2, \mathcal{O}/\varpi^{ne}) \\
\uparrow & & \uparrow & \nearrow_{\alpha^*} & \\
\mathrm{Hom}_{\mathcal{O}}(I/I^2, \mathcal{O}/\varpi^{ne}) & & \mathrm{Hom}_{\mathcal{O}}(I_{Q_n}/I_{Q_n}^2, \mathcal{O}/\varpi^{ne}), & &
\end{array}$$

and all we will need to know is that the top row is a complex (it is in fact exact). Proposition 3.9 shows that α^*, and thus θ^*, has image with length at least $ner - d_r$. Reinterpreting in Galois cohomology, the isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}_{Q_n}/\mathfrak{p}_{Q_n}^2, \mathcal{O}/\varpi^{ne}) \xrightarrow{\sim} H^1_{\mathcal{D}_{S\cup Q_n}}(\Gamma_{F,S\cup Q_n}, W_{ne}),$$

and analogously when $Q_n = \emptyset$, shows that

$$h^1_{\mathcal{D}_{S\cup Q_n}}(\Gamma_{F,S\cup Q_n}, W_{ne}) - h^1_{\mathcal{D}_S}(\Gamma_{F,S}, W_{ne}) \ge ner - d_r.$$

□

Lemma 3.11. *There is a constant c_S, depending only on the restrictions $\rho_{\Pi,\iota}|_{\Gamma_{F_v}}$ for $v \in S$, such that, for every $n \geq 1$ and every Taylor–Wiles set Q_n of level n,*

$$h^1_{\mathcal{D}_{S\cup Q_n}}(F, W_{ne}) - h^1_{\mathcal{D}^\perp_{S\cup Q_n}}(F, W^*_{ne}) - ne|Q_n| = c_{S,n},$$

where $0 \leq c_{S,n} \leq c_S$.

Proof. The Greenberg–Wiles formula gives

$$\begin{aligned} &h^1_{\mathcal{D}_{S\cup Q_n}}(F, W_{ne}) - h^1_{\mathcal{D}^\perp_{S\cup Q_n}}(F, W^*_{ne}) \\ &\quad = h^0(F, W_{ne}) - h^0(F, W^*_{ne}) + \sum_{v\in S\cup Q_n} \left(\lg \mathcal{D}_{S\cup Q_n, ne, v} - h^0(\Gamma_{F_v}, W_{ne})\right) \\ &\quad = \sum_{v\in S\setminus\{w|\infty\}} \left(\lg \mathcal{D}_{S\cup Q_n, ne, v} - h^0(\Gamma_{F_v}, W_{ne})\right) + ne|Q_n| - ne[F:\mathbb{Q}]. \end{aligned} \tag{6}$$

Here we have used the absolute irreducibility of $\bar\rho|_{\Gamma_{F(\zeta_p)}}$, the oddness of $\bar\rho$, and the usual local duality calculation at the Taylor–Wiles primes.

For each $v \in S$, let $\overline{R}^\square_v$ denote the following fixed-determinant lifting ring of $\bar\rho|_{\Gamma_{F_v}}$, with determinant $\kappa^{-1}\chi_\iota$. If $v \nmid p$, set

$$\overline{R}^\square_v = R^\square_{\bar\rho|_{\Gamma_{F_v}}}.$$

If $v \mid p$, let

$$\overline{R}^\square_v = R^{\square,\mathrm{pst}}_{\bar\rho|_{\Gamma_{F_v}}},$$

the lifting ring whose Artinian points are lifts occurring as subquotients of lattices in potentially semistable representations with Hodge–Tate weights in $[a, b]$ that become semistable over F'_v. A theorem of Tong Liu [Liu07] implies that the generic fiber $\overline{R}^\square_v[1/p]$ is isomorphic to the generic fiber of Kisin's potentially semistable deformation ring, which has the universal property of being the quotient of (the fixed determinant lifting ring) $R^\square_{\bar\rho|_{\Gamma_{F_v}}}$ whose points in finite E-algebras correspond to representations that become semistable over F'_v and have all Hodge-Tate weights in $[a, b]$ (see [Kis08, Theorem 2.7.6]).

Let $\mathfrak{p}_v$ be the kernel of the homomorphism $\overline{R}^\square_v \to \mathcal{O}$ defined by $\rho|_{\Gamma_{F_v}}$. Then

$$\lg \mathcal{D}_{S\cup Q_n, ne, v} = \lg \mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}_v/\mathfrak{p}_v^2, \mathcal{O}/\varpi^{ne}) - (\lg W_{ne} - h^0(\Gamma_{F_v}, W_{ne})).$$

Moreover,

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}_v/\mathfrak{p}_v^2, \mathcal{O}/\varpi^{ne}) \cong \mathrm{Hom}_{\mathcal{O}}(C_v, \mathcal{O}/\varpi^{ne}), \qquad C_v := \Omega_{\overline{R}^\square_v/\mathcal{O}} \otimes_{\overline{R}^\square_v, \rho} \mathcal{O}.$$

By local–global compatibility and the genericity of Π_v, the Weil–Deligne representation associated to $\rho|_{\Gamma_{F_v}}$ is generic; see [All16, Lemma 1.1.3]. Suppose that $v \mid p$, and let τ_v and $\mathbf{v}_v$ denote the Galois type and Hodge type of $\rho|_{\Gamma_{F_v}}$. By [All16, Theorem 1.2.7], the point defined by $\rho|_{\Gamma_{F_v}}$ is smooth on the generic fiber of the corresponding fixed-type deformation ring. It therefore lies on a unique component of that space. Since the Hodge and Galois types are uniquely determined at this point, this is also the unique component of $\mathrm{Spec}\, \overline{R}^\square_v[1/p]$ passing through it. Thus the completed local rings agree, and [All16, Lemma 1.2.5] identifies the image of the resulting Zariski tangent space in $H^1(F_v, W_E)$ with $H^1_g(F_v, W_E)$. For $v \nmid p$, the corresponding assertion follows from [All16,

Proposition 1.2.2]. Consequently, C_v has rank

$$d_v = \begin{cases} 3, & v \nmid p, \\ 3 + [F_v : \mathbb{Q}_p], & v \mid p. \end{cases}$$

Write

$$C_v \simeq \mathcal{O}^{d_v} \oplus \bigoplus_{i=1}^{t_v} \mathcal{O}/\varpi^{e_{v,i}}. \tag{7}$$

It follows that

$$\lg \mathcal{D}_{S\cup Q_n, ne, v} - h^0(\Gamma_{F_v}, W_{ne}) = \begin{cases} c_{v,n}, & v \nmid p, \\ c_{v,n} + ne[F_v : \mathbb{Q}_p], & v \mid p, \end{cases}$$

where

$$c_{v,n} = \sum_i \min\{ne, e_{v,i}\}, \qquad 0 \le c_{v,n} \le c_v := \sum_i e_{v,i}.$$

Since S contains all places above p,

$$\sum_{v|p} [F_v : \mathbb{Q}_p] = [F : \mathbb{Q}].$$

Equation (6) therefore gives

$$c_{S,n} = \sum_{v\in S} c_{v,n}, \qquad c_S = \sum_{v\in S} c_v,$$

which proves the lemma. □

Now by judicious choice of the sets Q_n, we obtain the main bound:

Proposition 3.12. *There is an integer $r \ge 0$ such that, for every $n \ge 1$, there is a set Q_n of Taylor–Wiles primes of level n, with $|Q_n| = r$, such that*

$$h^1_{\mathcal{D}_S}(F, W_{ne}) \le d_r + c_{S,n} \le d_r + c_S,$$

where d_r is the constant appearing in Proposition 3.9, and $c_{S,n}$ and c_S are the local error terms defined in Lemma 3.11.

Proof. Since $\bar\rho|_{\Gamma_{F(\zeta_p)}}$ is absolutely irreducible, $(\mathrm{ad}^0(\bar\rho)^*)^{\Gamma_F}$ vanishes, and inductively $(W_m^*)^{\Gamma_F} = 0$ for all $m \ge 1$. Thus for any $n \ge 1$, the map (induced from the kernel of multiplication by ϖ) $H^1(\Gamma_{F,S\cup Q}, W_1^*) \to H^1(\Gamma_{F,S\cup Q}, W_{ne}^*)[\varpi]$ is an isomorphism (for any Q). For any Taylor-Wiles prime v of level n, $H^1(\Gamma_{F_v}, W_1^*) \to H^1(\Gamma_{F_v}, W_{ne}^*)[\varpi]$ is an isomorphism (since $(W_{ne}^*)^{\Gamma_{F_v}} \to (W_{ne-1}^*)^{\Gamma_{F_v}}$ is surjective in this case). Thus (c.f. the diagram on [Wil95, p. 522]) if we can show there is a Taylor-Wiles set Q_n of level n such that $H^1(\Gamma_{F,S}, W_1^*) \to \prod_{v\in Q_n} H^1(\Gamma_{F_v}, W_1^*)$ is injective, then the same conclusion holds with W_{ne}^* coefficients. For W_1^*, this is standard, and the argument of [DDT94, Theorem 2.49] (see also [Kis09, Proposition 3.2.5]) shows we can always find such Q_n with $|Q_n| = r := h^1(\Gamma_{F,S}, W_1^*)$ (this relies on Assumption 3.1). In particular, for such Q_n, we have $h^1_{\mathcal{D}^\perp_{S\cup Q_n}}(\Gamma_F, W_{ne}^*) = 0$.

Combining Corollary 3.10 with Lemma 3.11, we find

$$\begin{aligned} h^1_{\mathcal{D}_S}(F, W_{ne}) &\le h^1_{\mathcal{D}_{S\cup Q_n}}(F, W_{ne}) - ner + d_r \\ &= h^1_{\mathcal{D}^\perp_{S\cup Q_n}}(F, W^*_{ne}) + c_{S,n} + d_r \\ &= c_{S,n} + d_r \\ &\le c_S + d_r. \end{aligned}$$

□

Remark 3.13. With a bit more care, we can decrease the value of $r = |Q_n|$ as follows. If the preimage of $\mathcal{D}^\perp_{S,ne,v}$ under $W_1^* \to W_{ne}^*$ were $\mathcal{D}^\perp_{S,1,v}$ for all $v \in S$, we could deduce that $H^1_{\mathcal{D}^\perp_{S,1}}(\Gamma_{F,S}, W_1^*) \to H^1_{\mathcal{D}^\perp_{S,ne}}(F, W_{ne}^*)[\varpi]$ is an isomorphism, and then as above the argument of [DDT94, Theorem 2.49] would apply to annihilate the dual Selmer group for W_{ne}^* with $r = h^1_{\mathcal{D}^\perp_S}(\Gamma_{F,S}, W_1^*)$ primes. This statement about the preimage is not, however, true in general. If we write $f\colon W_{ne} \to W_1$ and $f^*\colon W_1^* \to W_{ne}^*$ for the (dual) multiplication by ϖ^{ne-1} maps (identifications using the choice of ϖ left implicit), then a local class $\psi_v \in H^1(\Gamma_{F_v}, W_1^*)$ satisfies $(f^*\psi_v) \in (\mathcal{D}_{ne,v})^\perp$ if and only if $\psi_v \in (f(\mathcal{D}_{ne,v}))^\perp$. Since each of our local conditions $\mathcal{D}_{m,v}$ is given as (the image in cohomology of) $\mathrm{Hom}(C_v, \mathcal{O}/\varpi^m)$ (C_v as in the proof of Lemma 3.11), $\dim(\mathcal{D}_{1,v}/f(\mathcal{D}_{ne,v}))$ is for $n \gg 0$ exactly the number of torsion factors t_v appearing in Equation (7). Thus a global class $\psi \in H^1_{\mathcal{D}^\perp_{S,ne}}(F, W_{ne}^*)[\varpi]$ belongs to a Selmer group $H^1_{L'_S}(\Gamma_{F,S}, W_1^*)$ where for all $v \in S$, $\dim(L'_v/\mathcal{D}^\perp_{S,v,1}) = t_v$. In particular, we can take

$$r = h^1_{\mathcal{D}^\perp_{S,1}}(\Gamma_{F,S}, W_1^*) + \sum_{v\in S} t_v,$$

as asserted in Theorem 1.1.

Corollary 3.14. *Choose r as in Proposition 3.12. Then $H^1_{\mathcal{D}_S}(F, W_{E/\mathcal{O}})$ is finite with order bounded by $|k|^{c_S+d_r}$. Consequently, $H^1_g(\Gamma_{F,S}, W_E) = 0$.*

Proof. The first assertion follows immediately from Proposition 3.12 upon taking the direct limit. For the second, taking the inverse limit over the cofinal sequence $m = ne$, the same bound gives

$$\left(\varprojlim_m H^1_{\mathcal{D}_S}(F, W_m)\right) \otimes_{\mathcal{O}} E = 0.$$

By deformation theory, the left-hand side is naturally identified with

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, E),$$

where $\mathfrak{p}$ is the kernel of the augmentation of $R^{\mathrm{pst}}_{\bar\rho,S}$ corresponding to ρ. This space parametrizes self-extensions of $\rho \otimes_{\mathcal{O}} E$, with fixed determinant and unramified outside S, whose restrictions at $v \mid p$ contain lattices that are inverse limits of objects satisfying the torsion potentially semistable condition used above. As noted in the proof of Lemma 3.11, Liu's proof of Fontaine's conjecture [Liu07], applied after restriction to F'_v and a Tate twist, identifies this characteristic-zero local condition with potential semistability over F'_v with Hodge–Tate weights in $[a, b]$. We also observed in Lemma 3.11 that, for each $v \mid p$, [All16, Lemma 1.2.5] identifies the image in $H^1(F_v, W_E)$ of the tangent space of $\overline{R}^{\square}_v[1/p]$ at the point defined by $\rho|_{\Gamma_{F_v}}$ with $H^1_g(F_v, W_E)$. Since the local deformation conditions at the places in S away from p are unrestricted, it follows that

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, E) \simeq H^1_g(\Gamma_{F,S}, W_E),$$

and hence $H^1_g(\Gamma_{F,S}, W_E) = 0$. □

We now return to the original field F_0 to deduce our final theorem.

Theorem 3.15. *Restriction induces an injection*

$$H^1_g(\Gamma_{F_0}, \mathrm{ad}^0(\rho_{\Pi_0,\iota}) \otimes E) \hookrightarrow H^1_g(\Gamma_F, \mathrm{ad}^0(\rho_{\Pi,\iota}) \otimes E).$$

In particular, the Selmer group on the left vanishes. Moreover, if we define local conditions $\mathcal{D}_{S_0}$ for deformations of $\bar\rho_{\Pi_0,\iota}$ exactly as we did with $\mathcal{D}_S$ (see the discussion after Definition 3.5), then

$$\lg H^1_{\mathcal{D}_{S_0}}(F_0, \mathrm{ad}^0(\rho_{\Pi_0,\iota}) \otimes E/\mathcal{O}) \le c_S + d_r = [F : F_0] \sum_{v_0 \in S_0} c_{v_0} + d_r,$$

where c_{v_0} is the local error term as defined in Lemma 3.11 (equal to c_v for any $v|v_0$ since F/F_0 is split at v_0).

Proof. The first assertion follows from inflation-restriction. With integral coefficients, note that the local conditions $\mathcal{D}_{S_0,v_0}$ at $v_0 \in S_0$ restrict isomorphically to the local conditions $\mathcal{D}_{S,v}$ for all $v \mid v_0$, since F/F_0 is split at v_0. Since $[F : F_0]$ is either 1 or 2, and hence is prime to p, we see that for all $m \ge 1$ the restriction

$$H^1_{\mathcal{D}_{S_0}}(F_0, \mathrm{ad}^0(\rho_{\Pi_0,\iota}) \otimes \mathcal{O}/\varpi^m) \xrightarrow{\sim} H^1_{\mathcal{D}_S}(F, W_m)^{\mathrm{Gal}(F/F_0)},$$

is an isomorphism. The length of the source is at most that of the target before taking $\mathrm{Gal}(F/F_0)$-invariants. The result follows from Proposition 3.12. □

4. Implications and future directions

We hope to generalize this work in two natural directions: to higher-rank Π_0, under usual Taylor-Wiles hypotheses (adequacy as in [Tho12]), and to more degenerate images $\bar\rho_{\Pi_0,\iota}$. We comment briefly on each of these.

If Π_0 is a regular algebraic polarizable (in the sense of [BLGGT14]) automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ for a CM field F, with $\bar\rho = \bar\rho_{\Pi_0,\iota}$ irreducible and $\bar\rho|_{\Gamma_{F(\zeta_p)}}$ adequate, then we expect the template of the present paper to apply to prove vanishing of a suitable adjoint Bloch-Kato Selmer group (classifying polarized deformations). For this, one must combine the higher-rank arguments with Taylor-Wiles sets as in [Tho12] with estimates on congruence modules using the methods of [CHT08, §5, Appendix] (the desired lower bounds, as in part (5) of Proposition 3.6, should not require Ihara's lemma). Note that the methods of the present paper require the weak multiplicity 1 condition from Lemma 2.1, and in higher rank one makes a base-change (with some loss of integral information) to reduce the analysis to a unitary group where the relevant π has multiplicity 1, using [Lab11, Théorème 5.4]. The output of this analysis would be new quantitative bounds in higher rank for torsion adjoint Selmer groups, refining the vanishing after inverting p first proven in this setting by Allen [All16].

Of course, even for GL_2 over a totally real field, we have not treated cases where $\bar\rho$ is reducible or where Taylor–Wiles sets as in Proposition 3.12 do not exist. In this setting one must work with sets Q_n of "approximate" Taylor–Wiles primes: the Frobenius eigenvalues need not be distinct modulo ϖ, but are distinct modulo ϖ^M for some M depending on the image of $\rho_{\Pi,\iota}$ and independent of n. One must also replace $R^{\mathrm{pst}}_{\bar\rho,S}$ by a suitable pseudodeformation ring; Newton and Thorne developed the necessary techniques for cohomological control of these rings in [NT23, § 2].

A significant obstacle is that, for approximate Taylor–Wiles sets, parts (2)–(4) of Proposition 3.6 fail. Instead of an isomorphism

$$M_{Q_n}/I_{Q_n}M_{Q_n} \simeq \mathcal{O} \oplus A_\emptyset$$

with $A_\emptyset$ independent of n, one can probably only bound the exponent of $(M_{Q_n}/I_{Q_n}M_{Q_n})_{\mathrm{tor}}$. We have an approach to handling this weaker control by modifying Propositions 3.6 and 3.9, but it depends on the following hypothesis.

Hypothesis 4.1. *Let $\bar{I}_{Q_n}$ be the image of I_{Q_n} in*

$$\mathrm{Hom}_{\mathcal{O}}(M_{Q_n}/I_{Q_n}M_{Q_n}, I_{Q_n}M_{Q_n}/I_{Q_n}^2M_{Q_n}).$$

Then

$$\mathrm{Fitt}_{\mathcal{O}}(\bar{I}_{Q_n}) \subseteq \eta_{\lambda_{Q_n}}(M_{Q_n}).$$

More abstractly, let $\lambda\colon T \to \mathcal{O}$ and M be as in the setup of Section 2, put $I = \ker(\lambda)$, and set

$$\bar{I}_M := \mathrm{im}\left(I/I^2 \longrightarrow \mathrm{Hom}_{\mathcal{O}}(M/IM, IM/I^2M)\right).$$

One may ask whether

$$\mathrm{Fitt}_{\mathcal{O}}(\bar{I}_M) \subseteq \eta_\lambda(M). \tag{8}$$

For $M = T$, this is the standard elementary containment

$$\mathrm{Fitt}_{\mathcal{O}}(I/I^2) \subseteq \eta_\lambda(T).$$

In general, however, (8) is false.[4]

The automorphic modules M_{Q_n} considered here are self-dual as $\mathbb{T}_{Q_n}$-modules, and we know no counterexample to (8) for self-dual M. Whether or not the precise containment (8) holds in that case, one may ask more generally for lower bounds for $\lg(\bar{I}_M)$ in terms of invariants intrinsic to M, such as its congruence ideal $\eta_\lambda(M)$.

If Hypothesis 4.1 holds, or if a useful weakening of it can be established for the self-dual automorphic modules considered here, then our methods might extend beyond the traditional Taylor–Wiles setting. This could yield a new proof of the vanishing theorem of [NT23], together with explicit bounds on the corresponding torsion Bloch–Kato Selmer groups.

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[4]For example, take $\mathcal{O} = \mathbb{Z}_3$,

$$T = \mathcal{O}[x,y]/(x(x-3)(x-9)+3xy,\ y(y-3)(y-9)+3xy), \qquad \lambda(x) = \lambda(y) = 0,$$

and

$$M = \mathcal{O}\left\langle 1, \frac{x+6}{9}, \frac{x^iy^j}{9} : 0 \le i,j \le 2,\ (i,j) \notin \{(0,0),(1,0)\}\right\rangle \subset T[1/3].$$

Then T is a reduced finite-flat local complete intersection of $\mathcal{O}$-rank 9, and M is a T-stable lattice, generically free of rank one. For $I = \ker(\lambda)$, one computes $\bar{I}_M \simeq (\mathcal{O}/9)^2$ and $\eta_\lambda(M) = (3^5)$. Hence $\mathrm{Fitt}_{\mathcal{O}}(\bar{I}_M) = (3^4) \not\subseteq (3^5) = \eta_\lambda(M)$. The module M is not self-dual. This example was found over several interactions with ChatGPT.

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