so $x_m = tv\, x_m$, so $tv = 1$, a contradiction. ✓

Uniqueness: If $x = u_1 t^{n_1} = u_2 t^{n_2}$, wma $n_1 \geq n_2$, and then $t^{n_1 - n_2} = u_1^{-1} u_2 \in R^\times$, so $n_1 - n_2 = 0$ and thus $u_1 = u_2$ as well.

R is a PID because for $(0) \subsetneq I \subset R$, we set $n_I = \min\{n \in \mathbb{Z}_{\geq 0} \mid t^n \in I\}$. Then $I = (t^{n_I})$. ∎

Finally, given a DVR, we can define the valuation

$$v_R : \mathrm{Frac}(R)^\times \longrightarrow \mathbb{Z} \quad \text{by}$$

$v_R : R \setminus 0 \longrightarrow \mathbb{Z}_{\geq 0}$ is $v_R(x) =$ the $n \in \mathbb{Z}_{\geq 0}$: $x = t^n \cdot u,\ u \in R^\times$.

On $\mathrm{Frac}(R)^\times$, $v_R(a/b) = v_R(a) - v_R(b)$ for $a, b \in R$.

This satisfies
- group hom
- $v_R(x+y) \geq \min\{v_R(x), v_R(y)\}$

Ex: $v_p : \mathbb{Q}^\times \longrightarrow \mathbb{Z}$

$p^n \cdot \frac{a}{b} \longmapsto n$

$(ab, p) = 1$

$v_{(t-a)} : \bar{k}(t)^\times \longrightarrow \mathbb{Z}$, $a \in \bar{k}$

$(t-a)^n \cdot \frac{g(t)}{h(t)}$

$(gh, t-a) = 1$

Shangjie's talk

also have

$v_\infty : \bar{k}(t)^\times \longrightarrow \mathbb{Z}$

On $\bar{k}[t]$, this is $v_\infty(f) = -\deg(f)$

$\bar{k}[1/t] \subset \bar{k}(t)$ and $\bar{k}[1/t]_{(1/t)}$ is a DVR, and this is the associated valuation.

this corresponds to $\infty \in \mathbb{P}^1$,

so we have a bijection

$$\boxed{\mathbb{P}^1 \longleftrightarrow \text{surj. discrete valuations on } \bar{k}(t)}.$$

Now we prove the Prop: $P \in C \subset \mathbb{A}^n$ a nonsingular point on the curve $C \Rightarrow \mathcal{O}_{C,P}$ is a DVR. (next time)

(We've just seen some examples of this.)

Pf: $\mathcal{O}_{C,P}$ is a Noetherian local domain, so we must show $\mathfrak{m}_{C,P}$ is principal. $P \in C$ nonsingular (curve) tells us if $I(C) = (f_1, \dots, f_r)$, then

$$J(P) \rightarrow \mathrm{rk}\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}(P) \quad \text{is } n-1 \qquad \left(\frac{\partial (x_{i_0} - p_{i_0})}{\partial x_j}\right)_j$$

$$\downarrow i_0$$

Let $P = (p_1, \dots, p_n)$; $\exists\, i_0$ such that $(0, 0 \dots \underset{i_0}{1} \dots 0)$ together w/ the rows of $J(P)$ spans $\bar{k}^n$.

So for any $g \in \bar{k}[x_1, \dots, x_n]$ such that $g(P) = 0$, the Taylor expansion of g at P looks like

$$g(x_1, \dots, x_n) = \underbrace{\cancel{g(P)}}_{0} + \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(P)(x_j - p_j) + \text{higher order in } x_j - p_j$$

$$= \text{some combo of } \left(\frac{\partial f_i}{\partial x_j}(P)\right)_j \text{ (for } i = 1, \dots, r\text{) and } \left(\frac{\partial (x_{i_0} - p_{i_0})}{\partial x_j}\right)_j$$

so $\underset{\bar{k}[C]}{\bar{g}} \in (\underbrace{\overline{x_{i_0} - p_{i_0}}}_{t}) + \mathfrak{m}_{C,P}^2$ $\qquad$ $\mathfrak{m}_{C,P} = \mathfrak{m}_P$

<u>Conclude</u>: $\mathfrak{m}_{C,P} = (t) + \mathfrak{m}_{C,P}^2$

This forces $\mathfrak{m}_{C,P} = (t)$ by <u>Nakayama's Lemma</u>.

$X = \mathfrak{m}_{C,P}/(t)$ satisfies: X is a f.g. $\mathcal{O}_{C,P}$-module such that $\mathfrak{m}_{C,P} \cdot X = X$. (b/c $\mathfrak{m}_P \cdot \mathfrak{m}_P + (t) = \mathfrak{m}_P$)

This forces $X = 0$: else we take a <u>minimal</u> set of generators $m_1, \dots, m_s$ of X as $\mathcal{O}_{C,P}$-module.

As $m_s \in X$, we can write $m_s = \sum_{i=1}^{s} a_i m_i$ with $a_i \in \mathfrak{m}_P$.

$\rightsquigarrow (1 - a_s) m_s = \sum_{i=1}^{s-1} a_i m_i$. But $1 - a_s \in \mathcal{O}_{C,P}^\times$, so $m_s \in \sum_{i=1}^{s-1} \mathfrak{m}_P \cdot m_i$, contradicting minimality of $m_1, \dots, m_s$. ∎

Let's use the Prop to make some important constructions and observations.

$\boxed{\text{Let } C \text{ be a nonsingular projective curve}}$

General Case

Let $f \in \bar{k}(C)^*$

$\forall\, P \in C$, we have the discrete valuation $v_P : \bar{k}(C)^* \twoheadrightarrow \mathbb{Z}$

so can compute $v_P(f)$

We say f has a $\begin{cases} \text{zero} \\ \text{pole} \end{cases}$ at P if $\begin{matrix} v_P(f) > 0 \\ v_P(f) < 0 \end{matrix}$

Case $C = \mathbb{P}^1$

$f(t) \in \bar{k}(t)^*$, i.e.

$f(t) = \dfrac{g(t)}{h(t)}$ $\quad g, h \in \bar{k}[t]$ non-zero

$P = [a, 1]$, $a \in \bar{k}$ $\rightsquigarrow v_{t-a}(f)$

$P = [1, 0]$ "∞" $\rightsquigarrow v_\infty(f) = -\deg(f)$
$= -\deg(g) + \deg(h)$

<u>Prop</u>: • $v_P(f) = 0$ for all but finitely many $P \in C$.

• If $v_P(f) \geq 0$ $\forall\, P \in C$, then $f \in \bar{k}^*$

uses projective (global regular functions on a projective variety are constants)

a generalization of our result that if $f \in \bar{k}[x,y]$ irreducible, and $f \nmid g$, then $V(f, g)$ is finite.

<u>Pf for $\mathbb{P}^1$</u>: • for only finitely many $a \in \bar{k}$ does $t-a$ divide $g(t)$ or $h(t)$

• $f = g/h$. wma $(g, h) = 1$.
If $v_{t-a}(f) = v_{t-a}(g) - v_{t-a}(h)$ is ≥ 0 $\forall\, a \in \bar{k}$, then $h \in \bar{k}$ is constant, and $f \in \bar{k}[t] \setminus 0$. But $v_\infty(f) = -\deg(f) \geq 0$ forces f to be constant. ∎

We define the <u>divisor group</u> of C to be

$$\mathrm{Div}(C) = \bigoplus_{P \in C} \mathbb{Z}[P] \quad \text{(the free ab. gp. on } C\text{)}$$

$$= \left\{ \sum_{P \in C} c_P \cdot [P] \;\middle|\; c_P \in \mathbb{Z} \text{ and all but finitely many } c_P \text{ are } 0 \right\}$$

$\mathrm{Div}(\mathbb{P}^1)$

$$= \left\{ \sum_{[a_1, a_2] \in \mathbb{P}^1} c_{[a_1, a_2]} [a_1, a_2] \text{ where almost all } c\text{'s are } 0 \right\}$$

By the Prop, we have a well-defined group homomorphism

$$\mathrm{div}: \bar{k}(C)^\times \longrightarrow \mathrm{Div}(C)$$

$$\mathrm{div}(f) = \sum_{P\in C} v_P(f)\cdot[P],$$

and $\ker(\mathrm{div}) = \bar{k}^\times$

What about the image?

- Define the degree of a divisor

$$\deg: \mathrm{Div}(C) \longrightarrow \mathbb{Z}$$

$$\sum_{P\in C} c_P\cdot[P] \longmapsto \sum_{P\in C} c_P, \text{ and}$$

let $\mathrm{Div}^0(C) = \ker(\deg)$

Then $\forall f\in \bar{k}(C)^\times$,

$$\boxed{\deg(\mathrm{div}(f)) = 0}$$

(can prove by reduction to the case $C=\mathbb{P}^1$ using $f: C\to\mathbb{P}^1$)

Defn: ① $\mathrm{Pic}(C) = \mathrm{Div}(C)/\mathrm{div}(\bar{k}(C)^\times)$ (Picard)

② $\mathrm{Pic}^0(C) = \mathrm{Div}^0(C)/\mathrm{div}(\bar{k}(C)^\times)$

For $C\neq\mathbb{P}^1$, $\mathrm{Pic}^0(C)$ is very interesting!

Let $f(t) = \dfrac{g(t)}{h(t)}$

$$= \frac{\prod_{a\in\bar{k}}(t-a)^{n_a}}{\prod_{a\in\bar{k}}(t-a)^{m_a}}\cdot(\text{cst})$$

(almost all n_a and m_a are zero)

so $\mathrm{div}(f) = \mathrm{div}(g) - \mathrm{div}(h)$

$$= \sum_{a\in\bar{k}}(n_a-m_a)\cdot[a,1]$$

$$-\left(\sum_{a\in\bar{k}}(n_a-m_a)\cdot[1,0]\right)$$

so $\deg(\mathrm{div}(f))$
$= 0$

Prop: $\mathrm{Pic}^0(\mathbb{P}^1) = (0)$.

Pf: Let $D\in\mathrm{Div}^0(\mathbb{P}^1)$

$$D = \sum_{a\in\bar{k}} n_a\cdot[a] - \sum_{a\in\bar{k}} n_a\cdot[\infty]$$

Define

$$f(t) = \prod_{a\in\bar{k}}(t-a)^{n_a}$$

Then $\mathrm{div}(f) = D$ ∎

Cor: $\mathrm{Pic}(\mathbb{P}^1) \xrightarrow[\sim]{\deg} \mathbb{Z}$ is an isomorphism.

How do we understand $\mathrm{Pic}^0(C)$?
What are the possible zeroes and poles of elements of $\bar{k}(C)^\times$?

$\boxed{\text{Riemann–Roch Theorem.}}$ ((Continue to let C be a nonsingular irreducible projective curve.))

Define a partial order on $\mathrm{Div}(C)$ by

$$\sum c_P \cdot [P] \geq \sum d_P \cdot [P] \iff c_P \geq d_P \ \forall P \in C.$$

Ex: $\mathrm{div}(f) \geq 0 \iff f$ has no poles $\iff f \in \bar{k}^\times$.

Spaces of functions with bounded zeroes and poles

For $D \in \mathrm{Div}(C)$, let $L(D)$ be the $\bar{k}$-vector space (WHY?)

$$L(D) = \{ f \in \bar{k}(C)^\times \mid \mathrm{div}(f) \geq -D \} \cup \{0\}$$

Let $\ell(D) = \dim_{\bar{k}} L(D)$

Ex: $D = [P]$
$L(D) = \{ f \mid f$ has no poles away from P and a pole of order at most 1 at $P \}$

Lemma: $D \in \mathrm{Div}(C)$

1. If $\deg(D) < 0$, then $L(D) = \{0\}$
2. $L(D)$ is finite-dimensional.
3. $L(D - \mathrm{div}(f)) \xrightarrow{\sim} L(D) \quad \forall f \in \bar{k}(C)^\times$.

$$g \longmapsto g \cdot f$$

Pf: ① If $f \neq 0$ and $\mathrm{div}(f) \geq -D$, then "$\deg(\mathrm{div} f) \geq -\deg(D)$ > 0", a contradiction.

② For any $D, D' \in \mathrm{Div}(C)$ with $D \leq D'$, $L(D) \subseteq L(D')$, and we claim $\dim L(D')/L(D) \leq \deg(D'-D)$ (so is f.d.)

By induction, STP $\dim L(D+P)/L(D) \leq 1$ $\forall P \in C$.

Let c_P = the coeff. of $[P]$ in D, and let $t \in \mathcal{O}_{C,P}$ be a uniformizer. Define the linear map

$$\begin{array}{ccc} L(D+P) & \longrightarrow & \bar{k} \\ \cup\!\!\!\!\;\shortmid & & \cup\!\!\!\!\;\shortmid \\ f & \longmapsto & (t^{c_P+1} f)(P) \end{array}$$

whose kernel is $L(D)$. Now inductively reduce to the case $D = 0$. ∎ ③ - exercise (easy).

$\boxed{\text{Theorem}}$ C as before. There is an integer $g = g(C) \geq 0$ and a divisor $K \in \mathrm{Div}(C)$ such that $\forall D \in \mathrm{Div}(C)$,

$$\boxed{\ell(D) - \ell(K - D) = \deg(D) + 1 - g.}$$

(by the Lemma, any representative of $[K] \in \mathrm{Pic}(C)$ has this property).

K is called a canonical divisor on C.

g is called the genus of C

$\boxed{\text{Cor:}}$ ① $\ell(K)=g$: take $D=0$ and get

$\underbrace{\ell(0)}_{\dim(\text{cst fns})} - \ell(K) = 0+1-g$ so $\ell(K)=g$

② $\deg(K) = 2g-2$: take $D=K$ and get.

$\underbrace{\ell(K)}_{g} - \underbrace{\ell(0)}_{1} = \deg(K)+1-g$. , so $\deg(K)=2g-2$

(*) ③ If $\deg(D)>2g-2$, then $\ell(D)=\deg(D)+1-g$:
indeed, then $\deg(K-D)<0$, so (last lemma) $L(K-D)=\{0\}$. ☒

Eg: $C=\mathbb{P}^1$. We know $\mathrm{Pic}(C) \xrightarrow[\deg]{\sim} \mathbb{Z}$

For any $D\in \mathrm{Div}(C)$, $D=\deg(D)\cdot[\infty] + \mathrm{div}(f)$ for some $f\in \bar{k}(\mathbb{P}^1)^\times$.

Claim: Taking $g=0$ and $K=-2\cdot[\infty]$, the R-R formula holds

$\ell(D)-\ell(K-D) = \ell(\deg(D)\cdot[\infty]) - \ell((-2-\deg D)\cdot[\infty])$

For $n\geq 0$, $L(n\cdot[\infty]) = \{f\in \bar{k}(t)^\times \mid \mathrm{div}(f)\geq -n\cdot[\infty]\}$
$= \{$ degree $\leq n$ polynomials in $\bar{k}[t]\}$ ($\dim = n+1$).

For $n<0$, $L(n\cdot[\infty]) = (0)$.

So $\ell(D)-\ell(K-D) = \begin{cases} -[-2-\deg D+1] & \deg D\leq -2 \\ 0 & \deg D = -1 \\ 1+\deg(D) & \deg D\geq 0 \end{cases}$

$= \deg(D)+1-0$ ($0 = g$)

Conversely: Lemma: $g(C)=0 \Rightarrow C \cong \mathbb{P}^1$. (over $\bar{k}$)

Pf: Take $P, Q \in C$, $P \neq Q$. Apply R-R to $D = P - Q$. $\deg(D) > 2g-2 = -2$, so $\ell(D) = \deg(D) + 1 - g = 1$. Set $L(D) = \bar{k} \cdot f$

f is not constant ($f(Q)=0$), so it defines a nonconstant morphism

$$\begin{aligned} C &\longrightarrow \mathbb{P}^1 \\ \alpha &\longmapsto [f(\alpha), 1] \text{ if } f \in \mathcal{O}_{C,\alpha} \\ \alpha &\longmapsto [1, 0] \text{ if } f \notin \mathcal{O}_{C,\alpha} \end{aligned}$$

of degree 1, which is therefore an isomorphism.

Huh?! We haven't defined "morphisms" of varieties! I'll review a tiny bit, but you should read up — we won't use much, though.

But first I want to define g and K...

Differential Forms

Aim: For a curve C over k, define the $k(C)$-vector space $\Omega_{k(C)/k}$ of "meromorphic differentials" on C.

More generally, for any k-algebra R, we define:

a symbol for a basis element

$$\Omega_{R/k} := \bigoplus_{x\in R} R\cdot[x] \Big/ \left(\begin{array}{l}\text{the } R\text{-submodule generated by} \\ \bullet\ [x+y]-[x]-[y] \quad \forall\, x,y\in R \\ \bullet\ [c\cdot x]-c\cdot[x] \quad \forall\, c\in k, x\in R \\ \bullet\ [xy]-x[y]-y[x]\end{array}\right)$$

R-module

Write $dx \in \Omega_{R/k}$ for the image of $[x]$.

The idea here: we have in a universal way imposed the "differentiation rules" $d(x+y)=dx+dy$, $d(cx)=c\,dx$, $d(xy)=x\,dy+y\,dx$ (product rule)

Example: $R=k[t]\ni f(t)$. Then $d\,f(t)=f'(t)\cdot dt$

More generally, for $R=k[t_1,\dots,t_n]$,

$$\Omega_{R/k}=\bigoplus_{i=1}^{n} R\,dt_i \quad (\text{free } R\text{-module}).$$

since $d\,f(t_1,\dots,t_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i}(t_1,\dots,t_n) \cdot dt_i$

- $\Omega_{k(t_1,\dots,t_n)/k} = \bigoplus_{i=1}^{n} k(t_1,\dots,t_n) \cdot dt_i$ (vector space of rank n)

(combine last example with $d(f/g) = \frac{g\,df - f\,dg}{g^2}$).

Abbreviate $\Omega_{\bar{k}(C)/\bar{k}} =: \Omega_C$

<u>Prop:</u> ① For any curve C, Ω_C is a $\bar{k}(C)$-vector space of dimension 1.

② For $x \in \bar{k}(C) \setminus \bar{k}$, $\Omega_C = \bar{k}(C)dx$ if and only if $\bar{k}(C)/\bar{k}(x)$ is separable.

<u>The divisor of a differential form.</u>

<u>Aim:</u> Define $\operatorname{div}(\omega) \in \operatorname{Div}(C)$ for $\omega \in \Omega_C$ when C is a nonsingular projective curve.

$$\operatorname{div}(\omega) = \sum_{P \in C} v_P(\omega)\,[P]$$