1) Classical context of Bernoulli numbers : $1^m + 2^m + \dots + n^m$

2) $\zeta(2n)$, $n \in \mathbb{N}$, and B. numbers

3) Analytic properties of ζ

4) p-adic properties of B.#'s : denominator primes
numerator primes (Herbrand–Ribet ; don't prove)
Kummer congruences and p-adic interpolation of ζ.

Fix $m \in \mathbb{Z}_{\geq 1}$. Set $S_m(n) = 1^m + 2^m + \dots + (n-1)^m$

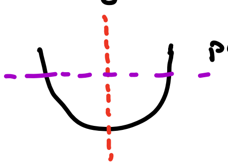
Closed formula? $S_1(n) = 1 + \dots + n-1 = \frac{(n-1)n}{2} = \frac{n^2}{2} - \frac{n}{2}$

$$S_2(n) = 1^2 + \dots + (n-1)^2 = \frac{(n-1)n(2n-1)}{6} = \frac{(n^2-n)(2n-1)}{6} = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$

$$S_3(n) = 1^3 + \dots + (n-1)^3 = \left[\frac{(n-1)n}{2}\right]^2 = \frac{n^4}{4} - \frac{n^3}{2} + \frac{n^2}{4}$$

$\vdots$?

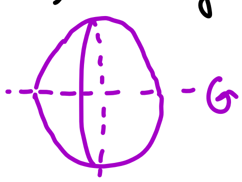
Before symbolic algebra, some special cases used to compute areas (Archimedes); a general inductive approach to determining $S_m(n)$ in terms of $S_{m-1}, \dots, S_1$ was given by Ibn al-Haytham (c. 1000). Some context:



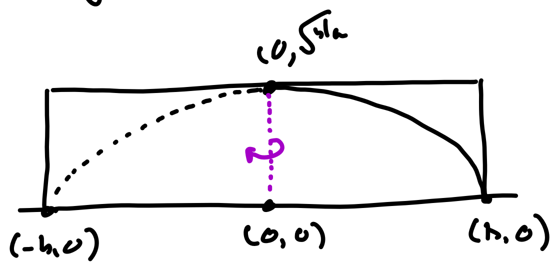
$\rightsquigarrow$ solid by rotating

Arch. computed these volumes

solid by rotating

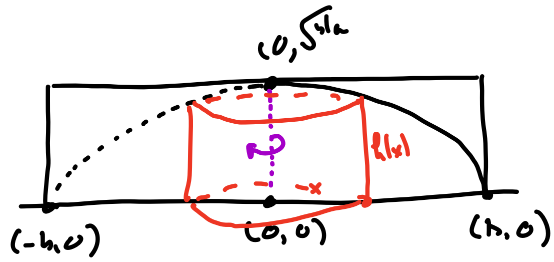


Ibn al-Haytham computed these (as 8/15 the volume of an inscribing cylinder)



$x - b = -ay^2$

Compute the integral by cylindrical shells



cylinder volume $= \sqrt{b/a} \cdot \pi b^2$

rotated parabola volume

$$= \int_0^b 2\pi x h(x)\,dx = \int_0^b 2\pi x \sqrt{-\frac{x}{a} + \frac{b}{a}}\,dx$$

Let $u = \sqrt{b-x}$ $\quad \frac{du}{dx} = \frac{-1}{2\sqrt{b-x}}$

$b - u^2 = x$

$$= \frac{2\pi}{\sqrt{a}} \int_{\sqrt{b}}^{0} (b-u^2)\, u\, (-2u)\,du$$

$$= \frac{4\pi}{\sqrt{a}} \int^{\sqrt{b}} bu^2 - u^4\,du$$

$$= \frac{4\pi}{\sqrt{a}}\left[b\frac{u^3}{3} - \frac{u^5}{5}\right]_0^{\sqrt{b}} = \frac{4\pi}{\sqrt{a}}\, b^{5/2}\cdot\left(\frac{1}{3}-\frac{1}{15}\right) = \frac{8\pi b^2}{15}\cdot\sqrt{b/a} = \frac{8}{15}\text{ cylinder.}$$

Note that to do the Riemann integral you would use a formula for $\sum_{i=1}^{n} i^4$, and Ibn al-Haytham computed inductively – we'll give modern version now.

$$S_m(n) = 1^m + \cdots + (n-1)^m$$

$(k+1)^{m+1} - k^{m+1} = \sum_{i=0}^{m}\binom{m+1}{i}k^i$. Sum $k=0,\dots,n-1$: (when $k=0$, RHS is 1, and then $S_0(n) := \sum_{k=0}^{n-1} 1 = n$)

$n^{m+1} = \sum_{i=0}^{m}\binom{m+1}{i}S_i(n)$, so $S_m(n)$ determined recursively:

$$S_m(n) = \frac{n^{m+1}}{m+1} - \frac{\sum_{i=0}^{m-1}\binom{m+1}{i}S_i(n)}{m+1}$$, and then we easily see by induction that $S_m(n)$ is a polynomial in n of degree $m+1$, leading coeff. $\frac{1}{m+1}$, and no constant term.

Example: $S_4(n) = \frac{n^5}{5} - \frac{1}{5}\left[n + 5S_1(n) + 10S_2(n) + 10S_3(n)\right]$

$$= \frac{n^5}{5} - \frac{1}{5}\left(n + 5\frac{n^2-n}{2} + 10\frac{n^3}{3} - 5n^2 + \frac{5n}{3} + \frac{5n^4}{2} - 5n^3 + \frac{5n^2}{2}\right)$$

$$= \frac{n^5}{5} - \frac{n^4}{2} + \underbrace{\left(-\frac{2}{3}+1\right)}_{1/3}n^3 + \underbrace{\left(-\frac{1}{2}+1-\frac{1}{2}\right)}_{0}n^2 + \underbrace{\left(-\frac{1}{5}+\frac{1}{2}-\frac{1}{3}\right)}_{-1/30}n$$

First Defn of Bernoulli #: $\forall\, m \geq 0$, let B_m = coefficient of n in $S_m(n)$ (with convention $S_0(n) = n$, so $B_0 = 1$).

Eg: $B_0 = 1 \quad B_1 = -1/2 \quad B_2 = 1/6 \quad B_3 = 0 \quad B_4 = -1/30, \dots$

By $n^{m+1} = \sum_{i=0}^{m}\binom{m+1}{i}S_i(n)$, $0 = \sum_{i=0}^{m}\binom{m+1}{i}B_i$ (compare coeff of n),

so $\boxed{(m+1)B_m = -\sum_{i=0}^{m-1}\binom{m+1}{i}B_i}$ and we obtain

2nd Defn: $B_0 = 1$, and given $B_0, \dots, B_{m-1}$, we define B_m by boxed formula.

$$B_5 = -\frac{1}{6}\left(B_0 + 6B_1 + 15B_2 + 20B_3 + 15B_4\right)$$
$$= -\frac{1}{6}\left(1 - 3 + \frac{5}{2} + 0 - 1/2\right) = 0.$$

The important observation (Jakob Bernoulli & Takaku Seki) is that *all* coefficients of $S_m(n)$ are expressible in terms of B_k's:

$\boxed{\text{Thm:}}$ For any $m \in \mathbb{Z}_{\geq 1}$,

$$(m+1)S_m(n) = \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}$$

Eg: $5S_4(n) = B_0 n^5 + 5B_1 n^4 + 10B_2 n^3 + 10B_3 n^2 + 5B_4 n$

$\to S_4(n) = \frac{n^5}{5} - \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ (but now much easier)

For a quick proof, note:

Lemma/ ~~3rd defn of B_m's~~: $\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} b_m \frac{t^m}{m!}$ for some b_m. Then $b_m = B_m$ $\forall m \in \mathbb{Z}_{\geq 0}$

(removable singularity at 0)

Pf: $t = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{m=0}^{\infty} b_m \frac{t^m}{m!}$. Equate coefficients:

$\overset{(t)}{1 = b_0}$, $\overset{(t^2)}{0 = \frac{1}{2!} b_0 + \frac{1}{1!} b_1}$, ... $\overset{t^{m+1}, m \geq 1}{0 = \sum_{j=0}^{m} \frac{b_j}{j!} \cdot \frac{1}{(m+1-j)!}}$, which $\cdot (m+1)!$

gives $0 = \sum_{j=0}^{m} \binom{m+1}{j} b_j$, the same recursion defining B_m. ∎

Cor: For $m \geq 1$, $B_{2m+1} = 0$.

Pf: $\frac{t}{e^t - 1} + \frac{t}{2} = 1 + \sum_{m=2}^{\infty} B_m t^m / m!$. LHS is an even function $\left(\frac{-te^t}{1-e^t} - \frac{t}{2} = \frac{-t(2e^t + 1 - e^t)}{2(1-e^t)} = \frac{t}{2}\left(\frac{e^t+1}{e^t-1}\right) = \frac{t}{2}\left(\frac{2}{e^t-1} + 1\right) = \text{LHS} \right)$, so the odd Taylor coeff's of LHS must be 0. ∎

Back to Thm.

Pf of Thm: For $k=0,1,\dots,n-1$, add the equations

$$e^{kt}=\sum_{m=0}^{\infty}\frac{k^m t^m}{m!},\quad \text{i.e.}$$

$$1+e^t+e^{2t}+\dots+e^{(n-1)t}=\sum_{m=0}^{\infty}S_m(n)\frac{t^m}{m!}$$

$$\frac{e^{nt}-1}{e^t-1}=\frac{e^{nt}-1}{t}\cdot\frac{t}{e^t-1}=\sum_{i=1}^{\infty}n^i\frac{t^{i-1}}{i!}\cdot\sum_{j=0}^{\infty}B_j\frac{t^j}{j!}$$

and comparing coeffs. of t^m, $\quad i-1=m-j$

$$\frac{S_m(n)}{m!}=\sum_{j=0}^{m}\frac{B_j}{j!}\frac{n^{m-j+1}}{(m-j+1)!}.$$ Multiply by $(m+1)!$

$$(m+1)S_m(n)=\sum_{j=0}^{m}\binom{m+1}{j}B_j\,n^{m-j+1}\quad\blacksquare$$

(*) The next major development, and really the starting-point for this course, is Euler's discovery of relation between B_{2n} and $\zeta(2n)$ $(n\in\mathbb{Z}_{\geq1})$.

Here's some craziness. For $k\geq 2$, let's compute

$$\sum_{n=1}^{\infty}n^{k-1}=\sum_{n=1}^{\infty}\left[\left(\frac{d}{dt}\right)^{k-1}e^{nt}\right]\Big|_{t=0}$$

$$= \left[\left(\frac{d}{dt}\right)^{k-1} \sum_{n=1}^{\infty} e^{nt}\right]\Bigg|_{t=0}$$

$$= \left[\left(\frac{d}{dt}\right)^{k-1}\left(\frac{1}{1-e^t}-1\right)\right]_{t=0}$$

$$= \left[\left(\frac{d}{dt}\right)^{k-1}\left(-\frac{1}{t}\cdot\left(\frac{t}{e^t-1}\right)\right)\right]_{t=0} \qquad \left(\frac{d}{dt}(1)=0\right)$$

$$= \left[\left(\frac{d}{dt}\right)^{k-1} \sum_{n=0}^{\infty} -\frac{1}{t}\cdot B_n \frac{t^n}{n!}\right]_{t=0}.$$

If you pretend the $n=0$ term is not there, you get

$$\left[\left(\frac{d}{dt}\right)^{k-1} \sum_{n=1}^{\infty} -\frac{B_n}{n}\cdot\frac{t^{n-1}}{(n-1)!}\right]_{t=0} = \frac{-B_k}{k}.$$

We'll show this is "correct," but it will take a lot of work.

<u>Defn:</u> Let $s \in \mathbb{C}$. The series

$\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely, and uniformly on compact subsets, in the domain $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.

For $\mathrm{Re}(s) > 1$, we define

$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ to be the resulting function.

Don't worry if complex exponential is not familiar.
For $a \in \mathbb{R}_{>0}$ and $s \in \mathbb{C}$, we define

$a^s = e^{\log a \cdot s}$ by the convergent series

$$\sum_{k=0}^{\infty} \frac{(\log a)^k s^k}{k!}$$

or equivalently by $s = \sigma + it$, $a^s = e^{\sigma \log a} \cdot \underbrace{e^{it \log a}}_{\cos(t \log a) + i \sin(t \log a)}$

Then $\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma}$, and so absolute convergence for $\sigma > 1$ is familiar.

$\boxed{\text{Euler's Theorem:}}$ For all $m \in \mathbb{Z}_{\geq 1}$,

$$\zeta(2m) = (-1)^m \pi^{2m} \frac{2^{2m-1}}{(2m-1)!} \left(-\frac{B_{2m}}{2m}\right)$$

($\in \pi^{2m} \cdot \mathbb{Q}$; we've written it this way for reasons that will become clear later).

Our crazy calculation was somehow "$\zeta(1-2m)$", which will take time to make sense of.

<u>What's ahead</u>: Give 2 proofs of E's thm, introducing some important tools along the way.

some complex analysis

Some Fourier analysis

§ Fourier analysis approach to the theorem

(Prereqs: basic theory of series; Riemann integration.)

There are many ways. We'll give two.

Let $f : [a,b] \to \mathbb{C}$ be Riemann integrable. Set $L = b-a$

<u>Defn:</u> $\forall n \in \mathbb{Z}$, $\hat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi i n x/L} \, dx$ is the n^{th} Fourier coefficient of f.

$\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x/L}$ is the Fourier series of f. No claim about convergence is implied — this is a formal expression now.

<u>Example:</u> $[a,b] = [-\pi, \pi]$, $L = 2\pi$.

$[-\pi,\pi] \xrightarrow{f} \mathbb{R}$ defined on $|x| \le \pi$ by $f(x) = |x|$

(so f factors through continuous $\mathbb{T} \to \mathbb{R}$)

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy = \frac{1}{2\pi} \int_0^{\pi} y e^{-iny} dy + \frac{1}{2\pi} \int_{-\pi}^{0} (-y) e^{-iny} dy$$

<u>$n=0$:</u> $\frac{1}{2\pi}\left[\pi^2/2 - \left(-\frac{\pi^2}{2}\right)\right] = \pi/2$

<u>$n \neq 0$:</u> Use $\int y e^{-iny} dy = y \frac{e^{-iny}}{-in} - \int \frac{e^{-iny}}{-in} dy = -y\frac{e^{-iny}}{in} + \frac{1}{n^2} e^{-iny}$

So
$$\hat{f}(n) = \frac{1}{2\pi}\left[\left(\frac{-\pi(-1)^n}{in} + \frac{1}{n^2}((-1)^n - 1)\right) \quad [0,\pi]\right.$$
$$\left. + \underbrace{\left[y\frac{e^{-iny}}{in} - \frac{1}{n^2} e^{-iny}\right]_{-\pi}^{0}}_{-\frac{1}{n^2} - \left[\frac{-\pi(-1)^n}{in} - \frac{1}{n^2}(-1)^n\right]} \right) \quad [-\pi, 0]$$

$$= \frac{1}{\cancel{2}\pi}\left(\frac{\cancel{2}}{n^2}\right)\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n^2} & n \text{ odd} \end{cases}$$

SUPPOSE that the Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$

converges to $f(x)$ in the sense that

$\lim_{N\to\infty} S_N(f)(x) = f(x)\ \forall\, x \in [-\pi,\pi]$, where $S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$.

That is,

$|x| = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{-\pi i n x}$. Taking $x=0$:

$(|x| \le \pi)$

$$0 = 2\sum_{\substack{n \text{ odd} \\ \geq 1}} \frac{-2}{\pi n^2} + \frac{\pi}{2}, \text{ or equivalently}$$

$$\sum_{\substack{n \text{ odd} \\ \geq 1}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$ Set $E = \sum_{\substack{n \text{ even} \\ \geq 1}} \frac{1}{n^2}$, $O = \sum_{\substack{n \text{ odd} \\ \geq 1}} \frac{1}{n^2}$

$\frac{1}{4}(E+O) = E$, so $E = \frac{O}{3}$, and $E + O = \pi^2\left(\frac{1}{8} + \frac{1}{24}\right) = \pi^2/6$

Conclusion: $\zeta(2) = \pi^2/6 = \pi^2 \cdot \frac{2^1}{1!} \frac{B_2}{2}$, as expected.

How do we ensure the convergence? Subtle problem in general, and Fourier analysis spends a great deal of effort deciding what classes functions have Fourier series that "converge" in one sense or another.

We use only the following simple criterion:

$\boxed{\text{Thm:}}$ Suppose $f: \underset{= [-\pi,\pi] \text{ uma}}{[a,b]} \to \mathbb{C}$ is continuous and $f(a) = f(b)$

(so f defines a cts function on circle). Also suppose $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$.

Then $\lim_{N\to\infty} (S_N f)(x) = f(x)$, uniformly in x

$\overset{\text{ii}}{\lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)}$

In our example, $\sum |\hat{f}(n)| < \infty$ since $\sum \frac{1}{n^2}$ converges, so the thm. completes the calculation of $\zeta(2)$.

Note that if $\{a_n\}_{n\in\mathbb{Z}}$ satisfies $\sum |a_n| < \infty$, then we can define $S_N(x) = \sum_{n=-N}^{N} a_n e^{inx}$ and check S_N converges uniformly to <u>some</u> continuous fn g, with $\hat{g}(n) = a_n$. Apply w/ $a_n = \hat{f}(n)$. We still need to know $\{\hat{f}(n)\}$ determines f.

Should we prove the theorem? It follows from

$\boxed{\text{Thm}}$ Let f be an integrable function on the circle. Suppose $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$. Then for any x_0 : f is continuous at x_0, $f(x_0) = 0$.

(<u>Rmk:</u> No generality lost by restricting to $[a,b] = [-\pi,\pi]$, ie $f : [-\pi,\pi] \to \mathbb{C}$ is integrable and $f(-\pi) = f(\pi)$; f is <u>not</u> assumed continuous. The "circle" here is $[-\pi,\pi]/_{-\pi\sim\pi}$, or $\mathbb{R}/_{2\pi\mathbb{Z}}$.)

<u>Cor:</u> Two <u>continuous</u> functions on circle with same Fourier coefficients are equal. In particular, if $f : [-\pi,\pi] \to \mathbb{C}$ is continuous & $f(-\pi) = f(\pi)$, and $\sum |\hat{f}(n)| < \infty$, then

$$g(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx} \quad \left(\text{really } \lim_{N\to\infty} (S_N f)(x)\right) \text{ is \underline{continuous}},$$

because $((S_N f)(x))_{N=1}^{\infty}$ converges <u>uniformly</u> to $g(x)$.

$\rightarrow$ by Weier. M-test, since $|\hat{f}(n)e^{inx} + \hat{f}(-n)e^{-inx}| \le |\hat{f}(n)| + |\hat{f}(-n)|$ and $\sum_{n\ge 1}^{\infty}$ ✓ converges. $\forall \epsilon\ \exists N : \forall p$ so $|S_{N+p}(x) - S_N(x)|$

Then also $\hat{g}(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx} \cdot e^{-imx}\, dx$

$$\overset{\text{we uniform conv.}}{=} \sum_{n\in\mathbb{Z}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(n) e^{i(n-m)x}\, dx = \hat{f}(m) \quad \forall m \in \mathbb{Z}.$$

By 1st sentence of the Cor, $f(x) = g(x)$ identically.

Another route to Euler's theorem: the partial fraction expansion of $\cot(z)$.

$\boxed{\text{Thm}}$ For all $z \in \mathbb{C} \setminus \pi\mathbb{Z}$,

$$\cot(z) = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{z}{z^2 - n^2\pi^2} \qquad \text{(absolute convergence)}$$

You can either restrict to $z \in \mathbb{R}$, all that's needed for Euler, or define $\cot(z) = \frac{\cos(z)}{\sin(z)} = \frac{e^{iz}+e^{-iz}}{2} \Big/ \frac{e^{iz}-e^{-iz}}{2i} = i\,\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$

$$\left(\begin{array}{l} e^{iz} = \cos(z) + i\sin(z) \\ e^{-iz} = \cos(z) - i\sin(z) \end{array}\right).$$

Thus $z\cot(z) = 1 + 2\sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2\pi^2} = 1 + 2\sum_{n=1}^{\infty} \frac{-z^2}{n^2\pi^2\left(1 - \left(\frac{z}{n\pi}\right)^2\right)}$

$$= 1 + 2\sum_{n=1}^{\infty} \frac{-z^2}{n^2\pi^2} \cdot \sum_{m=0}^{\infty} \left(\frac{z}{n\pi}\right)^{2m} = 1 - 2\sum_{m=1}^{\infty} \frac{z^{2m}}{\pi^{2m}} \cdot \zeta(2m)$$

But we also have

$(w = 2iz)$ $\frac{w}{e^w - 1} = \sum B_m w^m/m!$

$$z\cot z = iz\,\frac{e^{2iz}+1}{e^{2iz}-1} = iz + iz\,\frac{2}{e^{2iz}-1} = iz + \sum_{m=0}^{\infty} B_m \frac{(2iz)^m}{m!}$$
$$= 1 + \sum_{m=2}^{\infty} B_m \frac{(2iz)^m}{m!}$$

And so, by the miracle of comparing coefficients, we find

$$\frac{2^m i^m}{m!} B_m = -\frac{2\,\zeta(2m)}{\pi^{2m}}, \text{ i.e.}$$

Cor: (Euler's theorem) $\zeta(2m) = (-1)^m \frac{2^{2m-1}\pi^{2m}}{(2m-1)!}\left(-\frac{B_{2m}}{2m}\right)$.

Pf of Thm: (via Fourier analysis)

For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, consider $f : [-\pi, \pi]$ defined by $f(x) = \cos(\alpha x)$.

Note that $f(-\pi) = f(\pi)$ since $\cos(\alpha x)$ is even, and on the circle f is continuous.

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\alpha x)\, e^{-inx}\,dx$$

$$\int \cos(\alpha x) e^{-inx}\,dx = \int \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}\, e^{-inx}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{i(\alpha-n)x}}{i(\alpha-n)} + \frac{e^{-i(\alpha+n)x}}{-i(\alpha+n)}\right], \text{ so}$$

$$\hat{f}(n) = \frac{1}{4\pi}\left(\frac{(-1)^n e^{i\alpha\pi}}{i(\alpha-n)} + \frac{(-1)^n e^{-i\alpha\pi}}{-i(\alpha+n)} - \frac{(-1)^n e^{-i\alpha\pi}}{i(\alpha-n)} - \frac{(-1)^n e^{i\alpha\pi}}{-i(\alpha+n)}\right)$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n}{\cancel{i}(\alpha-n)}\sin(\alpha\pi)\cdot\cancel{i} - \frac{(-1)^n}{\cancel{i}(\alpha+n)}\sin(-\alpha\pi)\cancel{i}\right]$$

$$= \frac{(-1)^n}{2\pi}\left(\frac{\sin(\alpha\pi)}{\alpha-n} + \frac{\sin(+\alpha\pi)}{\alpha+n}\right) = \frac{(-1)^n\alpha}{\pi}\,\frac{\sin(\alpha\pi)}{\alpha^2-n^2}$$

Again ($\alpha \notin \mathbb{Z}$), $\sum |\hat{f}(n)| < \infty$, so since f is continuous, the basic convergence theorem shows that

(*) f is represented by its Fourier series at all $x \in [-\pi, \pi]$, $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$. Taking $x = \pi$, we get

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)(-1)^n = \cos(\alpha\pi)$$

$$\parallel$$

$$\frac{\sin(\alpha\pi)}{\pi\alpha} + 2\sum_{n=1}^{\infty} \frac{\alpha\sin(\alpha\pi)}{\pi(\alpha^2-n^2)}, \text{ so}$$

$$\cot(\alpha\pi) = \frac{1}{\pi\alpha} + 2\sum_{n=1}^{\infty} \frac{\alpha\pi}{(\alpha\pi)^2 - n^2\pi^2}, \text{ or equivalently,}$$

for $z \notin \pi\mathbb{Z}$, $\cot(z) = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{z}{z^2 - n^2\pi^2}$ ∎

(many other proofs)

Maybe now we have enough motivation to prove the point-wise convergence theorem.
Recall it follows from (go back up in notes)

$\boxed{\text{Thm}}$ f Riemann integrable and 2π-periodic s.t. $\hat{f}(n)=0$ $\forall n\in\mathbb{Z}$.
Let x_0 be a point where f is continuous. Then $f(x_0)=0$.

Pf: Define a trigonometric polynomial (on $[-\pi,\pi]$) to be a finite sum

$$p(x)=\sum_{n=-N}^{N} a_n e^{inx} \quad \left(\text{on other intervals use } e^{2\pi i n x/\text{length}}\right)$$

[Using the Euler identity $e^{inx}=\cos(nx)+i\sin(nx)$ and angle sum formulas, you can write these as polys in $\cos(x),\sin(x)$]

The assumption $0=\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \quad \forall n\in\mathbb{Z}$

$\Rightarrow$ $\forall$ trig. polys, $\displaystyle\int_{-\pi}^{\pi} f(x)p(x)\,dx=0.$

To prove the theorem, wma f is $\mathbb{R}$-valued. Assuming this case, in general write $f(x)=u(x)+iv(x)$ u,v periodic, integrable, $\mathbb{R}$-valued & cts at x_0. $\bar{f}(x)=u(x)-iv(x)$, and $f(x)+\bar{f}(x)=2u(x)$

so Fourier coefficients $\hat{u}(n)=\frac{1}{2}\left\{\hat{f}(n)+\hat{\bar{f}}(n)\right\}$

$$\hat{\bar{f}}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\bar{f}(x)e^{-inx}\,dx=\frac{1}{2\pi}\overline{\left(\int_{-\pi}^{\pi} f(x)e^{inx}\,dx\right)}=\overline{\hat{f}(-n)}$$

So all $\hat{f}(n)=0 \Rightarrow$ all $\hat{\bar{f}}(n)=0 \Rightarrow$ all $\hat{u}(n)=0$.

Likewise for $v(x)=\dfrac{f(x)-\bar{f}(x)}{2i}$, $\hat{v}(n)=0$ $\forall n\in\mathbb{Z}$. By the $\mathbb{R}$-case, $u(x_0)=v(x_0)=0$, so $f(x_0)=0$.

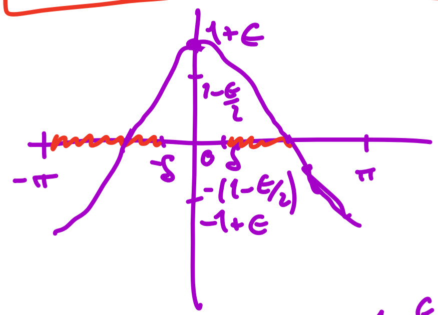
So assume $f:\mathbb{R}/2\pi\mathbb{Z}\to\mathbb{R}$, and for a contradiction assume $f(x_0)\neq 0$.

Wma $x_0=0$ and $f(0)>0$. We'll find a trig poly $q(x)$: $\int f(x)q(x)\,dx\neq 0$.

Idea: construct a family $p_1,p_2,\ldots$ of trig polys where as $k\to\infty$ p_k "spikes" more & more at 0 $(=x_0)$ but remains small away from 0.

f is cts at 0, $f(0)>0 \Rightarrow \boxed{\exists\, \delta>0 : |x|<\delta \Rightarrow f(x) > \frac{f(0)}{2}.}$

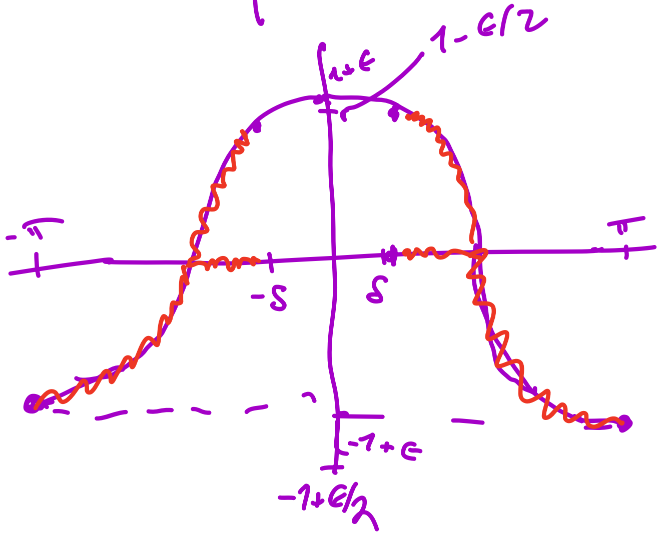
Consider $p(x) = \epsilon + \cos(x)$



Choose $\epsilon>0$ s.t. when $\delta \le |x| \le \pi$, we have $|p(x)| < 1 - \epsilon/2$ ("small")

and $p(x)>0$ for $|x|<\delta$

Also choose η, $0<\eta<\delta$, s.t. $|x|<\eta \Rightarrow p(x) \geq 1+\epsilon/2$



Set $p_k(x) = p(x)^k \ \forall\, k \geq 1$.

(Note $p_k(0) \to \infty$ as $k\to\infty$)

We claim $\int_{-\pi}^{\pi} f(x)\, p_k(x)\,dx \to \infty$ as $k \to \infty$. Then we win.

($\hat{f}(n)=0\ \forall n \Rightarrow$ all these $=0$).

Indeed, $\int_{-\pi}^{\pi} = \int_{|x|<\eta} + \int_{\eta<|x|<\delta} + \int_{\delta\le|x|\le\pi}$. f is bounded (it is Riemann integrable on $[-\pi,\pi]$, so by defn bounded), say $|f(x)| \le M\ \forall x$.

$$\left|\int_{\delta\le|x|\le\pi} f(x)\,p_k(x)\right| \le 2\pi M \left(1-\frac{\epsilon}{2}\right)^k \quad (\to 0 \text{ as } k\to\infty)$$

$$\int_{\eta\le|x|\le\delta} f(x)\,p_k(x) \geq 0 \quad \left(\text{for } |x|<\delta,\ f(x) > \frac{f(0)}{2} > 0, \text{ and } p_k(x)>0\right)$$

$$\int_{|x|\le\eta} f(x)\,p_k(x) \geq 2\eta\,\frac{f(0)}{2}\cdot\left(1+\frac{\epsilon}{2}\right)^k \quad (\to \infty \text{ as } k\to\infty)$$

The claim is now clear. ∎

§ Analytic properties of $\zeta(s)$

We recall some complex function theory: let $\Omega \subset \mathbb{C}$ be an open set (arbitrary union of open balls), and let $f : \Omega \to \mathbb{C}$ be a function.

Defn: f is holomorphic at $z_0 \in \Omega$: f

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \text{ exists (then denoted } f'(z_0)).$$

Crucially, $h \to 0$ in $\mathbb{C}$, so this means $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall$ $h \in \mathbb{C}$ with $|h| < \delta$, $\left| \frac{f(z_0+h)-f(z_0)}{h} - f'(z_0) \right| < \epsilon$.

Eg: $f(z) =$ polynomial of z ($\Omega = \mathbb{C}$),
sums, products, compositions and quotients when they make sense.

Non-eg: $f(z) = \bar{z}$: taking $h = x \in \mathbb{R} \to 0$, $\lim_{x\to 0} \frac{f(z_0+x)-f(z_0)}{x} = \lim_{\substack{x\to 0 \\ \in \mathbb{R}}} \frac{\bar{z}_0 + x - \bar{z}_0}{x} = 1$

Taking $h = iy \in i\mathbb{R} \to 0$, $\lim_{y \to 0} \frac{\bar{z}_0 - iy - \bar{z}_0}{iy} = -1$, so the $\lim_{h\to 0}$ DNE.

Key eg: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R. Here $0 \le R \le \infty$ is uniquely determined by the property

- if $|z| < R$, the series converges absolutely
- if $|z| > R$, " diverges

and explicitly $\frac{1}{R} = \limsup_n |a_n|^{1/n}$ (with $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$)

Then f is holomorphic on $D = \{|z| < R\}$; moreover, $\forall z \in D$,
$f'(z) = \sum_{n=0}^{\infty} a_n n z^{n-1}$, where RHS is another power series with the same radius of convergence R.

Eg: $f(z) = e^z := \sum_{n=0}^{\infty} z^n/n!$ has $R = \infty$, so is holomorphic on all of $\mathbb{C}$
(likewise $\cos z$, $\sin z$).

Pf: Just as in the theory of power series of a real variable.

The remarkable fact, expressing the rigid nature of holomorphic fns, is that the converse holds: SAY. OMIT REST OF PAGE or after Thm

Thm: (Cauchy) Suppose f is holomorphic on an open set $\Omega \subset \mathbb{C}$. Then all $f^{(n)}$ exist and are holomorphic on Ω. Moreover, for any disc $D = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ with closure $\overline{D} \subset \Omega$, f has a convergent power series expansion on D:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)^n \quad \forall \; z \in D.$$

could state

Cauchy proved this by finding a remarkable contour integral expression for f and its derivatives:

Let $C = \partial \overline{D} = \{|z - z_0| = r\}$ ($\overline{D} \subset \Omega$ as above), positively oriented.

Then $\forall \; z \in D$, $f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} d\xi$

and $f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$

(the meaning of an integral $\int_C g(\xi) d\xi$ is: parametrize C by a smooth curve $\gamma : [a,b] \to \mathbb{C}$, e.g. $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = z_0 + re^{2\pi i t}$, and set $\int_C g(\xi) d\xi := \int_a^b g(\gamma(t)) \gamma'(t) dt$. To be precise, we ask γ only differentiable on (a,b) and continuous at endpoints. We can also take piecewise smooth γ.)

Sketch Goursat / Morera in hol + cts derivative simple case?

After stating, SAY: the significance of this formula will be philosophical — we are going to extend ξ to $\mathbb{C} \setminus \{1\}$ — but what will characterize the extension? Holomorphicity condition uniquely characterizes it: two hol. fns Ω that agree on an open, or even just a subset with a limit point, are =

on all of Ω. This follows from power series results

$\boxed{\text{Thm:}}$ $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is holomorphic on $\mathrm{Re}(s) > 1$.

More generally, if a Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ $(f: \mathbb{N} \to \mathbb{C})$ converges absolutely on $\mathrm{Re}(s) > \sigma$, then it defines a holomorphic function in this region.

<u>Pf:</u> • $\sum \frac{f(n)}{n^s}$ converges absolutely & uniformly on every compact subset of $\{\mathrm{Re}(s) > \sigma\}$; (even on closed half-planes)

For $\mathrm{Re}(s) \geq \sigma + \epsilon$ (some $\epsilon > 0$), $\left|\sum_{n=N}^{\infty} \frac{f(n)}{n^s}\right| \leq \sum_{n=N}^{\infty} \frac{|f(n)|}{n^{\mathrm{Re}(s)}} = \sum_{n=N}^{\infty} \frac{|f(n)|}{n^{\sigma+\epsilon/2}} \cdot \frac{1}{n^{\underbrace{\mathrm{Re}(s)-\epsilon/2-\sigma}_{>0}}}$, so this is $\leq \frac{\ldots}{N^{\mathrm{Re}(s)-\frac{\epsilon}{2}-\sigma}}$

$\leq \left(\text{the constant } \sum_{1}^{\infty} \frac{|f(n)|}{n^{\sigma+\epsilon/2}}\right) \cdot N^{-\epsilon/2} \to 0$ as $N \to \infty$, uniformly in s. ($\mathrm{Re}(s) - \sigma - \epsilon/2 \geq \epsilon/2$, so $n^{-\epsilon/2} \geq n^{-(\mathrm{Re}(s)-\sigma-\epsilon/2)}$ $\forall\, n \geq 1$,

and $\forall\, n \geq N$, $n^{-\epsilon/2} \leq N^{-\epsilon/2}$)

• Each $\frac{f(n)}{n^s}$ is a holomorphic function, even on all of $\mathbb{C}$, so the same is true of the partial sums $\sum_{n=1}^{N} \frac{f(n)}{n^s}$. Thus what we need is

<u>Thm:</u> If $\{g_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on an open $\Omega \subset \mathbb{C}$, and $\{g_n\}$ converges to a function $g: \Omega \to \mathbb{C}$ <u>uniformly</u> on compacts $\subset \Omega$, then g is holomorphic on Ω.

KNOW THIS! (different from diff. fns of real variable) — there want derivs to converge uniformly e.g. $\sqrt{x^2+1/n} \to |x|$

<u>Pf:</u> Goursat's thm: f hol. on Ω, $T \subset \Omega$ a triangle with interior in Ω. Then $\int_T f(z)\,dz = 0$. (or polygon or piecewise-smooth path or ...)

$\left(= \int_a^b f(\gamma(t))\gamma'(t)\,dt \;\ldots\right)$

Conversely: Morera's theorem asserts that a continuous function on an open disc D is holomorphic provided $\int_T f(z)\,dz = 0$ for any triangle $T \subset D$ (or closed polygonal path)

Thus, for any disc D: $\overline{D} \subset \Omega$, and any $T \subset D$,

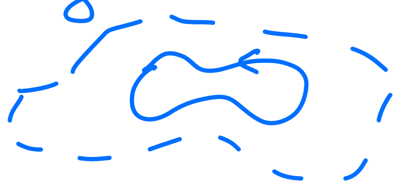
$\int_T g_n(z)\,dz = 0$. As $g_n \to g$ uniformly on $\overline{D}$, g is continuous

and $0 = \lim_{n \to \infty} \int_T g_n(z)\,dz = \int_T \lim_{n\to\infty} g_n(z)\,dz = \int_T g(z)\,dz$

$\forall\, T \subset D$. Thus g is holomorphic. ∎

Our $\sum_{n=1}^{N} f(n)/n^s$ are evidently holomorphic with <u>continuous</u> derivatives. This special case of Goursat / Morera is easier to prove

<u>Sketch:</u> $\Omega \subset \mathbb{C}$ open. $g: \Omega \to \mathbb{C}$ holomorphic with continuous derivative.



Let $\Gamma \subset \Omega$ be any "nice" curve with interior $\subset \Omega$

$\underbrace{\text{interior} \subset \Omega}_{\text{Forbids}}$

We'll just take "nice" = polygon

Write $z = x + iy$ and set $g(x,y) = g(z) = u(x,y) + i \cdot v(x,y)$, thinking of g as a function on $\Omega \subset \mathbb{R}^2$.

By assumption, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous on Ω, <u>and</u>:

<u>C-R equations</u> $\forall\, \overset{z_0 =}{(x_0,y_0)} \in \Omega$, $g'(z_0)$ exists, i.e.

$\lim_{h \to 0} \frac{g(z_0+h) - g(z_0)}{h}$ exists. Evaluate in two ways:

1) $= \lim_{\substack{\Delta x \to 0 \\ \in \mathbb{R}}} \frac{g(x_0+\Delta x, y_0) - g(x_0,y_0)}{\Delta x} =: \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right)(x_0,y_0)$

2) $= \lim_{\Delta y \to 0} \frac{g(x_0, y_0+\Delta y) - g(x_0,y_0)}{i \Delta y} =: \frac{1}{i}\left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right)(x_0,y_0)$.

$\boxed{h = i \cdot \Delta y}$

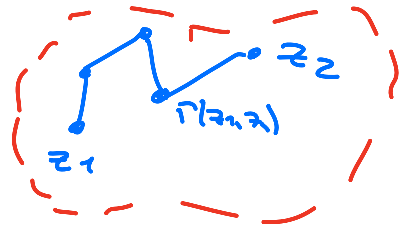
Thus $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$ and $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$.

Now, $\int_\Gamma g(z)\,dz = \int_\Gamma (u+iv)(dx + i\,dy)$

$= \int_\Gamma (u\,dx - v\,dy) + i \int_\Gamma (u\,dy + v\,dx)$, which by <u>Green's thm</u>, (using continuity of partials)

$$= \int\limits_{\left(\substack{\text{region bdd}\\ \text{by } \Gamma}\right)} \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dx\,dy + i \int\limits_{(\quad)} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0. \quad \blacksquare$$

That proves Goursat. For Morera, assume $\int_\Gamma g(z)\,dz = 0$ $\forall$ closed polygonal paths in Ω. $\forall$ $z_1, z_2 \in \Omega$, and any poly path $\Gamma(z_1, z_2)$ from z_1 to z_2 contained in Ω,



$\int_{\Gamma(z_1,z_2)} u\,dx - v\,dy$, $\int_{\Gamma(z_1,z_2)} u\,dy + v\,dx$ is ind. of choice of path.

Conclude $\exists$ $F, G : \Omega \to \mathbb{R}$ continuously differentiable w/

$(u, -v) = \nabla F$ $\left(= \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)$

$(u, v) = \nabla G$

Since $\underset{\sim}{\mathcal{C}}^1$, partials commute,

and $\frac{\partial^2 F}{\partial x \partial y} = -\frac{\partial v}{\partial x}$, $\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial v}{\partial x}$

$\quad \parallel \qquad\qquad\qquad\qquad \parallel$

$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial u}{\partial y}$ $\qquad \frac{\partial^2 G}{\partial y \partial x} = \frac{\partial u}{\partial y}$

. Hence $g = u + iv$ satisfies the C-R equations & has continuous partials.

Exercise: This implies g is holomorphic on Ω. $\blacksquare$

Much deeper:

$\boxed{\text{Thm}}$ There is a unique holomorphic extension of $\zeta(s)$, also denoted $\zeta(s)$, to $\mathbb{C}\setminus\{1\}$. It does not extend to a holomorphic function at 1, but $\zeta(s)\cdot(s-1)$ (hol. on $\mathbb{C}\setminus 1$) does.

This theorem now lets us make sense of "$\zeta(1-2m)$" ($m\geq 1$). We will describe (and prove) a refinement that lets us compute $\zeta(1-2m)$ in terms of our previous evaluation of $\zeta(2m)$.

Defn: (Gamma function) Let $s\in\mathbb{R}_{>0}$. Define

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}\,dt$$

(This improper integral is really $\lim_{\epsilon\to 0^+}\int_\epsilon^1 (\quad) + \lim_{N\to\infty}\int_1^N (\quad)$

$\int_0^1 e^{-t}t^{s-1}dt$ converges when $\int_0^1 t^{s-1}dt$ does, $\left.\frac{t^s}{s}\right]_\epsilon^1=\frac{1}{s}-\frac{\epsilon^s}{s}$, which converges as $\epsilon\to 0^+$ when $s>0$.

$\frac{e^{-t}t^{s-1}}{t^{-2}}\to 0$ as $t\to\infty$, so this $\int_1^\infty$ converges $\forall\, s\in\mathbb{R}$ since $\int_1^\infty 1/t^2\,dt$ does)

Lemma: The integral in fact defines $\Gamma(s)$ as a holomorphic function on $Re(s)>0$.

Pf: Since $|e^{-t}t^{s-1}|=e^{-t}t^{\sigma-1}$, $\sigma=Re(s)$, the integral converges.

We'll show it is a uniform limit (on compacts) of holomorphic functions.

Certainly $\Gamma(s)=\lim_{\epsilon\to 0^+}\underbrace{\int_\epsilon^{1/\epsilon} e^{-t}t^{s-1}dt}_{\Gamma_\epsilon(s)}$

$\Gamma_\epsilon(s) \to \Gamma(s)$ uniformly in vertical strips $0<\sigma_1<\mathrm{Re}(s)<\sigma_2$:

indeed, $|\Gamma(s)-\Gamma_\epsilon(s)| \le \int_0^\epsilon e^{-t}t^{\sigma-1}dt + \int_{1/\epsilon}^\infty e^{-t}t^{\sigma-1}dt$

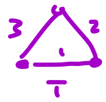
$$\le \frac{\epsilon^{\sigma_1}}{\sigma_1} + \int_{1/\epsilon}^\infty e^{-t}t^{\sigma_2-1}dt \le \frac{\epsilon^{\sigma_1}}{\sigma_1} + 2e^{-\frac{1}{2\epsilon}} \to 0 \text{ as } \epsilon\to0^+,$$

(for $0<\epsilon<1$, $\frac{\epsilon^{\sigma_1}}{\sigma_1} \ge \epsilon^\sigma/\sigma$)

$e^{-t}t^{\sigma_2-1} \le e^{-t/2}$ for $t \gg 0$ (i.e. $\epsilon \ll 1$), $\int e^{-t/2}dt = (-2)e^{-t/2}$

and evidently this is uniform in the vertical strip

Finally, each $\Gamma_\epsilon(s)$ is holomorphic: for any triangle T in $\{\mathrm{Re}(s)>0\}$,

$\int_T \Gamma_\epsilon(s)ds = \int_T\left(\int_\epsilon^{1/\epsilon} e^{-t}t^{s-1}dt\right)ds$  $= \int_1 + \int_2 + \int_3$

Each int. here is over some interval, so we have three $\int$'s of the form $\int_0^1\left(\int_\epsilon^{1/\epsilon} g(t_1,t_2)dt_1\right)dt_2$ where g is continuous on the compact set $[\epsilon,1/\epsilon]\times[0,1]$, hence is integrable and by Fubini's theorem $\int_T \Gamma_\epsilon(s)ds = \int_\epsilon^{1/\epsilon}\left(\int_T e^{-t}t^{s-1}ds\right)dt = 0$ since (Goursat) $e^{-t}t^{s-1}$ is hol. on open $\supset T$. We conclude $\Gamma_\epsilon(s)$ is hol. by Morera.

(Alternatively, show Γ_ϵ is a uniform lim of Riem sums that are hol.). ∎

Lemma: For $\mathrm{Re}(s)>0$, $\Gamma(s+1) = s\cdot\Gamma(s)$

Pf: $\int_\epsilon^{1/\epsilon} \underbrace{e^{-t}}_{dv}\underbrace{t^s}_{u} dt = -e^{-t}t^s\Big]_\epsilon^{1/\epsilon} - \int_\epsilon^{1/\epsilon} -e^{-t}st^{s-1}dt \xrightarrow{\epsilon\to0} s\cdot\Gamma(s)$

i.e. $\Gamma(s+1) = \lim_{\epsilon\to0^+}\Gamma_\epsilon(s+1) = s\cdot\Gamma(s)$. ∎

Prop: $\Gamma(s)$ uniquely extends to a holomorphic function on $\mathbb{C}\setminus\mathbb{Z}_{\le0}$ with simple poles at $\mathbb{Z}_{\le0}$, and never vanishing.

Pf: Inductively define $\Gamma(s)$ in $\mathrm{Re}(s)>-1,-2,-3,\ldots$

by: for $\mathrm{Re}(s)>-1$, $s\neq0$, $\Gamma(s) := \frac{\Gamma(s+1)}{s}$, holomorphic on $\{\mathrm{Re}(s)>-1\}\setminus0$

$\mathrm{Re}(s)>-2$, $s\neq-1$, $\Gamma(s) := \frac{\Gamma(s+1)}{s}$, hol. on $\{\mathrm{Re}(s)>-2\}\setminus\{0,-1\}, \ldots$

So for $\mathrm{Re}(s) > -n$ ($n \in \mathbb{Z}_{>0}$) we have

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)} = \cdots = \frac{\Gamma(s+n)}{s(s+1)\cdots(s+n-1)}$$

, and the claim about poles clear.

The residue at $1-n$ is $[s+(n-1)]\Gamma(s)\big|_{s=1-n} = \frac{\Gamma(1)}{(1-n)(2-n)\cdots(-1)}$

$$= \frac{(-1)^{n-1}}{(n-1)!} \cdot \int_0^\infty e^{-t}\,dt = \frac{(-1)^{n-1}}{(n-1)!}. \quad \blacksquare$$

Note lemma also shows for $n \in \mathbb{Z}_{>0}$, $\Gamma(n) = (n-1)!$.

$\Gamma(\frac{s}{2})$ is part of a "missing factor at ∞" of $\zeta(s) = \prod_p (1-p^{-s})^{-1}$.

$\boxed{\text{Thm}}$ Set $\xi(s) = \Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s)$, defined initially as a holomorphic function in $\mathrm{Re}(s)>1$. Then $\xi(s)$ extends (uniquely) to a holomorphic function on $\mathbb{C}\setminus\{0,1\}$, with simple poles at $0, 1$, and satisfying the functional equation

$$\xi(s) = \xi(1-s) \qquad (\forall s \neq 0,1).$$

We will both define & make the "pole" statement more precise when we give the proof.

We'll then deduce

$\boxed{\text{Cor:}}$ $\zeta(s) := \frac{\xi(s)}{\Gamma(\frac{s}{2})\pi^{-s/2}}$ extends to a holomorphic function on $\mathbb{C}\setminus\{1\}$. For $m \in \mathbb{Z}_{>0}$,

$$\zeta(-2m) = 0 \quad , \quad \zeta(0) = -1/2$$

$$\zeta(1-2m) = -\frac{B_{2m}}{2m}.$$

(SKIP TO THE calculation that reveals Θ-function)

(can phrase as $\zeta(1-n) = -\frac{B_n}{n}$ $\forall n \geq 1$ if we redefine $B_1 = +1/2$)

Fourier analysis for functions $f: \mathbb{R} \to \mathbb{C}$ is harder, but sometimes we get striking results by defining $F: \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by $F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$ & applying theory on $\mathbb{R}/\mathbb{Z}$.

- Convergence obviously an issue here. We'll restrict to the nicest possible setting:

Defn: The Schwartz space $\mathcal{S}(\mathbb{R})$ is the set (a $\mathbb{C}$-vect) of infinitely differentiable $f: \mathbb{R} \to \mathbb{C}$ s.t. $f, f', f'', \ldots$ all $f^{(\ell)}$, are rapidly decreasing: $\forall\, k, \ell \geq 0$,

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty$$

(all $f^{(\ell)}$ decay faster than any polynomial grows)

Important eg: $f(x) = e^{-x^2}$, $f^{(\ell)}(x) = (\text{some polynomial in } x) \cdot e^{-x^2}$ is clearly Schwartz

Easy observation: $f \in \mathcal{S}(\mathbb{R}) \Rightarrow f' \in \mathcal{S}(\mathbb{R})$

$$\Downarrow$$

$$x \cdot f \in \mathcal{S}(\mathbb{R})$$

There is a well-behaved Fourier transform of $f \in \mathcal{S}(\mathbb{R})$

Defn: For $\xi \in \mathbb{R}$, set $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(y) e^{-2\pi i \xi y}\, dy$.

(f is no longer periodic, so unreasonable to try to express it in terms of $e^{2\pi i n x}$ exponentials – the magic is that we can in terms of $\hat{f}(\xi)$).

Note that the integral converges for $f \in \mathcal{S}(\mathbb{R})$, $\forall\, \xi \in \mathbb{R}$.

Another relatively simple condition one could take on

f is $\exists A>0 : |f(x)| \leq \dfrac{A}{1+x^{2}}$ $\forall x \in \mathbb{R}$ (moderate decrease $\mathcal{M}(\mathbb{R})$)

or $1+\epsilon$, $\epsilon>0$

so f is bounded & decays like $1/x^2$ at ∞.

Then $\int_{-\infty}^{\infty} f(x)dx := \lim_{N\to\infty} \int_{-N}^{N} (\quad)$ exists, and so does

$\hat{f}(\xi)$ $\forall \xi \in \mathbb{R}$. It is even
- bounded
- continuous
- $\to 0$ as $|\xi| \to \infty$.

But $\hat{f}(\xi)$ may not be in $\mathcal{M}(\mathbb{R})$. ($\mathcal{S}(\mathbb{R})$ has this property)

(Aside)

Explanation of the shape of $\hat{f}$ in the two cases $\mathbb{R}/\mathbb{Z} \to \mathbb{C}$, $\mathbb{R} \to \mathbb{C}$:

For any locally compact abelian group G, set

$\boxed{\text{Problem 9: } G = \mathbb{Z}/m}$

$\hat{G} = \mathrm{Hom}_{cts}(G, S^1)$ equipped with the compact-open topology.

(It's a group!)

Given $f : G \to \mathbb{C}$ a (nice enough) function, define

$$\hat{f} : \hat{G} \to \mathbb{C} \quad \text{by}$$

$$\hat{f}(\xi) = \int_G f(y)\, \overline{\xi(y)}\, dy.$$

(dy a "Haar measure" on G — it is usual $\int$ when $G = (\mathbb{R}, +)$)

For $G = \mathbb{R}/\mathbb{Z}$, $\widehat{(\mathbb{R}/\mathbb{Z})} \xleftarrow{\sim} \mathbb{Z}$

$(y \mapsto e^{2\pi i n y}) \longleftarrow\!\shortmid\; n$

$\boxed{\text{(Problem 8)}}$

and for $G = \mathbb{R}$, $\hat{\mathbb{R}} \xleftarrow{\sim} \mathbb{R}$

$(y \mapsto e^{2\pi i y \xi}) \longleftarrow\!\shortmid\; \xi$

are isos of topological groups, identifying "abstract" $\hat{f}$ with the concrete examples.

A good example to think through (requiring no analysis) is $G = (\mathbb{Z}/n, +)$, with $\widehat{(\mathbb{Z}/n)} \xleftarrow{\sim} \mathbb{Z}/n$

$$\left(y \mapsto e^{\frac{2\pi i a y}{n}}\right) \longleftarrow\!\shortmid\ a$$

Lemma: $\widehat{(\ )} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, and $\forall f \in \mathcal{S}(\mathbb{R})$

$$\frac{d}{d\xi}\hat{f}(\xi) = \widehat{(-2\pi i x f)}(\xi).$$

Dually, $\widehat{(f')}(\xi) = 2\pi i \xi \hat{f}(\xi)$

(Fourier transform swaps differentiation & mult. by x).

Pf: $\hat{f}(\xi) = \int_{-\infty}^{\infty} \underbrace{f(x) e^{-2\pi i \xi x}}_{F(x,\xi)} dx$. Claim that $\frac{d\hat{f}}{d\xi}(\xi)$ exists and is given by $\int_{-\infty}^{\infty} \frac{\partial F}{\partial \xi}(x,\xi)\,dx = \int_{-\infty}^{\infty} -2\pi i x f(x) \cdot e^{-2\pi i \xi x} dx = \widehat{(-2\pi i x f)}(\xi)$

Let $g_n(\xi) = \int_{-n}^{n} F(x,\xi)\,dx$. Clearly $g_n \to \hat{f}$ pointwise. The key is to check each g_n' exists, and $g_n' \to \int_{-\infty}^{\infty} \frac{\partial F}{\partial \xi}(x,\xi)\,dx$ (as $n \to \infty$) uniformly on $\xi \in$ (any closed interval). Details an exercise — or see Körner Chpt. 53

OMIT:

Since $\frac{\partial F}{\partial \xi}$ exists and is continuous, each g_n is differentiable with

$\frac{dg_n}{d\xi}(\xi) = \int_{-n}^{n} \frac{\partial F}{\partial \xi}(x,\xi)\,dx$, and we further have

(*) $\left| g_n'(\xi) - \int_{-\infty}^{\infty} \frac{\partial F}{\partial \xi}(x,\xi)\,dx \right| = \left| \int_{|x| \geq n} \frac{\partial F}{\partial x}(x,\xi)\,dx \right| \to 0$ as $n \to \infty$ uniformly for $\xi \in$ any closed interval $[a,b]$.

Then $\lim g_n = \hat{f}$ is differentiable with derivative $\int_{-\infty}^{\infty} \frac{\partial F}{\partial \xi}(x,\xi)\,dx$.

For $\int_{-\infty}^{\infty} f'(x) e^{-2\pi i x \xi} dx$, integrate by parts: exercise.

Example: $f(x) = e^{-\pi x^2}$. $f'(x) = -2\pi x f(x)$. $\widehat{(f')} = 2\pi i \xi \hat{f}(\xi)$ and $= -i \cdot \frac{d\hat{f}}{d\xi}$,

using the two formulas from the lemma.

So $\frac{d\hat{f}}{d\xi}(\xi) = -2\pi \xi \hat{f}(\xi)$, the same DE f satisfies.

so $\left[(f/g)' = \frac{gf' - fg'}{g^2} = \frac{gfx - fgx}{g^2} = 0 \text{ for 2 solutions to } f' = xf \right]$

$\hat{f} = C \cdot f$, $C \in \mathbb{R}$.

$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1 = f(0)$, so $C = 1$,

and we conclude $\boxed{\hat{f} = f}$.

A simple COV shows that for $a \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$, $f_a(x) := f(ax)$, then

$$\widehat{(f_a)}(\xi) = \int_{-\infty}^{\infty} f(ax) e^{-2\pi i x \xi} dx \overset{y = ax}{=} \frac{1}{|a|} \int_{-\infty}^{\infty} f(y) e^{-2\pi i y \frac{\xi}{a}} dy$$

$$= \frac{1}{|a|} \hat{f}(\xi/a).$$

Thm: (Poisson summation) If $f \in \mathcal{S}(\mathbb{R})$, or more generally, if f and $\hat{f}$ are of moderate decrease, then

$$x \longmapsto \sum_{n\in\mathbb{Z}} f(x+n)$$

$$x \longmapsto \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

both converge, absolutely and uniformly on compacts, to the same continuous function of x.

Later we'll use this to study ζ's analytic ~~prop~~erties.

Now we use it to prove Euler's the~~orem~~

Computation of $\zeta(2m)$ ~~(Poisson kernel $P_t(x)$)~~

Let $f(x) = \frac{t}{\pi(x^2+t^2)}$ for $t>0$.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{t}{\pi(x^2+t^2)} \cdot e^{-2\pi i x \xi}\,dx$$

SKIP

Proof of Poisson summation, $f \in \mathcal{S}(\mathbb{R})$.

The sum defining $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ converges absolutely $\forall\, x\in\mathbb{R}$, & uniformly on compacts. Thus $F:\mathbb{R}\to\mathbb{C}$ is continuous and 1-periodic, so we may study its Fourier series.

We have also the continuous periodic function

$$G(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

, abs. conv. & unif. on compacts again b/c $\hat{f} \in \mathcal{S}(\mathbb{R})$

$\hat{f}$ is not periodic, so these are just values of F. Transform

By our basic pointwise convergence theorem, it will suffice to show F and G (continuous!) have same Fourier coefficients. Clearly $\hat{G}(m) = \hat{f}(m)$ $\forall$ m.

$$\hat{F}(m) = \int_0^1 F(x) e^{-2\pi i m x} dx = \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)\, e^{-2\pi i m x} dx$$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i m x} dx = \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x) e^{-2\pi i m x} dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i m x} dx = \hat{f}(m).$$

(Again, $\hat{F}$ means FS, $\hat{f}$ " FT) $\boxed{}$

We often use this in the form $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$.

Apply it to $f_a(x) = e^{-\pi a^2 x^2}$, where we know

$$\widehat{(f_a)}(\xi) = \frac{1}{|a|}\hat{f}(\xi/a) = \frac{1}{|a|} f(\xi/a) \quad \text{for } f(x) = e^{-\pi x^2}.$$

Thus $$\sum_{n\in\mathbb{Z}} e^{-\pi a^2 n^2} = \frac{1}{|a|}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/a^2}.$$ Letting $a^2 = y > 0$,

this says $\forall$ $y>0$, $$\sum_{n\in\mathbb{Z}} e^{-\pi y n^2} = \frac{1}{\sqrt{y}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/y},$$ i.e.

<u>Defn:</u> For $y \in \mathbb{R}_{>0}$, define the theta function

$$\Theta(y) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}.$$

<u>Cor:</u> $\forall$ $y>0$, $\Theta(y) = \frac{1}{\sqrt{y}}\Theta(1/y)$.

We'll apply this to ζ! Or more properly to
$\xi(s) = \Gamma(s/2)\pi^{-s/2}\zeta(s)$, which so far we know

is a holomorphic function on $\mathrm{Re}(s)>1$.

Key calculation:

$$\pi^{-s/2}\Gamma(s/2)\frac{1}{n^s} = \int_0^\infty \pi^{-s/2} n^{-s} e^{-y} y^{s/2}\frac{dy}{y} = \int_0^\infty e^{-y}\left(\frac{y}{\pi n^2}\right)^{s/2}\frac{dy}{y}$$

$$\underset{\substack{\uparrow \\ \text{Let } x=\frac{y}{\pi n^2}}}{=} \int_0^\infty e^{-\pi n^2 x} x^{s/2}\frac{dx}{x}.$$ When $\mathrm{Re}(s)>1$, we $\sum_{n=1}^\infty$ to get

$$\xi(s) = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 x} x^{s/2}\frac{dx}{x}.$$

If we replace the integrand with its absolute value,

$\sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 x} x^{\frac{\mathrm{Re}(s)}{2}}\frac{dx}{x} = \pi^{-\mathrm{Re}(s)/2}\Gamma\left(\frac{\mathrm{Re}(s)}{2}\right)\zeta(\mathrm{Re}(s))$ is finite,

so we may interchange $\sum$ & $\int$ to find ($\underbrace{\sum\int_0^a}_{\text{finite } \forall a \text{ and incr. in } a} \le \sum\int_0^\infty < \infty$; $\sum_{n=1}^N\int_0^\infty < \infty$; $=\int_0^\infty\sum_1^N$ incr. in N conv.)

$$\xi(s) \underset{\substack{\uparrow \\ \mathrm{Re}(s)>1}}{=} \int_0^\infty \boxed{\sum_{n=1}^\infty e^{-\pi n^2 x}}\, x^{s/2}\frac{dx}{x}$$

almost $\Theta(x)$! Namely it is $\frac{1}{2}[\Theta(x)-1]$

$$= \frac{1}{2}\int_0^\infty [\Theta(x)-1]\, x^{s/2}\frac{dx}{x}.$$

Let's argue formally for now. Let $\mathrm{Re}(s)>1$.

$$\xi(s) = \underbrace{\int_1^\infty \frac{(\Theta(x)-1)}{2} x^{s/2}\frac{dx}{x}} + \int_0^1 \frac{(\Theta(x)-1)}{2} x^{s/2}\frac{dx}{x}$$

certainly for $x\geq 1$, and for ANY $s\in\mathbb{C}$

$$\left|\frac{(\Theta(x)-1)}{2} x^{s/2-1}\right| = e^{-\pi x}\left(1+\sum_{n=2}^\infty e^{-\pi(n^2-1)x}\right) x^{\mathrm{Re}(s/2)-1} \le C_1 e^{-\pi x} x^{\mathrm{Re}(s/2)-1}$$

(1) $\le C_2\cdot\frac{1}{x^2}$ (or any higher degree poly you want)

It follows that $\int_1^\infty \frac{\Theta(x)-1}{2} x^{s/2}\frac{dx}{x}$ converges absolutely and uniformly in s, so defines a holomorphic function on all of $\mathbb{C}$.

② What about $\int_0^1 \frac{\Theta(x)-1}{2} x^{s/2} \frac{dx}{x}$? This is where we use $\Theta(x) = \frac{1}{\sqrt{x}} \Theta(1/x)$! Still for $\mathrm{Re}(s) > 1$,

$$\int_0^1 \frac{\Theta(x)-1}{2} x^{s/2} \frac{dx}{x}$$

(note that the sign is correct!)

subbing $y = 1/x$.

$$= -\frac{1}{2}\int_0^1 x^{s/2-1}\,dx + \int_1^\infty \frac{\Theta(1/y)}{2} y^{-s/2} \frac{dy}{y}$$

$$= -\frac{1}{\cancel{2}} \left.\frac{x^{s/2}}{\frac{s}{\cancel{2}}}\right]_0^1 + \int_1^\infty \frac{\sqrt{y}\,\Theta(y)\,y^{-s/2}}{2} \frac{dy}{y}$$

$$= -\frac{1}{s} + \int_1^\infty \frac{\Theta(y)-1}{2} y^{\frac{1-s}{2}} \frac{dy}{y} + \underbrace{\int_1^\infty \frac{y}{2}^{\frac{1-s}{2}-1} dy}_{\frac{1}{2} \left.\frac{y^{\frac{-s+1}{2}}}{\frac{-s+1}{2}}\right]_1^\infty}$$

$$= -\frac{1}{s} + \frac{1}{s-1} + \underbrace{\int_1^\infty \frac{\Theta(y)-1}{2} y^{1-s} \frac{dy}{y}}_{\text{holomorphic on all of } \mathbb{C}\text{, exactly as before.}}$$

Conclusion: For $\mathrm{Re}(s) > 1$,

$$\xi(s) = \int_1^\infty \frac{\Theta(y)-1}{2} \left(y^{\frac{s}{2}} + y^{\frac{1-s}{2}}\right) \frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}.$$

holomorphic on all of $\mathbb{C}$ hol. on $\mathbb{C}\setminus\{0,1\}$.

$\boxed{\text{Thm}}$ $\forall\, s\in\mathbb{C}\setminus\{0,1\}$, <u>define</u> $\xi(s)$ by the above formula; it is the unique holomorphic extension of $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ (defined initially on $\mathrm{Re}(s)>1$) to all of $\mathbb{C}\setminus\{0,1\}$. Moreover, $\forall\, s\in\mathbb{C}\setminus\{0,1\}$,

$$\xi(s)=\xi(1-s).$$

<u>Pf</u>: The symmetry is obvious from the formula, and we've proven the rest.

<u>Cor</u>: $\zeta(s)=\dfrac{\xi(s)}{\pi^{-s/2}\Gamma(s/2)}$ is holomorphic on $\mathbb{C}\setminus\{1\}$, and equals $\dfrac{1}{s-1}+(\text{hol. everywhere in }\mathbb{C})$.

$\forall\, n\in\mathbb{Z}_{\geq 1}$, $\zeta(1-n)=-\dfrac{B_n}{n}$, and

$$\zeta(0)=-1/2.$$

<u>Pf</u>: Recall $\Gamma(s)=\dfrac{\Gamma(s+1)}{s}$ for all $s\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. $\leftarrow$ simple pole $\Gamma(1)=1$.

Thus near 0, $\Gamma(s)=\dfrac{1}{s}+(\text{holomorphic})$, so near 0,

$$\zeta(s)=\frac{\text{hol.}-1/s}{\pi^{-s/2}\cdot\left(\frac{2}{s}+\text{hol}\right)}=\frac{s\cdot\text{hol}-1}{\pi^{-s/2}(2+s\cdot\text{hol})}$$ is hol, and $\zeta(0)=-1/2$ (take $\lim_{s\to 0}$)

We next need that $\Gamma(s)$ is never 0: this is not obvious but follows from $\boxed{\Gamma(s)\Gamma(1-s)=\dfrac{\pi}{\sin(\pi s)}}$ * (so $\dfrac{1}{\Gamma(s)}=\Gamma(1-s)\dfrac{\sin(\pi s)}{\pi}$ has no poles: $\Gamma(1-s)$ has poles for $s\in\mathbb{Z}_{\geq 1}$, but the zeros of $\sin(\pi s)$ cancel them), Thus $\zeta(s)$ is hol. on $\mathbb{C} \setminus \{1\}$.

$\xi(s) = \xi(1-s)$ gives

$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$, so

$\zeta(1-s) = \pi^{\frac{1}{2}-s}\dfrac{\Gamma(s/2)}{\Gamma(\frac{1-s}{2})}\zeta(s)$. Now we ⊛ and use pset.

$\Gamma(s)\Gamma(s+\frac{1}{2}) = \dfrac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s)$

to rewrite $\begin{cases} \Gamma(\frac{s}{2})\Gamma(\frac{1+s}{2}) = \dfrac{2\sqrt{\pi}}{2^s}\Gamma(s) \\ \Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2}) = \dfrac{\pi}{\sin(\pi(\frac{1-s}{2}))} = \dfrac{\pi}{\cos(\frac{\pi}{2}s)}, \end{cases}$

$1-(\frac{1-s}{2})$

hence

$$\boxed{\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\frac{2\sqrt{\pi}}{2^s}\Gamma(s)}{\frac{\pi}{\cos(\frac{\pi}{2}s)}}\cdot\zeta(s) = 2(2\pi)^{-s}\Gamma(s)\cos(\tfrac{\pi}{2}s)\zeta(s)}$$

Now for $n \in \mathbb{Z}_{\geq 1}$ odd,

$\zeta(1-n) = 0$ because $\cos(\frac{\pi}{2}n) = 0$ and rest of RHS is $\neq 0$ at n.

For $n \in \mathbb{Z}_{\geq 1}$ even,

$\zeta(1-n) = 2(2\pi)^{-n}\Gamma(n)\cos(\frac{\pi}{2}n)\zeta(n)$

$$\underset{\text{Euler}}{\overset{\uparrow}{=}} 2(2\pi)^{-n}(n-1)!(-1)^{n/2}\cdot(-1)^{n/2}\frac{2^{n-1}\pi^n}{(n-1)!}\left(-\frac{B_n}{n}\right)$$

Recall

$$\zeta(2m) = (-1)^m \frac{2^{2m-1}\pi^{2m}}{(2m-1)!}\left(-\frac{B_{2m}}{2m}\right). \qquad = -B_n/n\ !$$

<u>7/6:</u> Recall: we set $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ (for $\mathrm{Re}(s)>1$) could be written as

$$\xi(s) = \underbrace{-\frac{1}{s} + \frac{1}{s-1}}_{\text{hol. on } \mathbb{C}\setminus 0,1} + \underbrace{\int_1^\infty \frac{\Theta(x)-1}{2}\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)\frac{dx}{x}}_{\text{hol. on } \mathbb{C}}$$

where $\Theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$ and thus obtained the

- "analytic continuation" of ξ to $\mathbb{C}\setminus\{0,1\}$
- "functional equation" $\xi(s) = \xi(1-s)$.

Also deduced that for $n \in \mathbb{Z}_{\geq 1}$,

$$\zeta(1-n) = -\frac{B_n}{n}$$

where for $n=1$ we use $+1/2$ instead of $-1/2$ for B_1.

$$\zeta(-1) = 1+2+3+4+\dots$$
$$4\zeta(-1) = \quad 4 \quad + \quad 8 \quad + 12 + \dots$$

$$-3\zeta(-1) = 1-2+3-4+\dots = (1-1+1-1+\dots)(1-1+1-1+\dots) = \frac{1}{(1+1)^2} = 1/4$$

so $\zeta(-1) = -1/12$

$$= -\frac{B_2}{2}.$$

$$\frac{1}{(1-x)^2} = (1+x+x^2+\dots)(1+x+x^2+\dots)$$
$$1+2x+3x^2+4x^3+\dots$$

And the key thing we needed was

$$\Theta(y) = \frac{1}{\sqrt{y}}\Theta(1/y). \quad (y\in\mathbb{R}_{>0}).$$

All of this generalizes to

Defn: Let $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to S^1$ be a homomorphism (χ is a Dirichlet character). Extend χ to $\mathbb{Z}$ by

" $\{z \in \mathbb{C} \mid |z| = 1\}$

$$\begin{aligned} \chi : \mathbb{Z} &\longrightarrow \mathbb{C} \\ a &\longmapsto \chi(a \bmod m) \quad \text{if } (a,m) = 1 \\ a &\longmapsto 0 \quad \text{if } (a,m) > 1 \end{aligned}$$

(so $\chi : \mathbb{Z} \to \mathbb{C}$ is still multiplicative)

Eg: $m = p$ prime. $\chi = \left(\frac{\cdot}{p}\right) : (\mathbb{Z}/p\mathbb{Z})^\times \to \{\pm 1\} \subset S^1$, extending to $\left(\frac{\cdot}{p}\right) : \mathbb{Z} \to \{0, \pm 1\} \subset \mathbb{C}$.

Eg: $m = 1$. $\chi(a) = 1 \ \forall a \in \mathbb{Z}$. Write $\mathbb{1}$ for χ, the trivial character.

We will assume for simplicity that χ is primitive: there is no divisor $d \mid m$ s.t. χ factors

$$\begin{tikzcd} (\mathbb{Z}/m\mathbb{Z})^\times \arrow[r, two heads] \arrow[rr, bend right, "\chi"'] & (\mathbb{Z}/d\mathbb{Z})^\times \arrow[r, dashed, "\text{"}\bar{\chi}\text{"}"] & S^1 \end{tikzcd}$$

for some $\bar{\chi}$.

(i.e. $\chi(a)$ is not determined by $a \bmod d$)
(if it were, we'd replace m by d)

Defn: $L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$, the Dirichlet L-function.

$L(\chi, s)$ converges absolutely on $\mathrm{Re}(s) > 1$, unif. on cpcts, so is a holomorphic fn there, where also $L(\chi, s) = \prod_p (1 - \chi(p) p^{-s})^{-1}$.
When χ non-trivial, $L(\chi, s)$ converges (conditionally) on $\mathrm{Re}(s) > 0$,

but we won't use this. For χ = trivial char, $L(\chi,s) = \zeta(s)$

Recall that we used $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ to analyze ζ more clearly. Likewise here: define

$$\Lambda(\chi,s) = \left(\frac{m}{\pi}\right)^{s/2}\Gamma\left(\frac{s+p}{2}\right)\cdot L(\chi,s), \quad \text{where}$$

$\chi(-1) = (-1)^p$, $p \in \{0,1\}$. So for $\zeta(s)$, $\chi = \mathbb{1}$, $p=0$, $m=1$, and we get $\xi(s) = \Lambda(\mathbb{1},s)$

$\boxed{\text{Thm A}}$ Let χ be a nontrivial primitive Dirichlet character. Then $\Lambda(\chi,s)$ extends to a holomorphic function on all of $\mathbb{C}$ satisfying

$$\Lambda(\chi,s) = \frac{G(\chi)}{i^p\sqrt{m}}\Lambda(\overline{\chi},1-s)$$

note! complex conjugate

$G(\chi)$ is the Gauss sum

$$G(\chi) = \sum_{j=0}^{m-1}\chi(j)e^{2\pi i j/m}$$

(We could take $\chi = \mathbb{1}$ also; then $\Lambda(\mathbb{1},s) = \xi(s)$ is hol. on $\mathbb{C}\setminus\{0,1\}$, $\overline{\chi} = \chi$, $G(\chi) = 1$, $p=0$, $m=1$.)

Pf: See the problems. Sketch: $\Gamma\left(\frac{s+p}{2}\right) = \int_0^\infty e^{-y}y^{\frac{s+p}{2}}\frac{dy}{y}$

Set $y = \frac{\pi n^2 t}{m}$ $\quad = \int_0^\infty e^{-\frac{\pi n^2}{m}t}\left(\frac{\pi n^2 t}{m}\right)^{\frac{s+p}{2}}\frac{dt}{t}$

so $\left(\frac{m}{\pi}\right)^{\frac{s+p}{2}}\Gamma\left(\frac{s+p}{2}\right)n^{-s} = \int_0^\infty n^p e^{-\pi n^2 t/m}\cdot t^{\frac{s+p}{2}}\frac{dt}{t}$

Multiply by $\chi(n)$ & $\sum_{n=1}^\infty$:

$$\left(\frac{m}{\pi}\right)^{p/2}\Lambda(\chi,s) = \sum_{n=1}^\infty\int_0^\infty \chi(n)n^p e^{-\pi n^2 t/m}t^{s+p/2}\frac{dt}{t}$$

$$\overset{(\mathrm{Re}(s)>1)}{=} \int_0^\infty\sum_{n=1}^\infty \chi(n)n^p e^{-\pi n^2 t/m}\cdot t^{\frac{s+p}{2}}\frac{dt}{t}$$

Define $\Theta(\chi,y) = \sum'_{n\in\mathbb{Z}} \chi(n)\, n^p e^{-\pi n^2 y/m}$ (when n & p are both 0, take $n^p = 1$ here)

For $\chi = \mathbb{1}$, this is $\Theta(y)$ from before.

Since $\chi(-n)(-n)^p = \chi(n)n^p$, (this is why we need the n^p!)

$$\left(\frac{m}{\pi}\right)^{p/2}\Lambda(\chi,s) = \int_0^\infty \left(\frac{\Theta(\chi,y)-\chi(0)}{2}\right) y^{\frac{s+p}{2}}\frac{dy}{y}.$$ (For $\chi \neq \mathbb{1}$, $\chi(0)=0$).

As with ζ, we use Poisson summation to show for $y>0$ } exercise.

$$\Theta(\chi, 1/y) = \frac{G(\chi)}{i^p\sqrt{m}}\, y^{p+\frac{1}{2}}\,\Theta(\bar\chi, y)$$

(Take $\chi \neq \mathbb{1}$.)

$$\left(\frac{m}{\pi}\right)^{p/2}\Lambda(\chi,s) \overset{\text{Relabel}}{=} \int_0^\infty \frac{\Theta(\chi,y)}{2}\, y^{\frac{s+p}{2}}\frac{dy}{y} = \int_0^1 + \int_1^\infty = \int_1^\infty \frac{\Theta(\chi,1/y)}{2}\, y^{-(\frac{s+p}{2})}\frac{dy}{y}$$
$$+ \int_1^\infty \frac{\Theta(\chi,y)}{2}\, y^{\frac{s+p}{2}}\frac{dy}{y} = \frac{G(\chi)}{i^p\sqrt{m}}\int_1^\infty \frac{\Theta(\bar\chi,y)}{2}\, y^{p+\frac{1}{2}-(\frac{s+p}{2})}\frac{dy}{y} + \int_1^\infty \frac{\Theta(\chi,y)}{2}\, y^{\frac{s+p}{2}}\frac{dy}{y}$$

which is an entire function of s, and multiplied by $\frac{i^p\sqrt{m}}{G(\chi)}$ we get

$$\left(\frac{m}{\pi}\right)^{p/2}\frac{i^p\sqrt{m}}{G(\chi)}\Lambda(\chi,s) = \int_1^\infty \frac{\Theta(\bar\chi,y)}{2}\, y^{\frac{1-s+p}{2}}\frac{dy}{y} + \frac{i^p\sqrt{m}}{G(\chi)}\int_1^\infty \frac{\Theta(\chi,y)}{2}\, y^{\frac{s+p}{2}}\frac{dy}{y}$$, whereas

$$\left(\frac{m}{\pi}\right)^{p/2}\Lambda(\bar\chi,1-s) = \underbrace{\int_1^\infty \frac{\Theta(\bar\chi,y)}{2}\, y^{\frac{1-s+p}{2}}\frac{dy}{y}} + \int_1^\infty \frac{\Theta(\bar\chi,1/y)}{2}\, y^{-(\frac{1-s+p}{2})}\frac{dy}{y}$$

$$= \underbrace{\qquad\qquad} + \frac{G(\bar\chi)}{i^p\sqrt{m}}\int_1^\infty \frac{\Theta(\chi,y)}{2}\, y^{p+\frac{1}{2}-(\frac{1-s+p}{2})}\frac{dy}{y}$$

$$= \underbrace{\qquad\qquad} + \frac{m(-1)^p}{i^p G(\chi)\sqrt{m}}\int_1^\infty \frac{\Theta(\chi,y)}{2}\, y^{\frac{s+p}{2}}\frac{dy}{y} = \frac{i^p\sqrt{m}}{G(\chi)}\Lambda(\chi,s)$$

here we use $|G(\chi)|^2 = m$, i.e. $G(\chi)\overline{G(\chi)} = m$ (for a primitive χ), and

$\overline{G(\chi)} = \sum'_{k\in\mathbb{Z}/m} \overline{\chi(k)}\, e^{-2\pi i k/m} = \chi(-1)\sum' \overline{\chi(k)}\, e^{2\pi i k/m} = (-1)^p G(\bar\chi)$. Win! ■

STOP HERE & GO BACK TO POISSON SUMMATION.

(Not in lecture: here's the messy Poisson summation step.

$$\Theta_\mu(a,b,y) = \sum_{n\in\mu\mathbb{Z}} e^{-\pi(a+n)^2 y + 2\pi i b n}$$

$$\rightsquigarrow \quad \Theta_\mu(a,b,1/y) = \sum_{n\in\mathbb{Z}} e^{-\pi(a+n\mu)^2\cdot\frac{1}{y} + 2\pi i b n\mu}$$

Set $f_{\mu,a,b}(x) = e^{-\pi(a+\mu x)^2 + 2\pi i b\mu x}$ $\qquad h(x) = e^{-\pi x^2}$

Then $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi(a+\mu x)^2 + 2\pi i b\mu x}\cdot e^{-2\pi i x\xi}\,dx$ $\qquad x - \frac{a}{\mu}$ for x

$$= \int_{-\infty}^{\infty} e^{-\pi(\mu x)^2}\cdot e^{2\pi i b\mu(x-\frac{a}{\mu}) - 2\pi i(x-\frac{a}{\mu})\xi}\,dx$$

$$\widehat{(h_\mu)}(\xi) = \frac{1}{\mu}\hat{h}(\xi/\mu) = \frac{1}{\mu} h(\xi/\mu)$$

$$= e^{-2\pi i ab + 2\pi i \frac{a\xi}{\mu}}\cdot\int_{-\infty}^{\infty} h(\mu x)\, e^{2\pi i x(b\mu - \xi)}$$

$$= e^{-2\pi i ab + 2\pi i \frac{a\xi}{\mu}}\cdot\widehat{(h_\mu)}(\xi - b\mu)$$

$$= \underbrace{\qquad\qquad}_{}\cdot\frac{1}{\mu} h\left(\frac{\xi}{\mu} - b\right)$$

Now $\Theta_\mu(a,b,1/y) = \sum_{n\in\mathbb{Z}} f_{\frac{\mu}{\sqrt{y}},\frac{a}{\sqrt{y}},b\sqrt{y}}(n) \overset{\text{Poisson}}{=} \sum_{n\in\mathbb{Z}} \widehat{f_{\frac{\mu}{\sqrt{y}},\frac{a}{\sqrt{y}},b\sqrt{y}}}(n)$

$$= e^{-2\pi i ab}\cdot\frac{\sqrt{y}}{\mu}\cdot\sum_{n\in\mathbb{Z}} h\left(\frac{n\sqrt{y}}{\mu} - b\sqrt{y}\right)\cdot e^{2\pi i \frac{an}{\mu}}$$

$$= e^{-2\pi i ab}\frac{\sqrt{y}}{\mu}\sum_{n\in\mathbb{Z}} e^{-\pi(\frac{n}{\mu}-b)^2 y + 2\pi i a\frac{n}{\mu}}$$

$$= e^{-2\pi i ab}\frac{\sqrt{y}}{\mu}\,\Theta_{\frac{1}{\mu}}(-b,a,y). \qquad \blacksquare$$)

Onto special values of $L(\chi,s)$. We've now seen a little more abstract Fourier analysis. Problem 9 is an essential exercise, and we'll use the following consequence:

(Recall from problems: $\chi: \mathbb{Z}/m \to \mathbb{C}$ has Fourier transform

$$\hat{\chi}(a) = \frac{1}{m}\sum_{k\in\mathbb{Z}/m} \chi(k)e^{-2\pi i ak/m} = \frac{1}{m}\cdot G(\chi,-a)$$

$$\underset{\substack{\uparrow \\ \chi \text{ prim.} \\ \text{for } (a,m)>1}}{=} \frac{1}{m}\bar{\chi}(-a)\,G(\chi)$$, and Fourier inversion

$$\chi(k) = \sum_{a\in\mathbb{Z}/m}\hat{\chi}(a)e^{2\pi i ak/m} = \frac{1}{m}\sum_a \bar{\chi}(-a)G(\chi)e^{2\pi i ak/m}$$

$$= \frac{1}{m}G(\chi)\chi(-1)\sum_a \bar{\chi}(a)e^{2\pi i ak/m}$$

so $$\sum_{k=1}^{\infty}\chi(k)k^{-s} \underset{\mathrm{Re}(s)>1}{=} \frac{1}{m}G(\chi)\chi(-1)\sum_{a=1}^{m}\bar{\chi}(a)\sum_{k=1}^{\infty}\frac{e^{2\pi i ak/m}}{k^s}$$).

(Special values of $L(\chi,s)$. χ primitive non-trivial mod m. $\chi(-1)=(-1)^p$

Defn: The generalized Bernoulli numbers $B_{n,\chi}$, $n\in\mathbb{Z}_{\geq 0}$, are defined by

$$\underbrace{\sum_{a=1}^{m}\chi(a)\frac{te^{at}}{e^{mt}-1}}_{F_\chi(t)=} = \sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!}$$

For $\chi=\mathbb{1}$: $$\frac{te^t}{e^t-1} = \frac{t}{e^t-1}+t = \sum_{n=0}^{\infty}B_n\frac{t^n}{n!}+t,$$ so $$B_{n,\mathbb{1}} = \begin{cases} B_n, & n\neq 1 \\ \underbrace{B_1+1}_{1/2}, & n=1\end{cases}$$

(this normalization more convenient for special values)

We know $B_{\text{odd}\geq 3}=0$. Similarly,

$$\underbrace{\sum_a \chi(a)\frac{-te^{-at}}{e^{-mt}-1}}_{F_\chi(-t)} = \sum_{a=1}^{m}\underbrace{\chi(a)}_{\chi(-1)\cdot\chi(m-a)}\frac{te^{(m-a)t}}{e^{mt}-1} = \chi(-1)\cdot F_\chi(t) = (-1)^pF_\chi(t)$$

So $(-1)^nB_{n,\chi} = (-1)^pB_{n,\chi}$. Thus if $n\not\equiv p \bmod 2$, $B_{n,\chi}=0$.

(& we'll see $L(\chi,n)$ for $n\equiv p\ (2)$ along the way)

$\boxed{\text{Thm B}}$ For any $n\in\mathbb{Z}_{\geq 1}$,

$$L(\chi,1-n) = -\frac{B_{n,\chi}}{n} \qquad (=0 \text{ for } n\not\equiv p \bmod 2)$$

Pf: $L(\chi,s) \overset{Re(s)>1}{=} \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \overset{\boxed{\text{Exercise}}}{=} \frac{\chi(-1)G(\chi)}{m}\sum_{a=1}^{m}\bar{\chi}(a)\sum_{n=1}^{\infty}\frac{e^{2\pi i a n/m}}{n^s}$ — see last page

(and $\bar{\chi}(m)=0$)

Recall from exercises the Bernoulli polynomials given by $B_k(x) = \sum_{j=0}^{k}\binom{k}{j}B_j x^{k-j}$

We can relate the inner sums $\sum_{n=1}^{\infty}\frac{e^{2\pi i n x}}{n^k}$ for $k\in\mathbb{Z}_{\geq 1}$ and $x\in\mathbb{R}$

to 1-periodic extensions of ($B_k(x)$ restricted to $x\in[0,1]$)

Set $A_k(x) = \sum_{n\in\mathbb{Z}\setminus 0}\frac{e^{2\pi i n x}}{n^k}$, $x\in[0,1]$ (and 1-periodic)

We $\boxed{\text{claim}}$ for $k\geq 2$, $A_k(x) = -\frac{(2\pi i)^k}{k!}B_k(x)$, $\forall\, x\in[0,1]$

actually for $x\in(0,1)$ this holds for $k=1$, but OMIT.

Note $B_k(0)=B_k(1)$ for $k\geq 2$:

$$\sum B_n\frac{t^n}{n!}\cdot\sum\frac{(xt)^m}{m!} = \sum_n\sum_{k=0}^{n} n!\frac{B_k}{k!}\frac{x^{n-k}}{(n-k)!}\cdot\frac{t^n}{n!} = \sum_{n=0}^{\infty}\Big(\underbrace{\sum\binom{n}{k}B_k x^{n-k}}_{B_n(x)}\Big)\frac{t^n}{n!}$$

$$B_n'(x) = \sum_{k=0}^{n-1}(n-k)\binom{n}{k}B_k x^{n-k-1} = n\cdot\sum\frac{(n-1)!}{k!(n-1-k)!}B_k x^{n-k-1} = n\cdot B_{n-1}(x)$$

$n=1$: $B_1(x) = x+B_1 = x-1/2$

$n=2$: $B_2(x) = x^2-x+\frac{1}{6}$.

$$B_k(1)-B_k(0) = \sum_{j=0}^{k}\binom{k}{j}B_j - B_k = \sum_{j=0}^{k-1}\binom{k}{j}B_j \overset{k\geq 2}{=} 0 \quad \text{(defining recursion)}$$

Moreover, since $B_k'(x) = kB_{k-1}(x)$ (above or ex.),

Thus $0 \underset{k\geq 2}{=} B_k(1)-B_k(0) = \int_0^1 B_k'(x)\,dx = k\int_0^1 B_{k-1}(x)\,dx$, so $\int_0^1 B_n(x)\,dx = 0$ for $n\geq 1$.

Pf of $\boxed{\text{claim:}}$ $\quad A_k(0) = \sum_{n\in\mathbb{Z}\setminus 0} n^{-k} = \zeta(k)\cdot(1+(-1)^k) = \begin{cases} 2\zeta(k) & k \text{ even} \\ 0 & k \text{ odd}\end{cases}$

For $k \geq 2$

Also $\quad -\frac{(2\pi i)^k}{k!} B_k(0) = -\frac{(2\pi i)^k}{k!}\cdot B_k = \begin{cases} 0 & k \text{ odd} \\ 2\cdot\frac{(-1)^m \pi^{2m}}{(2m-1)!}\cdot 2^{2m-1}\cdot\left(\frac{-B_{2m}}{2m}\right) & k=2m \end{cases}$

$$\shortparallel$$

$$2\cdot\zeta(2m)$$

So the expressions agree at $x=0$.

$A_k(x)$ is differentiable and for $k\geq 3$,

$$A_k'(x) = \sum_{n\in\mathbb{Z}\setminus 0} \frac{2\pi i n e^{2\pi i n x}}{n^k} \qquad \forall\ x\in\mathbb{R}\ (\text{or } [0,1])$$

$$= 2\pi i\cdot A_{k-1}(x)$$

But $\frac{d}{dx}\left[-\frac{(2\pi i)^k}{k!}B_k(x)\right] = -\frac{(2\pi i)^k}{k!} k B_{k-1}(x)$

$$= 2\pi i\cdot\left[-\frac{(2\pi i)^{k-1}}{(k-1)!}\cdot B_{k-1}(x)\right].$$

Thus by induction and the Mean Value Theorem it will suffice to show $A_2(x) = -\frac{(2\pi i)^2}{2!}B_2(x)$

$$= 2\pi^2(x^2-x+1/6)$$

for $x\in[0,1]$.

$A_2(x) = \sum_{n\in\mathbb{Z}\setminus 0}^{\infty} \frac{e^{2\pi i n x}}{n^2}$ has

$$\hat{A}_2(m) = \int_0^1 A_2(x)e^{-2\pi i m x}dx = \int_0^1 \sum_{n\in\mathbb{Z}\setminus 0} \frac{e^{2\pi i(n-m)x}}{n^2}dx$$

$= \sum\int \_\_ = \frac{1}{m^2}$ for $m\neq 0$, and $=0$ for $m=0$

Meanwhile

$\frac{1}{3}-\frac{1}{2}+\frac{1}{6}=0$

$$\hat{B}_2(m) = \int_0^1 \left(x^2 - x + \frac{1}{6}\right) e^{-2\pi i m x}\,dx = 0 \qquad m=0$$

$$\int \overset{u}{x^2} \overset{dv}{e^{\alpha x}} = \frac{x^2 e^{\alpha x}}{\alpha} - \int \overset{u}{2x} \overset{dv}{\frac{e^{\alpha x}}{\alpha}}\,dx = \frac{x^2 e^{\alpha x}}{\alpha} - 2\left(\frac{x e^{\alpha x}}{\alpha^2} - \int \frac{e^{\alpha x}}{\alpha^2}\,dx\right)$$

$$\text{so } \int_0^1 x^2 e^{-2\pi i m x}\,dx = \frac{1}{\alpha} - 2\left(\frac{1}{\alpha^2} - 0\right)$$

$\alpha = -2\pi i m$

$$-\int x e^{-2\pi i m x}\,dx = -\left(\frac{1}{\alpha}\right)$$

$$\text{so } \hat{B}_2(m) = \frac{-2}{(-2\pi i m)^2} = \frac{1}{2\pi^2 m^2}, \text{ i.e. } 2\pi^2 \hat{B}_2(m) = \hat{A}_2(m)\ \forall m.$$

Since $\forall m$ these continuous periodic functions have the same Fourier coefficients, $B_2(x) = A_2(x)\ \forall x \in [0,1]$.

Now we can evaluate $L(\chi, k)$, at least for $k \geq 2$ ($k=1$ missing from our treatment), and hence $L(\chi \text{ or } \bar{\chi}, 1-k)$.
but the same holds: $B_1(x) = A_1(x)$, $x \in (0,1)$ exercise!

$\boxed{\text{Thm B}'}$ $\chi : (\mathbb{Z}/m\mathbb{Z})^* \longrightarrow \mathbb{C}^*$ primitive character, $m > 1$. Let $k \in \mathbb{Z}_{\geq 1}$ satisfy $k \equiv p \bmod 2$, i.e. $\chi(-1) = (-1)^k$. Then

$$L(\chi, k) = -\frac{\chi(-1)\, G(\chi) (2\pi i)^k}{2m \cdot k!} \sum_{j=1}^{m-1} \bar{\chi}(j)\, B_k\!\left(\frac{j}{m}\right)$$

$\boxed{B' \Rightarrow B}$ $\left(L(\chi, 1-n) = -\frac{B_{\chi,n}}{n} \right)$

Pf: Recall

$$L(\chi, k) = \frac{\chi(-1)\, G(\chi)}{m} \sum_{a=1}^{m} \bar{\chi}(a) \sum_{n=1}^{\infty} \frac{e^{2\pi i a n/m}}{n^k}$$

(and $\bar{\chi}(m)=0$)

relate to $A_k\left(\frac{a}{m}\right) = \sum_{n\in\mathbb{Z}\setminus 0} \frac{e^{2\pi i an/m}}{n^k}$

We have a and $-a$ terms in the $\sum_a$:

$$\bar{\chi}(a) \sum_{n=1}^{\infty} \frac{e^{2\pi i an/m}}{n^k} + \bar{\chi}(-1)\bar{\chi}(a) \sum_{n=1}^{\infty} \frac{e^{-2\pi i an/m}}{n^k}$$

$$= \bar{\chi}(a) \sum_{n=1}^{\infty} \frac{e^{2\pi i an/m} + \chi(-1) e^{-2\pi i an/m}}{n^k}.$$

If $\chi(-1) = (-1)^k$ (the assumption of the theorem), this equals

$\bar{\chi}(a) \cdot A_k(a/m)$.

Thus by duplicating the $\sum_a \sum_n$ we get

$$L(\chi,k) = \frac{\chi(-1)G(\chi)}{2m} \cdot 2 \cdot \sum_a \sum_m = \frac{\chi(-1)G(\chi)}{2m}\left(\sum_a \sum_m + \sum_{-a} \sum_m\right)$$

$$= \frac{\chi(-1)G(\chi)}{2m} \sum_{a=1}^{m} \bar{\chi}(a) A_k(a/m)$$ (by doubling we get $\sum_{a=1}^{m}$ rather than $\sum_{a=1}^{m/2}$).

$$= \frac{\chi(-1)G(\chi)}{2m} \sum_a \bar{\chi}(a) \frac{-(2\pi i)^k}{k!} B_k(a/m)$$ (we proved for $k \geq 2$; valid for $k \geq 1$)

$$= \frac{-\chi(-1)G(\chi)(2\pi i)^k}{2m \cdot k!} \sum_{a=1}^{m} \bar{\chi}(a) B_k(a/m)$$

Recall.

$$A_k(x) = -\frac{(2\pi i)^k}{k!} B_k(x) \quad x\in[0,1] \ (\text{or } (0,1) \ k=1)$$

$$\Lambda(\chi,s) = \frac{G(\chi)}{i^p\sqrt{m}} \Lambda(\bar{\chi}, 1-s)$$

$$\sum_{a=1}^{m} \chi(a) \frac{t e^{at}}{e^{mt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}$$

Now we use the functional equation.

$$L(\chi,1-k) = \frac{\Lambda(\chi,1-k)}{\left(\frac{m}{\pi}\right)^{\frac{1-k}{2}} \Gamma\left(\frac{1-k+p}{2}\right)}$$

$$= \frac{\Lambda(\bar{\chi},k)}{\left(\frac{m}{\pi}\right)^{\frac{1-k}{2}} \Gamma\left(\frac{1-k+p}{2}\right)} \cdot \frac{G(\chi)}{i^p\sqrt{m}}$$

$$= \frac{L(\bar{\chi},k)\left(\frac{m}{\pi}\right)^{\left(\frac{k}{2}\right)}\Gamma\left(\frac{k+p}{2}\right)}{\left(\frac{m}{\pi}\right)^{\left(\frac{1-k}{2}\right)}\Gamma\left(\frac{1-k+p}{2}\right)} \frac{G(\chi)}{i^p\sqrt{m}}$$

$$\left(\frac{m}{\pi}\right)^{k-\frac{1}{2}} \frac{\Gamma\left(\frac{k+p}{2}\right)}{\Gamma\left(\frac{1-k+p}{2}\right)} \cdot \frac{-\chi(-1)(2\pi i)^k}{2m\cdot k!} \frac{G(\bar{\chi})G(\chi)}{i^p\sqrt{m}} \cdot \sum_{a=1}^{m} \chi(a) B_k(a/m)$$

$p=0$: $s=k$

$$\frac{\Gamma(k/2)}{\Gamma(\frac{1-k}{2})} = \frac{2\sqrt{\pi}\,\Gamma(k)\cos(\frac{\pi}{2}k)}{2^k\,\pi}$$

$p=1$:

$$\left\{\begin{array}{l} \Gamma(\frac{s}{2})\Gamma(\frac{1+s}{2}) = \frac{2\sqrt{\pi}}{2^s}\Gamma(s) \\ \Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2}) = \frac{\pi}{\sin(\pi(\frac{1-s}{2}))} = \frac{\pi}{\cos(\frac{\pi}{2}s)} \end{array}\right.$$

$$\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{1-k+1}{2})} = \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{1-(k-1)}{2})}\cdot\left(\frac{k-1}{2}\right) \underset{s=k-1}{=} \frac{2\sqrt{\pi}}{2^{k-1}}\frac{\Gamma(k-1)}{\pi}\cos(\tfrac{\pi}{2}(k-1))\cdot\frac{k-1}{2}$$

combine

So in sum (for $k \equiv p \bmod 2$)

b/c $G(\chi)G(\bar{\chi}) = \chi(-1)\cdot m$

$$L(\chi,1-k) = \left(\frac{m}{\pi}\right)^{k-\frac{1}{2}}\cdot\frac{2\sqrt{\pi}}{2^k}\cdot\frac{\Gamma(k)}{\pi}\underbrace{\cos(\tfrac{\pi}{2}(k-p))}_{(-1)^{\frac{k-p}{2}}}\cdot\frac{-\chi(-1)(2\pi i)^k}{2\,i^p\,k!\,\sqrt{m}}\cdot(-1)^p$$

$$\cdot\sum_a \chi(a)B_k(a/m)$$

$$= -(-1)^p(-1)^p\, i^{k-p}\cdot(-1)^{\frac{k-p}{2}}\cdot\frac{2^{k+1}}{2^{k+1}}\cdot\frac{\pi^{k+1/2}}{\pi^{k+1/2}}\cdot\frac{(k-1)!}{k!}\cdot m^{k-1}\sum_a \chi(a)B_k(\tfrac{a}{m})$$

$$= \frac{-1}{k}\underbrace{m^{k-1}\sum_a \chi(a)B_k(a/m)}_{} = -\frac{B_{k,\chi}}{k} \quad \blacksquare$$

Lemma: This $= B_{k,\chi}$!

Proof of Lemma:

$$\sum_{n=0}^{\infty} m^{n-1} \sum_{a=1}^{m} \chi(a) B_n(a/m) \frac{t^n}{n!} = \sum_{a=1}^{m} \frac{\chi(a)}{m} \sum_{n=0}^{\infty} B_n(a/m) \frac{(mt)^n}{n!}$$

$$= \sum_{a=1}^{m} \frac{\chi(a)}{\cancel{m}} \frac{\cancel{m}t}{e^{mt}-1} \cdot e^{\frac{a}{m}\cdot mt}$$

$$= \sum_{a=1}^{m} \chi(a) \frac{t\cdot e^{at}}{e^{mt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}$$

The result follows by comparing coefficients: $B_{n,\chi} = m^{n-1} \sum_{a=1}^{m} \chi(a) B_n\left(\frac{a}{m}\right)$ ■

$$\sum_{a=1}^{m} \chi(a) \frac{t e^{at}}{e^{mt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}$$

$$\frac{t}{e^t-1} \cdot e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

checked earlier (as this is the defn)

Phew!

$$\sum B_n \frac{t^n}{n!} \cdot \sum \frac{(xt)^m}{m!} = \sum_n \sum_{k=0}^{n} n! \frac{B_k}{k!} \frac{x^{n-k}}{(n-k)!} \cdot \frac{t^n}{n!} = \sum_{n=0}^{\infty} \Big(\underbrace{\sum \binom{n}{k} B_k x^{n-k}}_{B_n(x)} \Big) \frac{t^n}{n!}$$

New unit: p-adic interpolation and p-adic L-functions

Recall: $\zeta(1-n) = -B_n/n \quad \forall\ n \in \mathbb{Z}_{\geq 1}$ (taking $B_1 = 1/2$ instead of $-1/2$)

$L(\chi, 1-n) = -B_{n,\chi}/n \quad \forall\ n \in \mathbb{Z}_{\geq 1}$

The B_n's are rational numbers. The $B_{n,\chi}$'s live in the finite degree extension $\mathbb{Q}[\{\text{values of } \chi\}]/\mathbb{Q}$.

Question: What can we say about these rational numbers? Namely, about their prime divisibility properties?

- what p divide numerators?
- " " " denominators?
- any congruence relation between different values?

Thm: Let p be prime. $n \in \mathbb{Z}_{\geq 1}$

① If $n \not\equiv 0 \bmod (p-1)$, then $-\frac{B_n}{n} = \zeta(1-n) \in \mathbb{Z}_{(p)} = \{a \in \mathbb{Q} \mid a = \frac{b}{c},\ b, c \in \mathbb{Z}\ (b,c) = 1 \text{ and } p \nmid c\}$

①' If $n \equiv 0 \bmod (p-1)$, and n even, then $p \cdot B_n \equiv -1 \bmod p$

(i.e. $B_n \in -\sum_{\substack{p: \\ p-1 \mid n}} \frac{1}{p} + \mathbb{Z}$)

② Kummer congruences: Let $n \equiv n' \bmod (p-1)$ and $n \not\equiv 0 \bmod (p-1)$ (case ①). Then

$$-\frac{B_n}{n} = \zeta(1-n) \equiv \zeta(1-n') = -\frac{B_{n'}}{n'} \underbrace{\bmod p}_{\text{ie mod } p\mathbb{Z}_{(p)}}$$

More generally, if $n \equiv n' \bmod (p-1)\cdot p^r$ $r \in \mathbb{Z}_{\geq 0}$, then

$$(1-p^{n-1})\zeta(1-n) \equiv (1-p^{n'-1})\zeta(1-n') \bmod p^{r+1}.$$

Example: $p = 7$, $n = 4$, $n' = 10$ $\quad n' \equiv n \bmod (p-1)$

$\zeta(1-4) = -B_4/4 = \frac{1}{4\cdot 30}$ and $\frac{1}{4\cdot 30} + \frac{5}{66\cdot 10} = \frac{1}{60}\left(\frac{1}{2}+\frac{5}{11}\right)$

$\zeta(1-10) = -B_{10}/10 = -\frac{5}{66\cdot 10}$ $\qquad = \frac{1}{60}\cdot\frac{11+10}{22} \equiv 0 \bmod 7\cdot\mathbb{Z}_{(7)}$

!!! The heuristic here is

$$(1-p^{k-1})\zeta(1-k) = \sum_{\substack{n=1\\(n,p)=1}}^{\infty} n^{k-1} \underset{\substack{\Uparrow\\ k\equiv k' \bmod (p-1)\cdot p^r\\ p\nmid n}}{\equiv} \sum (n')^{k'-1} = (1-p^{k'-1})\zeta(1-k') \bmod p^{r+1}$$

where the congruence holds term-by-term

③ Kummer criterion. K. was trying to prove Fermat's Last Theorem; in the process he developed the foundations of alg. # theory.

$x,y,z \in \mathbb{Z}$: $x^p + y^p = z^p \Rightarrow \prod_{j=0}^{p-1}(x+\zeta_p^j y) = z^p$, an equality in the ring $\mathbb{Z}[\zeta_p]$.

$\zeta_p = e^{2\pi i/p}$

When factors on LHS are relatively prime, does this force them to be p^{th} powers (up to units $\in \mathbb{Z}[\zeta_p]^\times$). Yes if $\mathbb{Z}[\zeta_p]$ is UFD, and if $p \nmid \# Cl(\mathbb{Z}[\zeta_p])$, no in general

Kummer's criterion: TFAE:

Ⓐ $p \nmid \# Cl(\mathbb{Z}[\zeta_p])$

Ⓑ For all even m, $2 \le m \le p-3$, $p \nmid \zeta(1-m)$

(i.e. the numerator of B_m is not divisible by p).

(Refinement: Herbrand-Ribet Theorem).

Example: $\underset{p}{691} \mid -\frac{B_{12}}{12} = \zeta(-11)$, so $691 \mid \# Cl(\mathbb{Z}[\zeta_{691}])$

$m=12$

We see that ①+①' determine denominators of all $\zeta(1-m)$, but that the numerators lie very deep!

The natural context for all of these results is <u>p-adic</u> L-functions;

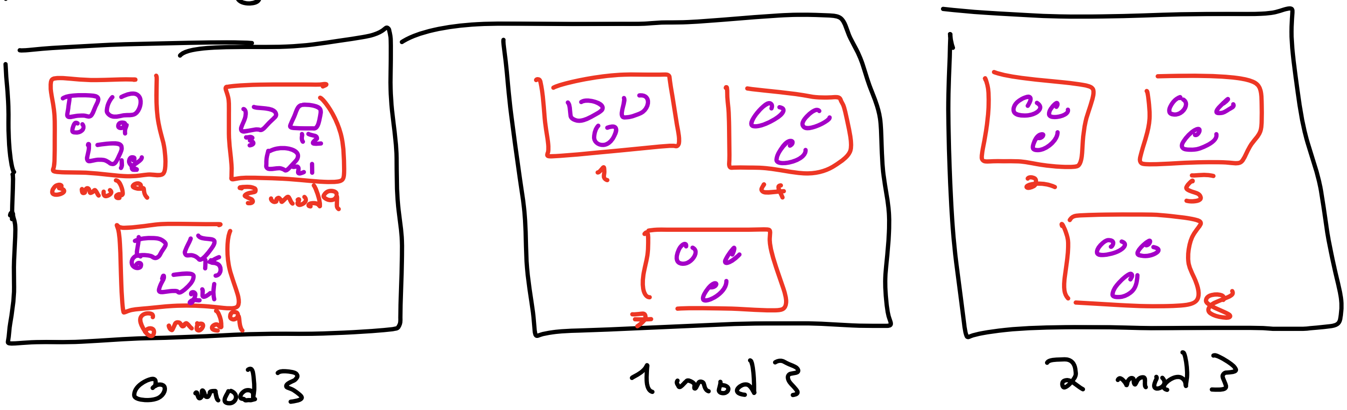
$\boxed{\S}$ The p-adic numbers

$p = 3$. Put integers into mod 3 boxes []

Then mod 9 boxes

Then mod 27 boxes

etc.



The 3-adics will be a number system with a notion of <u>distance</u> s.t. integers are "close" if they lie in a small box in this picture, ie if $\equiv$ mod (high power of p).

<u>Defn:</u> Fix a prime p. Let $a \in \mathbb{Q}$. Write $a = p^r \cdot \frac{u}{v}$ where $u, v \in \mathbb{Z}$ are not divisible by p. Define the <u>p-adic valuation</u> $v_p(a) = r$.

So $v_p : \mathbb{Q} \setminus 0 \to \mathbb{Z}$. We set $v_p(0) = $ "∞" for convenience

<u>Lemma:</u> (1) $\forall\, a, b \in \mathbb{Q}$, $v_p(ab) = v_p(a) + v_p(b)$

(with convention $\infty + ? = \infty$)

(2) $v_p(a+b) \geq \min(v_p(a), v_p(b))$, and equality holds if $v_p(a) \neq v_p(b)$. (again convention $\infty \geq$ any $n \in \mathbb{Z}$ or ∞)

Pf - easy and essential exercise.

Eg: Consider the sequence $\left\{ a_n = \sum_{k=0}^{n-1} p^k = 1 + p + \cdots + p^{n-1} \in \mathbb{Z} \right\}_{n \geq 1}$.

$\frac{1}{1-p} - a_n = \frac{1}{1-p} - \frac{1-p^n}{1-p} = \frac{p^n}{1-p}$ thus $v_p\left(\frac{p^n}{1-p}\right) \to \infty$ as

$n\to\infty$. The p-adics will be a metric space where the sequence $\{a_n\}_{n\geq 1}$ converges to $\frac{1}{1-p}$. We need to convert v_p into a notion of distance.

Defn: For $a\in\mathbb{Q}$, define $|a|_p = p^{-v_p(a)}$ $(=0 \text{ for } a=0)$.

The p-adic distance between $a,b\in\mathbb{Q}$ is $d_p(a,b) = |a-b|_p$, so a,b are close when $\equiv \bmod p^{\text{high}}$. Ex: $|a_n-\frac{1}{1-p}|_p \to 0$ as $n\to\infty$ in the example above.

Lemma: ① $\forall\, a,b\in\mathbb{Q}$, $|ab|_p = |a|_p\cdot|b|_p$

② $|a+b|_p \leq \max(|a|_p,|b|_p)$ $\leq |a|_p+|b|_p$ with equality if $|a|_p\neq|b|_p$.

③ $|a|_p=0$ iff $a=0$. Thus d_p is a metric on $\mathbb{Q}$.

Pf: Immediate from last lemma.

Note: For the "usual" $|\cdot|:\mathbb{Q}\to\mathbb{R}_{\geq 0}$, ① & ③ hold, and the weaker version $|a+b|\leq|a|+|b|$ (triangle inequality) holds.

In the $\{a_n = 1+p+\dots+p^{n-1}\}$ example, the a_n got close to $\frac{1}{1-p}\in\mathbb{Q}$. But sometimes a sequence of $a_n\in\mathbb{Q}$ will appear to be approaching a limit not in $\mathbb{Q}$.

Example: $p=5$. Recursively solve $a_1=2$, $a_n^2\equiv -1 \bmod 5^n$

(possible b/c: given $a_n^2 = -1+5^n\cdot k$, solve $(a_n+5^nb)^2 = -1+5^{n+1}\ell$

$= -1+5^nk + 2(-1+5^nk)5^nb + 5^{2n}b^2$

to get $-1 \bmod 5^{n+1}$ solve

$k + \underbrace{2(-1+5^nk)}_{\in(\mathbb{Z}/5)^\times} b \equiv 0 \bmod 5$

for b, possible b/c $\in(\mathbb{Z}/5)^\times$.

Clearly for $n\geq m$ $|a_n-a_m|_5 \leq 5^{-m} \xrightarrow{m\to\infty} 0$

so $\{a_n\}$ is a Cauchy sequence

but can't be approaching $\alpha\in\mathbb{Q}$

b/c then $\alpha^2=-1$!

Dfn: $\mathbb{Q}_p :=$ the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

Recall: $\mathbb{R} :=$ " " " " " " " $|\cdot|$

means: $\mathbb{R} = \{\text{Cauchy sequences } (a_1,a_2,\dots)\in\mathbb{Q}^{\mathbb{N}} \text{ w.r.t. } |\cdot|\}/\sim$

where Cauchy means

$\forall\ \epsilon>0\quad \exists\ N_\epsilon:\ \forall\ m,n\geq N_\epsilon\ ,\ |a_n-a_m|<\epsilon$

and $(a_n)_{n\geq 1}\sim(b_n)_{n\geq 1}$ means

$\forall\ \epsilon>0\quad \exists\ N_\epsilon:\ \forall\ n\geq N_\epsilon,\quad |a_n-b_n|<\epsilon.$

$\mathbb{Q}_p$ is defined exactly the same way, with $|\cdot|_p$ in place of $|\cdot|$.

Lemma ① $\mathbb{Q}_p$ is a ring with operations

$$[(a_n)_{n\geq 1}]+[(b_n)_{n\geq 1}]=[(a_n+b_n)_{n\geq 1}]$$
$$[(a_n)_n]\cdot[(b_n)_n]=[(a_n\cdot b_n)_n]$$

(write $[\cdot]$ for the equivalence class of $\cdot$)

which are well-defined (ind. of choice of reps of $\sim$ class)

② $\mathbb{Q}_p$ is a field containing $\mathbb{Q}$ via $\mathbb{Q}\to\mathbb{Q}_p$, $a\mapsto(a,a,a,\dots)$

③ $|\cdot|_p$ extends to $|\cdot|_p:\mathbb{Q}_p\to\mathbb{R}_{\geq 0}$ $(\{0\}\cup p^{\mathbb{Z}})$ via $|a|_p=p^{-v_p(a)}$, where for $a\neq 0$, $v_p(a)$ is the eventually constant value of the sequence $(v_p(a_n))_{n\geq 1}$. Equivalently, $|a|_p=\lim_{n\to\infty}|a_n|_p$

④ $\mathbb{Q}_p$ is complete w.r.t. $|\cdot|_p$.

Pf: easy and essential exercise. Let $a=[(a_n)]\in\mathbb{Q}_p$. Set $|a|_p=\lim_{n\to\infty}|a_n|_p$. This exists: if $a=0$, then by defn $\lim|a_n|_p=0$

If $a\neq 0$, then $\neg(\forall\epsilon\ \exists N:\ \forall\ j\geq N\ |a_j|_p<\epsilon)$

$=\ \exists\ \epsilon>0:\ \forall\ N\geq 1\ \ \exists\ j_N>N:\ |a_{j_N}|_p\geq\epsilon$

For this ϵ, choose N: $|a_n-a_{n'}|_p<\epsilon$ $\forall\ n,n'>N$. In particular

$|a_n-a_{j_N}|_p<\epsilon$ and $|a_{j_N}|_p\geq\epsilon$, so $|a_n|_p=|a_{j_N}|_p$ by strong Δ ineq.

$\quad |a_n|_p = |a_n-a_{j_N}+a_{j_N}|_p$

That is, $|a_n|_p$ is constant ($\Leftrightarrow v_p(a_n)$ is constant) for all $n>N$

More concrete way to think about p-adics:

Let $\mathbb{Z}_p=\{a\in\mathbb{Q}_p\mid |a|_p\leq 1\}$, the p-adic integers.

For $a\in\mathbb{Z}_p$, we can uniquely choose a Cauchy sequence (a_n) representing

a to satisfy $a_n \in \mathbb{Z}$, $0 \le a_n < p^n$, $a_{n+1} \equiv a_n \bmod p^n$

or equivalently by division algorithm $\exists\ b_0, b_1, b_2, \ldots \in \{0, 1, \ldots, p-1\}$ such that $a = \underbrace{b_0 + b_1 p + b_2 p^2 + \cdots}_{\text{convergent infinite sum.}}$

Quick proof: $\mathbb{Z} \longrightarrow \mathbb{Z}_p$ induces $\mathbb{Z}/p^n \longrightarrow \mathbb{Z}_p/p^n\mathbb{Z}_p$, which we claim is an iso. Surj: let $x \overset{(x_n)}{\in} \mathbb{Z}_p \setminus 0$, so $\exists N: \forall n > N$ $|x_n|$ is constant and ≤ 1. That is, $x_n = \frac{a_n}{b_n}$ with $a_n, b_n \in \mathbb{Z}$ & $(b_n, p) = 1$.

Solve $b_n \cdot u_n \equiv a_n \bmod p^n$, $u_n \in \mathbb{Z}$. Then $|x_n - u_n| = \left|\frac{a_n}{b_n} - u_n\right| = \left|\frac{a_n - u_n b_n}{b_n}\right| \le p^{-n}$, and $|x - u_n| = |x - x_n + x_n - u_n| \le \max\underbrace{\{|x - x_n|, |x_n - u_n|\}}_{\text{both} \to 0 \text{ as } n \to \infty}$

so $x = \lim_{n\to\infty} u_n$ is now represented by the $\mathbb{Z}$ Cauchy seq. (u_n).

Now for any fixed n_0, $\exists\, u \in \mathbb{Z}$: $|x - u| \le p^{-n_0}$, so $x - u = p^n \cdot (y \in \mathbb{Z}_p)$.

This shows $\mathbb{Z} \longrightarrow \mathbb{Z}_p/p^n\mathbb{Z}_p$ surjective. The kernel is $\{a \in \mathbb{Z} \mid v_p(a) \ge n\} = p^n\mathbb{Z}$. $\blacksquare$

With these isos in hand, for $a \in \mathbb{Z}_p$ we reduce mod p^n, represent by $\bar{a}_n \in \mathbb{Z}/p^n$, and let $a_n \in \{0, \ldots, p^n - 1\}$ represent $\bar{a}_n$. Then $a = [(a_n)]$ as desired. $\blacksquare$ Cor: $\mathbb{Z}_p \xrightarrow{\sim} \varprojlim_n \mathbb{Z}/p^n$.

Then $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$, and more precisely $\forall\ a \in \mathbb{Q}_p$, $(a \ne 0)$ $a = p^{v_p(a)} \cdot u$ where $|u|_p = 1$ (ie $u \in \mathbb{Z}_p^\times$).

Lemma: (How to do p-adic analysis) Let $\alpha_1, \alpha_2, \ldots \in \mathbb{Q}_p$ be a sequence with $|\alpha_n|_p \to 0$ as $n \to \infty$. Then $\sum_{n=1}^{\infty} \alpha_n$ converges in $\mathbb{Q}_p$.

Pf: Set $S_N = \sum_{n=1}^{N} \alpha_n$. $\forall\ M > N$, $|S_M - S_N|_p = \left|\sum_{n=N+1}^{M} \alpha_n\right|_p \le \max\{|\alpha_{N+1}|_p, \ldots, |\alpha_M|_p\} \longrightarrow 0$ as $N \to \infty$. Thus $\{S_N\}_N$ is a Cauchy sequence and so converges ($\mathbb{Q}_p$ is complete).

Essential example:

$\boxed{\text{Thm}}$ Let $f(X) \in \mathbb{Z}_p[X]$ be a polynomial. Let $a_0 \in \mathbb{Z}_p$ be a simple root of $\bar{f}(x) \in \mathbb{F}_p[x]$, i.e.

$f(a_0) \equiv 0 \bmod p$ and $f'(a_0) \not\equiv 0 \bmod p$.

Then $\exists!\ a \in \mathbb{Z}_p$ s.t. $a \equiv a_0 \bmod p$ and $f(a) = 0$.

Pf: Exercise: inductively define $a_{n+1} = a_n - \dfrac{f(a_n)}{f'(a_n)}$ ∎

("Newton's method").

Example: $f(x) = x^{p-1} - 1 \in \mathbb{Z}_p[x]$. For any

$a_0 \in \mathbb{Z} \setminus p\mathbb{Z}$, $f(a_0) \equiv 0 \bmod p$, and

$f'(a_0) = (p-1)a_0^{p-2} \not\equiv 0 \bmod p$,

so $\exists!\ a \in \mathbb{Z}_p$, $a \equiv a_0 \bmod p$ s.t. $a^{p-1} = 1$.

(a depends only on $a_0 \bmod p$, so we could restrict to $a_0 \in \{1, 2, \ldots, p-1\}$).

Thus, $\underbrace{\{(p-1)^{st} \text{ roots of } 1\}}_{\mu_{p-1}} \subset \mathbb{Z}_p^\times$.

For $a \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$, write $\omega(a) \in \mathbb{Z}_p^\times$ for the unique elt of μ_{p-1} $\omega(a) \equiv a \bmod p$.

The Kummer congruences are a shadow of the fact that there is a continuous (better than – really know ζ's & ξ's) function $\mathbb{Z}_p \to \mathbb{Q}_p$ interpolating the values $\zeta(1-r)$, for r in a fixed non-zero congruence class mod $p-1$.

Our earlier mock version of this = used $n^{r-1} \equiv n^{r'-1}$ mod p^{N+1} when $p \nmid n$ and $r \equiv r' \bmod (p-1)p^N$

So let's think about "the function $s \mapsto n^s$" for $s \in \mathbb{Z}_p$

$s = [(s_i)]$ for a Cauchy sequence of $s_i \in \mathbb{Z}_{\geq 0}$, so would want "$n^s = \lim_{i\to\infty} n^{s_i}$" But this DNE in general:

we would need $|s-s'|$ small $\Rightarrow$ $|n^s - n^{s'}|$ small, and

(i) if $n = p$, $|s-s'| = p^{-N}$, $|n^s - n^{s'}| = |p^s||1-p^{s'-s}| = p^{-s}$
$N \geq 1$

Imagine s fixed, say $s=0$; then making s' closer & closer to s doesn't make $p^{s'}$ any closer to p^s.

(ii) If $n \not\equiv 0,1 \bmod p$ (0 we already saw was bad), the $|n^s - n^{s+p^N}| = \underset{\overset{\shortparallel}{1}}{|n^s|} \cdot |1 - \underset{\underset{n \bmod p}{\shortparallel}}{n^{p^N}}| = 1$

bad again.

We fix (i) by avoiding $p|n$. One way to fix (ii) is restrict to $n \equiv 1 \bmod p$, say $n = 1+pm$ $m \in \mathbb{Z}$. Then

$$|n^s - n^{s+ap^N}| = |n^s| \cdot |1 - n^{ap^N}| = |1 - (1+pm)^{ap^N}|$$

$$= \left| pm \cdot ap^N + (pm)^2 \cdot \binom{ap^N}{2} + (pm)^3 \cdot \binom{ap^N}{3} - \cdots (-1)^{p^N} \cdot (pm)^{ap^N} \right| \leq \frac{1}{p^{N+1}}$$

and so we can extend to a cts fn of $s \in \mathbb{Z}_p$.

More explicitly, we can define for $n = 1+pm$, $s \in \mathbb{Z}_p$, p odd.

$$n^s = (1+pm)^s := \sum_{j}^{\infty} \frac{s(s-1)\cdots(s-j+1)}{j!} (pm)^j$$

$j=0$

Get useful further control by noting

The terms $\to 0$ for $s \in \mathbb{Z}_p$, since $\binom{s}{j} \in \mathbb{Z}_p$: $v_p\left(\frac{p^j}{j!}\right) = j - v_p(j!)$

and $v_p(j!) = \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{j}{p^2} \right\rfloor + \dots \leq j\left(\frac{1}{p} + \frac{1}{p^2} + \dots\right) = \frac{j}{p \cdot (1-\frac{1}{p})}$

$= \frac{j}{p-1}$.

This will imply that for p odd, $s \longmapsto n^s$, $n \equiv 1 \bmod p$, is an analytic function of $s \in \mathbb{Z}_p$

(For $p=2$ take $n \equiv 1 \bmod 4$ to get convergent power series rewriting of this series.)

Note that these expressions are not power series in s — some rewriting needed for that

IGNORE

But $\mathbb{Z}_2 \xrightarrow{f} \mathbb{Q}_2$

$s \longmapsto (1+2m)^s$ still exists by first argument. Its Mahler expansion is $\sum a_n \binom{s}{n}$ where

$$a_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (1+2m)^j$$

$$= (1+2m-1)^n = (2m)^n, \text{ so still}$$

$$(1+2m)^s = \sum_{n=0}^{\infty} (2m)^n \binom{s}{n}$$

Mahler series $\sum_n a_n \binom{s}{n}$ - - - - $\sum_n b_n s^n$.

This is the basic trick: write down series expansion & control their coefficients.

For general $a \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$, we then have a function $s \longmapsto \left(\frac{a}{\omega(a)}\right)^s$ from $\mathbb{Z}_p \to \mathbb{Q}_p$

Here $\frac{a}{\omega(a)} \equiv 1 \bmod p$, so the expression makes sense $\forall\, s \in \mathbb{Z}_p$, where it is given by a power series in s. $\boxed{\text{Exercise!}}$

Standard notion: $\langle a \rangle = \frac{a}{\omega(a)}$, so $s \longmapsto \langle a \rangle^s$ is (for fixed $a \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$) an analytic function of $s \in \mathbb{Z}_p$ (= given by convergent power series).

Here's the big theorem

$\boxed{\text{Existence of the p-adic L-function (Kubota-Leopoldt, Iwasawa)}}$

Fix a primitive Dirichlet character $(\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$

($m=1$, $\chi = \mathbb{1}$ allowed)

We'll need to regard χ and ω as either valued in $\mathbb{C}$ or in an extension of $\mathbb{Q}_p$. Choose a field embedding

$$\begin{array}{c} \mathbb{Q}[\zeta_{\varphi(m)}, \zeta_{p-1}] \hookrightarrow (\text{finite degree field extension of } \mathbb{Q}_p) \\ \cap \\ \mathbb{C} \end{array}$$

This will allow us to make sense of an equality between a complex number & a p-adic number provided they lie in the common subfield $\mathbb{Q}[\zeta_{\varphi(m)}, \zeta_{p-1}]$.

Recall (psct) that for $K/\mathbb{Q}_p$ finite, $\exists!$ extension of $|\cdot|_p : \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ to an absolute value on K, which we also write $|\cdot|_p$.

$$\begin{array}{c} \mathbb{Q}_p \\ \cap \\ K \end{array}$$

$\boxed{\text{Theorem}}$

- For $\chi \neq \mathbb{1}$, there is a (unique) analytic function (= given by a convergent power series) $L_p(\chi, s)$,

$$L_p(\chi, \ ) : \mathbb{Z}_p \longrightarrow \mathbb{Q}_p[\zeta_{\varphi(m)}] \quad \text{such that}$$

(actually analytic on the larger disc $|s| < q \cdot p^{-1/p-1}$ in $\mathbb{C}_p$, $q = \begin{cases} p & p \neq 2 \\ 4 & p = 2 \end{cases}$)

$$\forall n \in \mathbb{Z}_{\geq 1}, \quad L_p(\chi, 1-n) = (1 - (\chi\omega^{-n})(p)\, p^{n-1}) \cdot \underbrace{L(\chi\omega^{-n}, 1-n)}_{-\frac{B_{n,\chi\omega^{-n}}}{n}}.$$

(Note: if $\chi(-1) = -1$, then $(-1)^n \neq (\chi\omega^{-n})(-1)$, so $L_p(\chi, 1-n) = 0$, hence $L_p(\chi, s) = 0$). ONLY EVEN CHARACTERS

- The same holds for $\chi = \mathbb{1}$ except

$$L_p(\mathbb{1}, s) = \frac{1}{s-1} \cdot (\text{analytic function})$$

(just like ζ!)

We need to parse this! The $L(\chi, 1-n) = -\frac{B_{n,\chi}}{n}$ held for χ primitive. Here $\chi\omega^{-n}$ must be interpreted as the (unique) primitive character underlying $\chi\omega^{-n}$. Eg if $\chi = \omega^n$, this is not the trivial char mod p (the modulus of ω^n for $n \not\equiv 0 \mod p-1$) but the $\mathbb{1}$ mod 1. Note that ~~for $n \not\equiv 0 \mod p-1$~~ not needed yet.

$L_p(\omega^n, 1-n) = (1-p^{n-1}) \cdot -\frac{B_n}{n}$

$\left| L_p(\omega^{n+1}, 0) = \underbrace{(1-\omega^n(p)p^{1-1})}_{=1} - B_{1,\omega^n} \right.$
$n \not\equiv -1 \bmod p-1$

$\boxed{\text{For } \forall\, m \equiv n \bmod (p-1)\cdot p^r}$, $n \not\equiv 0 \bmod (p-1)$ and $(r \in \mathbb{Z}_{\geq 0})$, $\omega^n = \omega^m$, so the following corollary implies Kummer ≡ :

apply with $\chi = \omega^n = \omega^m$.

Cor: Suppose $\chi \neq \mathbb{1}$ is a primitive character of conductor m. Assume $p^2 \nmid m$ (p odd) or $8 \nmid m$ ($p=2$). Then

$L_p(\chi, s) = a_0 + a_1(s-1) + a_2(s-1)^2 + \dots$ for $a_i \in \mathbb{Q}_p[\chi]$

satisfying $|a_0|_p \leq 1$ and $|a_i|_p \leq 1/p$ $\forall\, i \geq 1$ and $|a_i|_p \to 0$ as $i \to \infty$.

(or $p | a_i$ in $\{\alpha \in \mathbb{Q}_p[\chi] \mid |\alpha|_p \leq 1\} = \mathcal{O}_\chi$)

(this is stronger than the convergence b/c it controls $a_0, a_1 \dots$ "small" :) $B_{2,f}$)

(Note it also shows for $n \not\equiv 0 \bmod p-1$, $B_n \in \mathbb{Z}_{(p)}$.)

Cor ⇒ Kummer: Let $m \equiv n \bmod (p-1)p^r$ be positive even integers $\not\equiv 0 \bmod p-1$. Then $(1-p^{n-1})\left(-\frac{B_n}{n}\right) \equiv (1-p^{m-1})\left(-\frac{B_m}{m}\right) \bmod p^{r+1}$.

Pf: We saw $L_p(\omega^m, 1-m) = (1-p^{m-1})\left(-\frac{B_m}{m}\right)$, and likewise for n.

$L_p(\omega^m, s) = L_p(\omega^n, s)$, so

$$L_p(\omega^m, 1-m) = a_0 + a_1(-m) + a_2(-m)^2 + \dots$$

$$\equiv a_0 + a_1(-n) + a_2(-n)^2 + \dots \bmod (p^{r+1})$$ since $p | a_i$, $i \geq 1$ and $m \equiv n$ (p^r)

$$= L_p(\omega^n, 1-n). \quad \blacksquare$$

A related consequence:

Cor: For $n \geq 1$ odd, $n \not\equiv -1 \bmod (p-1)$, $B_{1,\omega^n} \equiv \frac{B_{n+1}}{n+1} \bmod p$ in $\mathcal{O}_\chi$ (both sides in $\mathcal{O}_\chi$).

$L(\omega^r, 0)$
$= -B_{1,\omega^n} \equiv \zeta(1-(r+1)) \over r+1$

Pf: $n+1$ even $\not\equiv 0 \bmod p-1$, so

$$(1-p^n)\cdot\frac{-B_{n+1}}{n+1} = L_p(\omega^{n+1}, -n) \underset{\text{modulo } p}{\equiv} L_p(\omega^{n+1}, \underset{1-1}{0}) = (1-\omega^n(p)p^{1-1})\cdot\frac{-B_{1,\omega^n}}{1}$$

$$\equiv \frac{-B_{n+1}}{n+1}$$

$$= -B_{1,\omega^n}$$

$\omega^n \neq$ trivial, so $\omega^n(p)=0$

clear from power series: any $s\in\mathbb{Z}_p$ give values $\equiv$ mod p, and the p-integrality also clear from power series

---

There are many ways to construct $L_p(\chi, s)$; we'll give an explicit elementary construction (not optimal for continuing in the subject, but good p-adic number practice).

Prop: (Denominators of Bernoulli numbers) For m even $\in\mathbb{Z}_{>0}$,

$$B_m + \sum_{\substack{p:\\ p-1\mid m}} \frac{1}{p} \in \mathbb{Z},$$

Pf: Way back we showed $S_m(n) = \sum_{k=0}^{n-1} k^m$ satisfies

$$(m+1)S_m(n) = \sum_{j=0}^{m}\binom{m+1}{j}B_j\, n^{m-j+1} \qquad \binom{m+1}{j} = \frac{(m+1)\cdot m!}{(m+1-j)\cdot(m-j)!\,j!}$$

so
$$S_m(n) = \sum_{j=0}^{m}\binom{m}{j}\frac{B_j}{m+1-j}\cdot n^{m+1-j}$$

$$= \sum_{k=0}^{m}\binom{m}{k}B_{m-k}\frac{n^{k+1}}{k+1}$$

We'll check $\boxed{\forall\ m\in\mathbb{Z}_{\geq 1},\ pB_m\in\mathbb{Z}_{(p)} \text{ and for } m \text{ even } pB_m \equiv S_m(p) \bmod p}$

This will suffice because

$$S_m(p) = 1^m + 2^m + \dots + (p-1)^m \equiv 1^m + g^m + g^{2m} + \dots + g^{(p-2)m} \bmod p$$

for a generator g of $(\mathbb{Z}/p)^\times$ $= \dfrac{g^{(p-1)m}-1}{g^m-1}$

If $p-1 \nmid m$, we see $S_m(p) \equiv 0 \bmod p$, and if $(p-1)|m$, then $S_m(p) \equiv 1+1+\dots+1 \equiv -1 \bmod p$.

Thus (m even) $pB_m \equiv S_m(p) \bmod p$ tells us

(i) $B_m \in \mathbb{Z}_{(p)}$ for $p-1 \nmid m$

(ii) for $p-1|m$, $pB_m \equiv -1 \bmod p$, $pB_m \in \mathbb{Z}_{(p)}$

Thus $\forall \ell$, $B_m + \sum_{p-1|m} \frac{1}{p} \in \mathbb{Z}_{(\ell)}$, done by $\bigcap_{\text{primes } \ell} \mathbb{Z}_{(\ell)} = \mathbb{Z}$.

For the claim we induct on m. Clear for $m=1$. Suppose $m>1$.

$$\mathbb{Z} \ni S_m(p) = \sum_{k=0}^{m} \binom{m}{k} B_{m-k} \frac{p^{k+1}}{k+1}$$ $k=0$ term is $B_m \cdot p$

The other terms are in $\mathbb{Z}_{(p)}$: by induction, for $k=1,\dots,m$, $p \cdot B_{m-k} \in \mathbb{Z}_{(p)}$, and clearly $\frac{p^k}{k+1} \in \mathbb{Z}_{(p)}$ $\forall k \geq 1$

Thus $pB_m \in \mathbb{Z}_{(p)}$ $\forall m \in \mathbb{Z}_{\geq 1}$.

Similarly, for $k \geq 2$, $\frac{p^k}{k+1} \in p\mathbb{Z}_{(p)}$, so

$$pB_m \equiv -m \cdot B_{m-1} \frac{p^2}{2} + S_p(m) \bmod p$$

$$= \begin{cases} S_p(m) & \text{for } m \geq 4 \text{ even} \\ -2\left(-\frac{1}{2}\right)\frac{p^2}{2} + S_p(m) & \text{for } m=2 \end{cases}$$

$\equiv S_p(m) \bmod p$. $\blacksquare$

$\boxed{\text{Construction of } L_p(\chi,s)}$ χ - primitive conductor m.

For simplicity take p odd. Let F be any multiple of $m\cdot p$.
($F = mp$ is fine).

Define

$$L_p(\chi,s) = \frac{1}{F}\cdot\frac{1}{s-1}\sum_{\substack{a=1\\ p\nmid a}}^{F}\chi(a)\langle a\rangle^{1-s}\sum_{j=0}^{\infty}\binom{1-s}{j}B_j\left(\frac{F}{a}\right)^j \quad !$$

#, not poly.

Values: $(n\geq 1)$ $L_p(\chi,1-n) = \frac{1}{F}\cdot\frac{1}{-n}\sum_a \chi(a)\frac{a^n}{\omega(a)^n}\sum_j\binom{n}{j}\cdot B_j\cdot\left(\frac{F}{a}\right)^j$

Exercise: $B_{n,\chi} = F^{n-1}\sum_{a=1}^{F}\chi(a)\underbrace{B_n\left(\frac{a}{F}\right)}_{\sum\binom{n}{j}B_j\left(\frac{a}{F}\right)^{n-j}}$ (we showed for $F=m$) Wash. 4.1

$$L_p(\chi,1-n) = \sum_{\substack{a=1\\ p\nmid a}}^{F}\chi(a)\omega^{-n}(a)\underbrace{\sum_{j=0}^{n}\binom{n}{j}B_j\cdot\left(\frac{a}{F}\right)^{n-j}}_{B_n(a/F)}\cdot F\cdot -\frac{1}{n}$$

$$= \sum_{\substack{a=1\\ p\nmid a}}^{F}\chi(a)\omega^{-n}(a)B_n\left(\frac{a}{F}\right)\cdot\frac{F^{n-1}}{-n} \overset{@}{=} \sum_{\substack{p\nmid a\\ a=1}}^{F}(\chi\omega^{-n})(a)B_n\left(\frac{a}{F}\right)\frac{F^{n-1}}{-n}$$

$$= \sum_{a=1}^{F}(\chi\omega^{-n})(a)B_n\left(\frac{a}{F}\right)\frac{F^{n-1}}{-n} - \sum_{b=1}^{F/p}(\chi\omega^{-n})(pb)B_n\left(\frac{b}{F/p}\right)\frac{F^{n-1}}{-n}$$

@ Wash Ex 3.7c unless both $\chi_1(a)=0=\chi_2(a)$, $\chi_1(a)\chi_2(a)=(\chi_1\chi_2)(a)$

Case 1: $p\mid \mathrm{cond}(\chi\omega^{-n})$. Then all $=0$, and we get

$$-\frac{B_{n,\chi\omega^{-n}}}{n} = L(\chi\omega^{-n},1-n) = \Big(1-\underbrace{(\chi\omega^{-n})(p)}_{0}p^{n-1}\Big)L(\chi\omega^{-n},1-n) \quad ✓$$

Case 2: $p\nmid\mathrm{cond}(\chi\omega^{-n})$. Then $\mathrm{cond}(\chi\omega^{-n})\mid F/p$, and we get

$$-\frac{B_{n,\chi\omega^{-n}}}{n} - p^{n-1}\cdot\left(\frac{F}{p}\right)^{n-1}(\chi\omega^{-n})(p)\sum_{b=1}^{F/p}(\chi\omega^{-n})(b)B_n\left(\frac{b}{F/p}\right)\cdot -\frac{1}{n}$$

$$= \frac{-B_{n,\chi\omega^{-n}}}{n} - p^{n-1}(\chi\omega^{-n})(p)\cdot\left(\frac{-B_{n,\chi\omega^{-n}}}{n}\right)$$

$$= \left[1-(\chi\omega^{-n})(p)p^{n-1}\right]\cdot\frac{-B_{n,\chi\omega^{-n}}}{n} \qquad \checkmark$$

Convergence:

$$L_p(\chi,s) = \frac{1}{F}\cdot\frac{1}{s-1}\sum_{\substack{a=1\\ p\nmid a}}^{F}\chi(a)\langle a\rangle^{1-s}\sum_{j=0}^{\infty}\binom{1-s}{j}B_j\left(\frac{F}{a}\right)^j$$

STP $\langle a\rangle^{1-s}$ analytic, $\sum_{j=0}^{\infty}\binom{1-s}{j}B_j\cdot\left(\frac{F}{a}\right)^j$ analytic $s\in\mathbb{Z}_p$

$\boxed{\text{Take } p\neq 2 \text{ for simplicity}}$

$$\langle a\rangle^s = (1+(\langle a\rangle-1))^s = \sum_{j=0}^{\infty}\binom{s}{j}[\langle a\rangle-1]^j$$

converges for $s\in\mathbb{Z}_p$, since then $\left|\binom{s}{j}[\langle a\rangle-1]^j\right| \leq p^{-j}\to 0$ as $j\to\infty$.

Similarly, $\left|\binom{s}{j}B_j\left(\frac{F}{a}\right)^j\right| \leq |B_j|\,|F|^{-j}$ and vS-C shows along with $|F|\leq 1/p$ $(p\mid F)$ that $\leq p\cdot p^{-j}\to 0$.

Analytic? write $\langle a\rangle^s = \sum_{j=0}^{\infty}\binom{s}{j}\alpha^j$ $\qquad \alpha=\langle a\rangle-1\in p\mathbb{Z}_p$, $|\alpha|\leq 1/p$

$\circledast = \sum_{n=0}^{\infty}a_n s^n$ where $a_n = \frac{\alpha^n}{n!} + \frac{\alpha^{n+1}\cdot(\ldots)}{(n+1)!} + \ldots$

and $\left|\frac{\alpha^{n+k}}{(n+k)!}\right| \leq p^{-n-k}\cdot p^{\frac{n+k}{p-1}} \to 0$ $(p\neq 2)$, so the sum defining a_n exists. Moreover,

$$|a_n| \leq \max_{k\geq 0}\left|\frac{\alpha^{n+k}}{(n+k)!}\right| = \max p^{-(n+k)(1-\frac{1}{p-1})} = p^{-n(1-\frac{1}{p-1})}$$

so $\sum_{n=0}^{\infty}a_n s^n$ is a convergent power series for

$$\left\{s\in\mathbb{Q}_p \;\middle|\; |s|^n\cdot p^{-n(1-\frac{1}{p-1})}\to 0\right\}$$

$= \{ s \mid |s| \le p^{1-\frac{1}{p-1}-\epsilon}, \text{any } \epsilon > 0 \}$, in particular for $s \in \mathbb{Z}_p$.

The same argument applies to show $\sum_{j=0}^{\infty} \binom{1-s}{j} B_j \left(\frac{F}{a}\right)^j$ analytic on same domain (using $\sqrt{S-c}$ as before).

Some details for ⊗. (see Washington Prop 5.8)

The Corollary that we used to justify Kummer congruences requires a more careful tracking of coefficients: ($p \neq 2$ for simplicity)

$\chi \neq 1$. $p^2 \nmid \text{cond}(\chi)$. Then we can take F: $p \mid F$, $p^2 \nmid F$.

(and $\text{cond}(\chi) \mid F$).

$$L_p(\chi, s) = \frac{1}{F} \cdot \frac{1}{s-1} \sum_{\substack{a=1 \\ p \nmid a}}^{F} \chi(a) \langle a \rangle^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} B_j \cdot \left(\frac{F}{a}\right)^j$$

wma $\chi(-1) = 1$ (else $L_p(\chi, s) = 0$).

$$\left| \frac{1}{F} \binom{1-s}{j} B_j \cdot \left(\frac{F}{a}\right)^j \right| = \left| (1-s)(1-s-1)\cdots(1-s-(j-1)) \cdot \frac{B_j F^{j-1}}{a^j j!} \right| \le p \cdot p^{j/p-1} \cdot p^{-j+1}$$

$$= p^{-j(1-\frac{1}{p-1})} \cdot p^2 \underset{p \ge 3}{\le} p^{-j/2+2} \underset{(j \ge 6)}{\le} \frac{1}{p}$$

so the $(s-1)^6$ and higher coeffs are div. by p.

$(s-1)^{3,4,5}$ check this case by case

For instance, $(s-1)^5$ coefficient is (div. by p)

$$+ \cancel{\frac{B_5 \cdot F^4}{a^5 \cdot 5!}} \; 0$$

$(s-1)^4$ coeff. is (div. by p) + $\underbrace{\frac{B_4 F^3}{a^4 \cdot 4!}}$ =

$|\cdot| = \left| \frac{1}{30} \cdot \frac{F^3}{4!} \right| = \frac{p^{-3}}{|30 \cdot 4!|} \le 1/p$

in all cases $p \neq 2$.

$(s-1)^2$, $(\ )^1$, $(\ )^0$ coeff

($B_2 = 1/6$)

$B_1 = -1/2$

$j = 2$ always div. by p: $\frac{B_2 F}{a^2 \cdot 2!}$

unless $p=3$, then $\in \mathbb{Z}_3$

$j = 1$ $\in \mathbb{Z}_p$

$j=0$ have p in the denominator

So to complete the proof, since all coefficients of the power series $\langle a\rangle^{1-s}$ in $s-1$ are p-integral, and the $(s-1)^2$ and higher coefficients are div. by p^2 when we write $L_p(\chi,s)=\sum_{n=0}^{\infty} a_n(s-1)^n$, $p \mid a_{n\geq 3}$ is immediate, and it suffices to scrutinize the low-degree terms:

$$\frac{1}{F}\cdot\frac{1}{s-1}\sum_{\substack{a=1\\p\nmid a}}^{F}\chi(a)\underbrace{\left[\langle a\rangle^{1-s}\right]}_{\substack{\text{cut off}\\ \text{at } (\ )^0+(s-1)^1}}\cdot\sum_{j=0}^{2}\binom{1-s}{j}B_j\left(\frac{F}{a}\right)^j$$

$$=\frac{1}{s-1}\cdot\sum_{\substack{a=1\\p\nmid a}}^{F}\chi(a)\Big(1+(1-s)\left[\langle a\rangle-1\right]\Big)\cdot\left(\frac{1}{F}-\frac{1-s}{2a}+\frac{(1-s)(1-s-1)}{12a^2}\cdot F\right)$$

For $\chi\neq\mathbb{1}$, $\frac{1}{F}\sum_{\substack{a=1\\p\nmid a}}^{F}\chi(a)=0$ (certainly $\sum_{a=1}^{F}\chi(a)=0$, and if $p\nmid\mathrm{cond}(\chi)$, $\sum_{b=1}^{F/p}\chi(pb)=\chi(p)\cdot\underbrace{\sum_{b=1}^{F/p}\chi(b)}_{=0}$ where $\mathrm{cond}\,\chi \mid F/p$

and if $p\mid\mathrm{cond}(\chi)$, then all terms $\chi(pb)=0$.)

So there is no $\frac{1}{s-1}$ term for $\chi\neq\mathbb{1}$.

$$a_0=\sum_{\substack{a=1\\p\nmid a}}^{F}\chi(a)\left(\frac{\langle a\rangle-1}{F}-\frac{1}{2a}-\frac{F}{12a^2}\right)+(\text{div. by } p)$$

$$\in\mathbb{Z}_p\quad(p^2\nmid F)$$

$$a_1 = \sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a)\left[-\frac{\langle a\rangle -1}{2a} - \frac{(\langle a\rangle -1)\cdot F}{12a^2} + \frac{F}{12a^2}\right] + \text{div. by } p$$

$\in p\mathbb{Z}_p$ clearly for $p>3$, and for $p=3$ $\quad (a^2 \equiv 1 \bmod p)$

$$\sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a)a^{-2} \equiv \sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a) \equiv 0 \bmod p$$

$$a_2 = \sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a)\cdot -\frac{(\langle a\rangle -1)\cdot F}{12a^2} + \text{div. by } p$$

$\in p\mathbb{Z}_p$ clearly. $\blacksquare$

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p-adic L-function is even better

Fix $u \in 1+\underbrace{2p}_{q}\mathbb{Z}_p \simeq \mathbb{Z}_p$ a topological generator (ignore 2 for p odd)

so $\{u^\alpha\}_{\alpha\in\mathbb{Z}_p} = 1+p\mathbb{Z}_p$ (let's ignore $p=2$)

Thm: 1 $\exists\ G_\chi(T) \in \mathrm{Frac}(\mathcal{O}_\chi[[T]])$:

$$L_p(\chi,s) = G_\chi(u^s-1) \quad \forall\ s\in\mathbb{Z}_p$$

2 $\mathrm{cond}(\chi) \neq 1, p^n\ (n\geq 2) \Rightarrow G_\chi(T) \in \mathcal{O}_\chi[[T]]$

In this case 2, $L_p(\chi,s) = \sum_{n=0}^{\infty} a_n\cdot(u^s-1)^n$

$a_n\in\mathcal{O}_\chi$

$u = 1+pv \quad v\in\mathbb{Z}_p$

$u^s-1 = \sum_{n=1}^{\infty}\binom{s}{n}(pv)^n$, and so the integrality properties needed for Kummer congruences are clear.

③ $(\mu=0)$ χ as in ② $\quad G_\chi(T)$ is not div. by $\varpi \in \mathcal{O}_\chi$ (unif)

K — K_n, $\mathbb{Q}$ — $\mathbb{Q}_n = \mathbb{Q}(\mu_{p^n})^\Delta$, $\mathbb{Z}/p^n$

$L_n/\mathbb{Q}_n$ max'l abelian unr. p-extension or K_n for Cl_n

$$\begin{array}{ccc} A_{\mathbb{Q}_n} := Cl(\mathbb{Q}_n)_p & \xrightarrow{\sim} & Gal(L_n/\mathbb{Q}_n) \\ \downarrow \text{Nom} & & \downarrow \text{res} \\ Cl(\mathbb{Q}_m)_p & \xrightarrow{\sim} & Gal(L_m/\mathbb{Q}_n \cap L_m = \mathbb{Q}_m) \end{array}$$

for $n \geq m$

$X := \varprojlim_{\text{norm}} A_{\mathbb{Q}_n}$, $L_\infty = \bigcup_{n \geq 1} L_n$, so $Gal(L_\infty/\mathbb{Q}_\infty) \equiv \varprojlim Gal(L_n/\mathbb{Q}_n)$

hence $X \xrightarrow{\sim} Gal(L_\infty/\mathbb{Q}_\infty)$, and this is an iso
$\circlearrowleft$ $Gal(\mathbb{Q}_\infty/\mathbb{Q})$ of $\mathbb{Z}_p[\![Gal(\mathbb{Q}_\infty/\mathbb{Q})]\!]$-mods $\quad = \Gamma$

Thm: $X = X_{\mathbb{Q}_\infty}$ is a f.-g. torsion $\mathbb{Z}_p[\![\Gamma]\!]$-mod; thus it has a characteristic ideal

Main Conjecture: for $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ a prim Dir. char "of 1st kind" ($v_p(N) \leq 1$ and via $Gal(\mathbb{Q}(\mu_{Np^\infty})/\mathbb{Q}) \simeq (\mathbb{Z}/pN_0)^\times \times \mathbb{Z}_p$ $(1+p\mathbb{Z}_p)$, χ is a character of 1st factor) $\quad N_0 = N/p^{v_p(N)}$

$K_n := \mathbb{Q}(\mu_{N_0 p^n})$
$K = K_1 = \mathbb{Q}(\mu_{N_0 p})$

$$\left(\Delta = Gal(K/\mathbb{Q})\right) \times \left(\Gamma = Gal(K_\infty/K)\right) = Gal(K_\infty/\mathbb{Q})$$
$$\downarrow \chi$$
$$\mathbb{C}^\times$$

$\Lambda_{N_0} := \mathbb{Z}_p[\![\Delta \times \Gamma]\!]$
$\quad \downarrow \varphi_\chi$ (induced by χ on Δ)
$\Lambda_\chi = \mathcal{O}_\chi[\![\Gamma]\!]$

$$X_\chi := X_{K_\infty} \otimes_{\Lambda_{N_0}, \varphi_\chi} \Lambda_\chi$$

Thm: $\operatorname{char}(X_\chi) = (G_{\chi^{-1}\omega}(T))$ as ideals of $\Lambda_\chi \simeq \mathcal{O}_\chi[\![T]\!]$
(for $\chi \neq \omega$. X_ω in fact is 0)

Herbrand-Ribet: $p \mid Cl(\mathbb{Q}(\mu_p))[p^\infty]_{\omega^{1-i}} \iff p \mid \zeta(1-i) \iff p \mid B_i$

$\boxed{i \text{ even} > 0}$

$\iff p \mid B_{1,\omega^{i-1}}$

$(i+1 \not\equiv 0 \bmod p-1)$

so $i = 2, 4, \ldots, p-3$

691 $\quad \zeta(1-12)$

B_{12}

Explicitly, j odd, $1 < j < p-1$

$$\# A_{\mathbb{Q}(\mu_p)}^{\omega^j} \text{ (or } \omega^{1-r}) = \# \mathbb{Z}_p / L(\omega^{-j}, 0) = \# \mathbb{Z}_p / L_p(\omega^{1-j}, 1-1)$$

$= L_p(\omega^r, 0)$

$= \zeta(1-r)$

$r > 0$,

$\boxed{1 - r \equiv j \ (p-1)}$

$$L_p(\omega^{-j+1}, 1-1) = (\quad) \cdot L(\omega^{-j+1} \cdot \omega^{-1}, 1-1)$$