Problems for Ross 2022 Bernoulli Numbers

1. Work out the volume calculation done by Ibn al-Haytham: for a parabola rotated about an "ordinate" (say $x - b = -ay^2$, $x \in [0, b]$, rotated about the $x = 0$ line), the volume of the resulting solid is $\frac{8}{15}$ times the volume of the inscribing cylinder.

2. Compute B_{2m} for $m = 1, \ldots, 8$, and factor into primes the numerator and denominator. Do you have any conjectures about what primes can appear in the denominator of B_{2m}? (The numerators will probably look very mysterious!)

3. Prove that for $\mathrm{Re}(s) > 1$,
$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$
where the product is taken over all primes p.

4. Suppose $f : \mathbb{R} \to \mathbb{C}$ is Riemann-integrable on any closed interval, and that f is periodic with period L. Show that the Fourier coefficients of f do not depend on which interval of length L is used to compute them. Compute the Fourier coefficients of the 2π-periodic function $f(x) = e^{imx}$, $m \in \mathbb{Z}$.

5. Check that when $f(x)$ is twice continuously differentiable on the circle $\mathbb{R}/2\pi\mathbb{Z}$, it satisfies the hypothesis $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$in our pointwise-convergence result for Fourier series. Along the way, find a formula for $(\hat{f'})(n)$.

6. Complete the calculations sketched in class of the Fourier coefficients $\hat{f}(n)$ for:
   (a) $f(x) = |x|$, $x \in [-\pi, \pi]$.
   (b) $f(x) = \cos(\alpha x)$, $x \in [-\pi, \pi]$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. What happens when $\alpha \in \mathbb{Z}$?

7. Define the Bernoulli polynomials $B_n(x)$ $n \in \mathbb{Z}_{\geq 0}$ by the generating function
$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$
Check the following:
   (a) $B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$.
   (b) For $k \geq 2$, $B_k(0) = B_k(1)$.
   (c) For $k \geq 1$, $B_k'(x) = kB_{k-1}(x)$, and $\int_0^1 B_k(x)dx = 0$.

   Regard $B_k(x)$ as a function on $[0, 1]$, and compute its Fourier coefficients. Using the theorem on convergence of Fourier series stated in class, derive from this another calculation of $\zeta(2m)$, $m \in \mathbb{Z}_{\geq 1}$.

8. (a) Let G be a topological group: this means that G is a group and a topological space in such a way that the group operation $G \times G \to G$ and inversion $G \to G$ are both continuous maps. Examples (given as pairs $(G, \cdot)$, where G is the space and $\cdot$ is the group operation): $(\mathbb{R}, +)$, $(\mathbb{R}^\times, \times)$, $(\mathbb{C}, +)$, $(\mathbb{C}^\times, \times)$, with the usual topologies on the real line or complex plane, or any group $(G, \cdot)$, where G is a discrete topological space. Write down some other examples.
   (b) Our study of Fourier analysis for periodic functions on $\mathbb{R}$ can be thought of as the study of functions on the topological group $(\mathbb{R}/2\pi\mathbb{Z}, +)$, which via the complex exponential map $x \mapsto e^{ix}$ is isomorphic as topological group (what does that mean?) to the unit circle $(S^1, \cdot)$ with the operation multiplication ($S^1 = \{z \in \mathbb{C} : |z| = 1\}$).

(c) For $f\colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$, why is $\mathbb{Z}$ the right parameter space for Fourier coefficients, and for $n \in \mathbb{Z}$ what is the special role of the functions $x \mapsto e^{inx}$? One answer to this question is that they are the (unitary) *characters* of $\mathbb{R}/2\pi\mathbb{Z}$, i.e. they are precisely the continuous group homomorphisms $\chi\colon \mathbb{R}/2\pi\mathbb{Z} \to S^1$. The word "unitary" here refers to the fact that the homomorphisms land in S^1; more generally for a topological group G, a character is a continuous homomorphism $G \to \mathbb{C}^\times$; we will write $\widehat{G}$ for the set of continuous homomorphisms $\widehat{G} \to S^1$.

i. How can you make $\widehat{G}$ into a group? (If you are familiar with the compact-open topology on function spaces, you can elaborate on this exercise to make $\widehat{G}$ a topological group.)

ii. Check that these $x \mapsto e^{inx}$ are indeed all the characters of $\mathbb{R}/2\pi\mathbb{Z}$. Thus identifying $\mathbb{Z} \xrightarrow{\sim} \widehat{\mathbb{R}/2\pi\mathbb{Z}} = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{R}/2\pi\mathbb{Z}, S^1)$ (as what? sets? groups? topological groups?), we can think of the collection of Fourier coefficients $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ as the function $\hat{f}\colon \widehat{\mathbb{R}/2\pi\mathbb{Z}} \to \mathbb{C}$ given by

$$\hat{f}(\chi) = \frac{1}{2\pi}\int_{\mathbb{R}/2\pi\mathbb{Z}} f(x)\overline{\chi(x)}dx.$$

(The bar indicates complex conjugation. It's not essential to include it, but if we don't we would have to redo our past calculations to get normalizations right.)

iii. "Dually," determine the character group $\widehat{\mathbb{Z}}$.

(d) For another example, replace $\mathbb{R}/2\pi\mathbb{Z}$ with $\mathbb{R}$. What are the characters (valued in $\mathbb{C}^\times$) and the unitary characters (valued in S^1) of $\mathbb{R}$? (Again, depending on your background, you may determine $\widehat{\mathbb{R}}$ either as group or as topological group.) These will appear when we discuss the Fourier transform on $\mathbb{R}$.

9. We continue with the abstraction of the previous exercise in the case $G = \mathbb{Z}/m\mathbb{Z}$, the cyclic group of order $m \in \mathbb{Z} > 1$. Identify the set (group!) $\widehat{G}$ of characters (= unitary characters—why?) of $\mathbb{Z}/m\mathbb{Z}$ with m^{th} roots of unity $\mu_m = \{z \in \mathbb{C} : z^m = 1\}$. For a function $f\colon G \to \mathbb{C}$, we define its Fourier transform to be the function on $\widehat{G}$, $\hat{f}\colon \widehat{G} \to \mathbb{C}$, given by

$$\hat{f}(\zeta) = \frac{1}{m}\sum_{a\in\mathbb{Z}/m\mathbb{Z}} f(a)\overline{\zeta(a)}$$

We have written this thinking of $\zeta \in \widehat{G}$ as an abstract character so that the formal analogy with problem 8.c.ii is clear; but using the identification $\widehat{G} \xrightarrow{\sim} \mu_m$ that sends ζ to $\zeta(1)$, and then writing $\zeta(1) = e^{2\pi i k/m}$ for some $k \in \mathbb{Z}/m\mathbb{Z}$, we can regard $\hat{f}$ as the function $\hat{f}\colon \mathbb{Z}/m \to \mathbb{C}$ given by

$$\hat{f}(k) = \frac{1}{m}\sum_{a\in\mathbb{Z}/m\mathbb{Z}} f(a)e^{-2\pi i ak/m}$$

(a) Show that f can be expressed in terms of its "Fourier series":

$$f(a) = \sum_{k\in\mathbb{Z}/m\mathbb{Z}} \hat{f}(k)e^{2\pi i ak/m}$$

(b) Now let $\chi\colon (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character, extended to a function $\chi\colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$ by setting $\chi(a) = 0$ if $(a, m) > 1$. Relate the Fourier transform $\hat{\chi}(a)$ to the *Gauss sum*

$$G(\chi, a) = \sum_{k\in\mathbb{Z}/m\mathbb{Z}} \chi(k)e^{2\pi i ak/m}.$$

(c) We say that χ is a primitive character modulo m if there is no proper divisor d of m such that χ agrees on $(\mathbb{Z}/m\mathbb{Z})^\times$ with the composition $(\mathbb{Z}/m\mathbb{Z})^\times \to (\mathbb{Z}/d\mathbb{Z})^\times \xrightarrow{\bar{\chi}} \mathbb{C}^\times$ for a Dirichlet character modulo d $\bar{\chi}$. Show that if χ is primitive, then $G(\chi, a) = \bar{\chi}(a)G(\chi, 1)$ for every $a \in \mathbb{Z}/m\mathbb{Z}$ (note: without assuming primitivity, this will be true when $(a, m) = 1$; show $G(\chi, a)$ vanishes when $(a, m) > 1$).

(d) Combine parts (a)-(c) to show that for a primitive character χ, $|G(\chi,1)|^2 = m$. (This can be shown directly as well.) We will use this calculation in our later discussion of Dirichlet L-functions.

10. A little practice with complex functions.

(a) Compute the path integrals

$$\int_\Gamma z^n dz$$

where Γ is the path traversing the unit circle once counter-clockwise given by $\gamma(t) = e^{2\pi i t}$ for $t \in [0,1]$ and $n \in \mathbb{Z}$. What n is exceptional? Can you account for this in light of Goursat's theorem?

(b) Show that $f(z) = \bar{z}$ (complex conjugate) is not holomorphic (anywhere?!).

(c) Generalize the previous part to establish the *Cauchy-Riemann equations*: identify $\mathbb{C} = \mathbb{R}^2$ via $x + iy \mapsto (x, y)$ and regard a function $f\colon \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ an open subset, as a function $f(x,y) = u(x,y) + iv(x,y)$ for $(x,y) \in \Omega \subset \mathbb{C} = \mathbb{R}^2$, where u and v are now real-valued functions of two variables. Show that if f is holomorphic at a point $z_0 = (x_0, y_0) \in \Omega$, then the partial derivatives satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

at the point (x_0, y_0). (The following converse is also true: if u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on an open set Ω, then $u + iv$ is holomorphic on Ω.) Show that $f(z) = e^z$, or explicitly $f(x+iy) = e^x(\cos(y) + i\sin(y))$ (and thus $\cos(z)$ and $\sin(z)$), and also any polynomial in z satisfy the Cauchy-Riemann equations at every point of $\mathbb{C}$ (with continuous partial derivatives, so the converse mentioned would show they are holomorphic on all of $\mathbb{C}$).

(d) Let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function on an open subset $\Omega \subset \mathbb{C}$. Further assume that $f'\colon \Omega \to \mathbb{C}$ is continuous. Use Green's theorem and the Cauchy-Riemann equations to show that $\int_\Gamma f(z)dz = 0$ (as in Goursat's theorem) for any "nice" closed path $\Gamma \subset \Omega$ such that the "inside" of Γ is entirely contained in Ω (use whatever notions of "nice" and "inside" you are comfortable with; for instance, you could take Γ to be a polygon whose interior is also contained in Ω

(e) Fix $z_0 \in \mathbb{C}$. Generalize arguments you've seen for real power series to show that if $f(z) = \sum_{n=0}^\infty a_n(z - z_0)^n$ is a power series with $a_n \in \mathbb{C}$ that converges absolutely on a disc $\{|z - z_0| < R\}$, then f defines a holomorphic function on this disc. (This provides another proof that e^z, defined by its power series, is a holomorphic function on all of $\mathbb{C}$.)

11. In class we reduced the extension to a holomorphic function on $\mathbb{C}$ and functional equation of Dirichlet L-series to an identity for generalized theta series. This exercise asks you to prove this identity. Let $\chi\colon (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a non-trivial primitive Dirichlet character modulo $m > 1$, and as in class define for $y > 0$

$$\theta(\chi, y) = \sum_{n\in\mathbb{Z}} \chi(n) n^p e^{-\pi n^2 y/m},$$

where $p \in \{0,1\}$ is characterized by $\chi(-1) = (-1)^p$.

(a) Check that $\theta(\chi, y) = \sum_{a=0}^{m-1} \chi(a) \sum_{n\in m\mathbb{Z}} (a+n)^p e^{-\pi(a+n)^2 y/m}$.

(b) We consider a generalization of the inner sum in part (a): for $a, b \in \mathbb{R}$, $\mu \in \mathbb{R}_{>0}$, let

$$\theta_\mu(a, b, y) = \sum_{n\in\mu\mathbb{Z}} e^{-\pi(a+n)^2 y + 2\pi i b n}.$$

Show that this series converges absolutely for $y > 0$ and uniformly on compact subsets of $\mathbb{R}_{>0}$. Use Poisson summation to show that

$$\theta_\mu(a, b, 1/y) = e^{-2\pi iab} \frac{\sqrt{y}}{\mu} \theta_{1/\mu}(-b, a, y).$$

(Hint: for $f(\mu, a, b, x) = e^{-\pi(a+\mu x)^2 + 2\pi ib\mu x}$, compute the Fourier transform in the x variable to be (omitting the μ, a, b parameters from the notation for f)

$$\hat{f}(\xi) = e^{2\pi i\frac{a}{\mu}\xi - 2\pi iab} \frac{1}{\mu} e^{-\pi(\frac{\xi}{\mu}-b)^2}.$$

Check that $\theta_\mu(a, b, 1/y) = \sum_{n\in\mathbb{Z}} f(\mu/\sqrt{y}, a/\sqrt{y}, b\sqrt{y}, n)$, and apply Poisson summation to finish the proof.)

(c) Show that as a function of a, $\theta_\mu(a, b, y)$ is differentiable, and its derivative can be computed by differentiating each term of the infinite sum. Conclude that for $p \in \{0, 1\}$,

$$\theta_\mu^{(p)}(a, b, 1/y) = \frac{1}{i^p e^{2\pi iab}\mu} y^{p+\frac{1}{2}} \theta_{1/\mu}^{(p)}(-b, a, y),$$

where the superscript in $\theta^{(p)}$ indicates differentiation p times with respect to the variable a.

(d) Combine parts (a) and (c) to deduce the desired formula, that

$$\theta(\chi, 1/y) = \frac{G(\chi)}{i^p\sqrt{m}} y^{p+\frac{1}{2}} \theta(\bar{\chi}, y).$$

(You will need to use that χ is primitive and in particular problem 9(c).)

12. In class we omitted the modifications to the proof needed to evaluate $L(\chi, 1)$, for χ a primitive Dirichlet character with $\chi(-1) = -1$. Complete the proof.

13. Using our results on special values of Dirichlet L-functions (including the previous problem), evaluate the sums

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots,$$
$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots,$$
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \cdots,$$

14. Check the lemma from class that $v_p(ab) = v_p(a) + v_p(b)$ and $v_p(a+b) \geq \min(v_p(a), v_p(b))$, with equality if $v_p(a) \neq v_p(b)$; deduce the corresponding multiplicative version ($|\cdot|_p$ is multiplicative and satisfies the strong triangle inequality $|a + b|_p \leq \max(|a|_p, |b|_p)$).

15. Let $p = 5$. Set $a_1 = 2$, and recursively solve (in $\mathbb{Z}$) the equation $a_n^2 \equiv -1 \pmod{5^n}$, with $a_n \equiv a_{n-1} \pmod{5^{n-1}}$. Show that $(a_n)_n$ is a Cauchy sequence with respect to $|\cdot|_5$, i.e. for all $\epsilon > 0$ there exists N such that for all $m, n > N$, $|a_n - a_m|_5 < \epsilon$.

16. Generalize the previous exercise as follows (one form of Hensel's lemma).

(a) Let $f(x) \in \mathbb{Z}[x]$, and let $a_1 \in \mathbb{Z}$ satisfy $v_p(f(a_1)) > 0$ and $v_p(f'(a_1)) = 0$ ($f'(x)$ is the derivative, computed using the usual formulas for polynomials; note this can be done for polynomials over any ring). Show that the sequence $(a_n)_{n\geq 1}$ of rational numbers

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$$

is a Cauchy sequence with respect to $|\cdot|_p$, and $|f(a_n)|_p \to 0$ as $n \to \infty$.

(b) Once we have (one 7/20) defined $\mathbb{Z}_p$, reformulate the result of part (a) as a statement about solving $f(x) = 0$ in $\mathbb{Z}_p$ given a mod p solution (also check that we may, more generally, take $f(x) \in \mathbb{Z}_p[x]$).

17. Let K be a field. A function $v\colon K \to \mathbb{Z} \cup \infty$ (here $\mathbb{Z}$ can be replaced with any totally ordered abelian group isomorphic to $\mathbb{Z}$) satisfying the properties of v_p in Problem 14 is called a discrete valuation.

    (a) Let K be a field extension of $\mathbb{Q}_p$ of finite degree. Show that $v_p\colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ extends uniquely to a discrete valuation $v_p\colon K \to \frac{1}{e}\mathbb{Z} \cup \{\infty\}$ for some integer e.

    (b) Let $K = \mathbb{Q}[i]$. Does $v_3\colon \mathbb{Q}^\times \to \mathbb{Z}$ extend uniquely to K? What about v_5? What is going on?

18. Verify the rest of the lemma that we partially proved in class: component-wise addition and multiplication yield well-defined (independent of representative Cauchy sequences) ring operations on $\mathbb{Q}_p$; $\mathbb{Q}_p$ is a field; $\mathbb{Q}_p$ is complete with respect to $|\cdot|_p$.

19. Consider the usual power series expansions for the exponential and logarithm functions:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

    If we attempt to use these power series to define $\exp(x)$ and $\log(1+x)$ as $\mathbb{Q}_p$-valued functions of a p-adic variable $x \in \mathbb{Q}_p$ (giving the p-adic exponential and logarithm), what are their radii of convergence? (Recall a series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence r if it converges for all $|x| < r$ and diverges for all $|x| > r$; here of course we should take $|\cdot|_p$ for the relevant "absolute value.") In calculus, the exponential and logarithm functions are inverse homomorphisms $\mathbb{R} \to \mathbb{R}_{>0}$ (the group operations are addition on the source and multiplication on the target): what analogue is true for our p-adic exponential and p-adic logarithm functions?

20. How many quadratic (degree 2) field extensions, up to isomorphism, does $\mathbb{Q}_p$ have for p odd? (Use Problem 16.) What about for $p = 2$? (This is more subtle; you can try generalizing the results of Problem 16 to include some cases when $v_p(f'(a_1)) > 0$.)

21. Let $\chi\colon (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character. Let F be any integer multiple of m. Show (generalizing the argument we gave in class for $F = m$) that for all $n \in \mathbb{Z}_{\geq 0}$,

$$B_{n,\chi} = F^{n-1} \sum_{a=1}^{F} \chi(a) B_n\left(\frac{a}{F}\right)$$

    (here $B_n(a/F)$ is the value of the Bernoulli polynomial at a/F, *not* B_n times a/F).

22. Let χ_1 and χ_2 be primitive Dirichlet characters modulo m_1 and m_2. Let $(\chi_1\chi_2)$ denote the *primitive* Dirichlet character underlying the naïve product character $\psi(a) = \chi_1(a)\chi_2(a)$, which is a well-defined Dirichlet character modulo $\mathrm{lcm}(m_1, m_2)$. (Stop and make sure you understand the distinction being drawn here!)

    (a) Give an example where $\psi \neq (\chi_1\chi_2)$. Give an example where $\psi = (\chi_1\chi_2)$.

    (b) Show that unless $\chi_1(a) = \chi_2(a) = 0$, $\chi_1(a)\chi_2(a) = (\chi_1\chi_2)(a)$.

23. In class we established the basic properties of the p-adic L-function $L_p(\chi, s)$, restricting for simplicity to the case $p \neq 2$. Make the necessary case-by-case adjustments to extend our results to the case $p = 2$.

Further reading: Here are some references that you might enjoy as follow-ups or companions to this course.

1. Ireland-Rosen, *A Classical Introduction to Modern Number Theory*, chapters on Bernoulli numbers and Dirichlet L-functions (including: a different proof of the analytic continuation of $L(\chi, s)$—somewhat easier than what we did—but not working out the functional equation; an elementary proof of the Kummer congruences; a proof of Herbrand's theorem).

2. Körner, *Fourier Analysis*: a wide-ranging essayistic introduction to Fourier analysis, with some fun history and applications.

3. Shakarchi-Stein, *Fourier Analysis*: a wonderfully clear Fourier analysis book written for an audience who has just taken a first real analysis course.

4. Washington, *Introduction to Cyclotomic Fields*: a more advanced book on Iwasawa theory. Our treatment of the p-adic L-function follows Chapter 5 (the book later gives other, more sophisticated, constructions of the p-adic L-function). Other parts of the book assume more background in complex analysis, algebra, and algebraic number theory.

5. Kato-Kurokawa-Saito, *Number Theory I: Fermat's Dream*: a beautiful introduction to several fundamental topics in modern number theory, written for an audience with some algebra and complex analysis background but otherwise quite accessible. Chapter 3 gives a different treatment of our work on L-functions and Bernoulli numbers, and chapter 4 develops the first connections to algebraic number theory. (This is the first book in a series of 3 number theory texts that are commonly used in Japan; the third book discusses Iwasawa theory and the p-adic L-function.)

6. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions.* Chapter 1 is a careful introduction to p-adic numbers. Chapter 2 constructs the p-adic zeta function (a special case of the p-adic L-function) using a different method than the one we gave in class (similar to but at a more accessible level than the construction given in Washington Chapter 12).