References for each lecture

Lecture 1: 6/16/23.

- We first looked quickly at an ancient Egyptian calculation of the volume of a truncated square pyramid. For the image from the Moscow Mathematical Papyrus, see
  - https://www.maa.org/press/periodicals/convergence/mathematical-treasure-the-rhind-and-moscow-mathematical-papyri

  For a translation, and much other context, see
  - Katz, Victor (ed.); with contributions by Imhausen, Annette; Robson, Eleanor; Dauben, Joseph; Plofker, Kim; and Berggren, Lennart, *The Mathematics of Egypt, Mesopotamia, China, India, and Islam, a Sourcebook*, Princeton University Press, 2007. (p. 33)
- We then worked through the calculation of the volume of a sphere in *The Nine Chapters on the Mathematical Art* (*Jiuzhang suanshu*), with the commentaries of Liu Hui (3rd c. CE) and Li Chunfeng (7th c. CE). The latter reported on Zu Geng's successful calculation (5th or 6th c. CE) based on comparing the cross-sections of a sphere inscribed in a cube with those of the "double umbrella" (*mouhefanggai*) obtained by intersecting two orthogonal inscribed cylinders. The portion of text we looked at is translated in:
  - Katz, Victor (ed.); with contributions by Imhausen, Annette; Robson, Eleanor; Dauben, Joseph; Plofker, Kim; and Berggren, Lennart, *The Mathematics of Egypt, Mesopotamia, China, India, and Islam, a Sourcebook*, Princeton University Press, 2007. (pp. 256-259)
- We ended by starting to look at Euclid's approach to the volume of a sphere (*Elements* Prop. XII.18), which required first a precise formulation of the meaning of a "ratio of magnitudes;" to that end, we touched on the distinction between *number* (Books VII-IX of the *Elements*) and *magnitude* (introduced in Book V of the *Elements*) in Euclid's mathematics, (in)commensurability of magnitudes, and anthyphairesis ("Euclidean algorithm") applied to magnitudes. The manuscript we looked at was the MS d'Orville 301:
  - `https://www.claymath.org/library/historical/euclid/`[1]

  The text of the *Elements* we looked at was:
  - `http://aleph0.clarku.edu/~djoyce/java/elements/elements.html` (see especially Book V Defn. 5 for the definition of ratio of magnitudes, Book X for commensurability, and Book XII for the volume of the sphere).

Lecture 2: 6/23/23.

- We began by going carefully through *Elements* XII.P2 (areas of circles are as the squares on their diameters), using the "method of exhaustion" grounded on *Elements* X.P1. Because of time I skipped a discussion of a new result on the perimeter of the circle in Archimedes: see *Measurement of the Circle* P1.
- After a little background on the ancient Greek theory of conics, we looked through Archimedes' *Quadrature of Parabolas* P18-P24, which by the method of exhaustion (based on iterated inscription of triangles in smaller and smaller parabolic segments) proves that a parabolic segment determined by points Q and q on the parabola, with associated diameter[2] passing through P on the parabola, has area equal to $\frac{4}{3}$ the area of triangle PQq. One notable feature of this calculation is Archimedes' summing of, in our terms, a geometric series.

[1] See also `https://digital.bodleian.ox.ac.uk/objects/d4a23501-0b98-4aff-acd6-fe06fe9b62e3/`

[2] This is the unique line parallel to the main axis of the parabola but bisecting $\overline{Qq}$.

- After some historical background on the Archimedes Palimpsest ("Codex C"), we then looked at Archimedes' dazzling heuristic proof in *The Method* P1 of the same result on areas of parabolic segment, relying on "balancing" (as in the law of the lever) infinitesimal cross-sections of the parabola and the triangle on a suitable lever. (Archimedes took up the mathematical theory of levers—and related center of gravity problems—at length in two books on *Planes in Equilibria*.)
- Having seen Archimedes the rigorous geometer and Archimedes the intuitive mathematical physicist, we ended with glimpses of two texts by Newton, showing some of the range of his mathematical temperament: we saw the formal geometry of the *Principia* as well as the more intuitive arguments with infinitesimals in *A Treatise on the Methods of Series and Fluxions*. I wanted to spend more time on this and will do so at the start of class next time, as the latter text provides a good reference point for how far mathematical language has come since antiquity. Algebra, the topic of the next couple classes, is the key to this development.

Lecture 3: 6/30/23.

- We began by looking at Newton's descriptions of differentiation, antidifferentiation, and their relationship in the fundamental theorem of calculus in Problems I, II, and IX of *The Method of Fluxions and Infinite Series* (as translated from Latin and published in 1736—Newton's unpublished text dates from 1671). We noted too Newton's free use of power series in the examples in this text.
- We then looked back and saw the simple geometric idea behind the fundamental theorem, without a full appreciation of its force as a computational tool, in Isaac Barrow's *Geometrical Lectures* (Lecture X, Prop. 11): the difference is largely Newton's algebraic mindset, against Barrow's more classical geometric mindset.
- So we turned back to some of the origins of algebra, looking at al-Khwarizmi's *Compendium on Calculation by Completion and Reduction* ("completion" or "restoring" being "*al-jabr*") and the role played there by the decimal place value system. See pages 542 ff. of Katz, et al., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook* (see Lecture 1). The quotation about the decimal place value system from al-Uqlidisi can be found on pg. 269 of:
  - Katz, Victor, *A History of Mathematics*, 3rd edition, Addison-Wesley, 2009.

  (This is a very helpful reference, from what I can tell the most reliable and up-to-date history of mathematics textbook available.) We continued with Abu Kamil's algebraic solution to the problem of inscribing a regular pentagon in a square: see pg. 552 of the *Sourcebook* above.
- We then read a text, the preface to the Banu Musa's edition of Apollonius's *Conics* (*Sourcebook* pg. 521-523), that documents the 9th c. CE struggle to interpret and translate the most advanced works of Greek geometry. We ended with a text from about 100 years later, written within a tradition that has now fully mastered and is going beyond what it has inherited from Greek geometry: ibn al-Haytham's confident preface to his *Completion of the Conics* (*Sourcebook* pg. 523-524).

Lecture 4: 7/7/2023.

- We continued our discussion from last time with a quick look at the *Algebra* of Omar Khayyam, where he systematically explains how to find a solution to any cubic equation by realizing it geometrically as a point of intersection of two conics (see the *Sourcebook* above, pg. 556-558). Although this solution is geometric, Khayyam's introduction continues the theme of liberating the terms of "algebra" from the ancient Greek operations on geometric magnitudes. Khayyam notes that a "numerical" solution is still out of reach but that "perhaps someone else, who will come after us, will know [how to do] it." (*loc. cit.* pg. 557)

- After a gradual assimilation of the Islamic algebra (particularly the works of al-Khwarizmi and Abu Kamil—it is unclear to what extent the more advanced texts were transmitted) in European mathematics (key transmission texts being those of Fibonacci around 1200), we saw how Khayyam's question was finally resolved in Italy in the 16th century, first by Tartaglia (in part preceded by Scipione del Ferro) and then systematized and published in 1545 in Cardano's *Ars Magna*. We looked at Cardano's solution in the case "cube and things equals numbers" (Chapter XI). A digital scan of Cardano's text is available from the Linda Hall Library:
  - `https://catalog.lindahall.org/discovery/delivery/01LINDAHALL_INST:LHL/1286504780005961`
- After looking at the primitive algebraic symbolism in Cardano, we took a quick tour of the development of algebraic symbolism from 1500-1650, with examples from Cardano, Viète, Harriot, and Descartes (having seen Newton's quite modern symbolism last week). See pg. 471 of Katz's *A History of Mathematics* cited in Lecture 3.
- We then turned to Descarte's work La Géométrie, one of the foundational works of modern "analytic geometry":
  - Descartes, René, *The Geometry of Rene Descartes, with a facsimile of the first edition*, trans. Smith, David and Latham, Marcia, Dover, 1954.

  We focused on the discussion (pg. 26-37 of *loc. cit.*, 309-314 in the original, which was an appendix to Descarte's *Discourse on the Method*), where Descartes gives a novel "analytic" solution to "the problem of four lines," with a generalization to any number of lines in the plane. (The case of 4 lines had been solved by Apollonius in his *Conics*.) On page 29 we see Descarte's first introduction of coordinate axes, with lengths of all segments in the problem being referred to one points coordinate with respect to fixed (not necessarily rectangular) axes.
- Also prominent in Descarte's *Geometry*, starting from the very first page, is how straight-edge and compass constructions correspond to arithmetic operations. We picked this thread up with a discussion of constructible numbers, the classical construction problems, and their 19th century resolutions using[3] modern developments in algebra. We started talking about Gauss's answer to which regular polygons are constructible (at the very end of the *Disquisitiones Arithmeticae*) and will likely continue a little more with this next week.

Lecture 5: 7/14/2023. We returned in detail to some of Gauss's calculations with "periods" in the last chapter of the *Disquisitiones*. Here's a quick guide to the notation if you are trying to look at the text. For a prime n (we used p in class), a factorization $n - 1 = ef$, a primitive root g modulo

[3]With the exception of irrationality of $\sqrt{\pi}$.

n, a fixed root r (we wrote ζ) of $\frac{x^n-1}{x-1} = 0$, and any integer λ (mod n), Gauss sets

$$(f, \lambda) = \sum_{k=0}^{f-1} r^{\lambda g^{ek}}.$$

He also abbreviates $[\lambda] := r^\lambda$. (See §342-345 for the definitions and for some elementary properties of these sums, including the important product formula decomposing $(f, \lambda) \cdot (f, \mu)$ into a sum of f periods of length f.)

We showed in the case $n = 17$ how to determine a quadratic equation with $\mathbb{Q}$ coefficients satisfied by $(8, 1)$ and $(8, 3)$, and then how to determine a quadratic equation with coefficients in $\mathbb{Q}((8, 1))$ satisfied by $(4, 1)$ and $(4, 9)$. Full details of Gauss's calculation of how to realize explicitly (including defining equations) the field extension $\mathbb{Q}(r)/\mathbb{Q}$ as a tower of four quadratic extensions appear in §354. We discussed how to interpret Gauss's calculation in terms of Galois theory, of which it is one of the important precursors.[4]

We returned to general primes n and computed the quadratic equations over $\mathbb{Q}$ satisfied by the periods $(\frac{n-1}{2}, 1)$ and $\frac{n-1}{2}, g)$. As a corollary, we computed the square of what we now call the Gauss sum associated to the Legendre symbol. (See §356.) We ended by deducing quadratic reciprocity! While Gauss did derive quadratic reciprocity from his work on Gauss sums (initially using his more refined calculation of the sign of the quadratic Gaus sum), our proof, given in Chapter 6 of Ireland and Rosen, *A Classical Introduction to Modern Number Theory*, is essentially that of Eisenstein in his article *La loi de récriprocité tireé des formules de Mr. Gauss, sans avoir déterminé préalablement le signe du radical*, available at `https://eudml.org/doc/147237`.

At the end of class, I mentioned that the "next two" (non-existent) classes would have been, returning to our Euclidean roots, on the classification of regular polyhedra (from Euclid to Euler) and on the need to use the method of exhaustion in Prop. XII.5 of Euclid's *Elements*[5] (from Euclid to Hilbert and Max Dehn). See Hartshorne's book *Geometry: Euclid and Beyond* (the chapter on non-Euclidean geometry would be a third excellent follow-up topic).

[4]Another, at least as important, is Lagrange's theory of resolvents, which unfortunately we did not have time to discuss. Harold Edward's book *Galois Theory* gives an elementary and historical account of Galois theory.

[5]Triangular pyramids with equal bases and equal heights have equal volume.