Dimension, Bases, and the Extension and Contraction Theorems

Throughout this note, V denotes a vector space.

Definition 1. A vector space is *finite dimensional* if there is a finite subset $S \subset V$ such that $\text{span}(S) = V$. We define the *dimension* of V, denoted $\dim(V)$, to be the minimal number of elements $|S|$ of a set S which spans V.

From here on, V will denote a finite dimensional vector space. We will need the following theorem and corollary for the proofs of the extension and contraction theorems:

Theorem 2. *Let $S \subset V$.*

- *If S spans V, then $\dim(V) \leq |S|$.*
- *If S is linearly independent, then $|S| \leq \dim(V)$.*

Notice that the first statement above follows immediately from the definition of $\dim(V)$.

Proof for $n+1$ vectors in $V = \mathbb{R}^n$. Suppose we have $n+1$ vectors $v_1, \ldots, v_{n+1}$ in $\mathbb{R}^n$. Form the matrix

$$A := \left(\, v_1 \, \middle| \, \cdots \, \middle| \, v_{n+1} \, \right),$$

which has rank at most n. This means if we row-reduce A, we get at most n pivots, so there must be a non-trivial solution to $A\vec{x} = \vec{0}$. Thus the columns of A are linearly dependent. This means if $S \subset \mathbb{R}^n$ has more than n elements, then S is linearly dependent. The contrapositive of this statement is that if S is linearly independent, S has at most n elements. □

Definition 3. A *basis* for V is a subset $B \subset V$ which both is linearly independent and spans V.

Exercise 4. Let $S = \{v_1, \ldots, v_n\} \subset V$ be a spanning set for V. This means that every vector in V can be written as a linear combination of the elements of S. Show that the following are equivalent.

- S is also linearly independent, so that S is a basis.
- Every vector in V can be *uniquely* expressed as a linear combination of the elements of S.
- The zero vector $\vec{0} \in V$ can be *uniquely* expressed as a linear combination of the elements of S.

Corollary 5. *If B is a basis for V, then $|B| = \dim(V)$.*

Proof. As B spans V, $\dim(V) \leq |B|$ by Theorem 2. As B is linearly independent, $|B| \leq \dim(V)$ by Thereom 2. We conclude that $|B| = \dim(V)$. □

The extension and contraction theorems play spanning sets off linearly independent sets for strong results in dual ways. The idea of contraction is more fundamental as a finite dimensional vector space assumes the existence of a finite spanning set.

One should think of linearly independent sets as "small" and spanning sets as "big," and the sets exactly in the middle are bases.

Exercise 6. Let $S_1 \subset S_2 \subset V$.

(1) If S_2 is linearly independent, then so is S_1.
(2) If S_1 is linearly dependent, then so is S_2.
(3) If S_1 spans V, then S_2 spans V.
(4) If S_2 does not span V, then neither does S_1.

The proofs of the Contraction and Extension Theorems are very similar in structure.

Lemma 7 (Contraction Lemma)**.** *Suppose* $S = \{v_1, \ldots, v_n\}$ *spans* V*. If* $v_k \in \operatorname{span}(S \setminus \{v_k\})$*, then* $S \setminus \{v_k\}$ *spans* V*.*

Proof. Since $v_k \in \operatorname{span}(S \setminus \{v_k\})$, we may express v_k as a linear combination of $v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$:

$$v_k = \sum_{i \neq k}^{n} \lambda_i v_i.$$

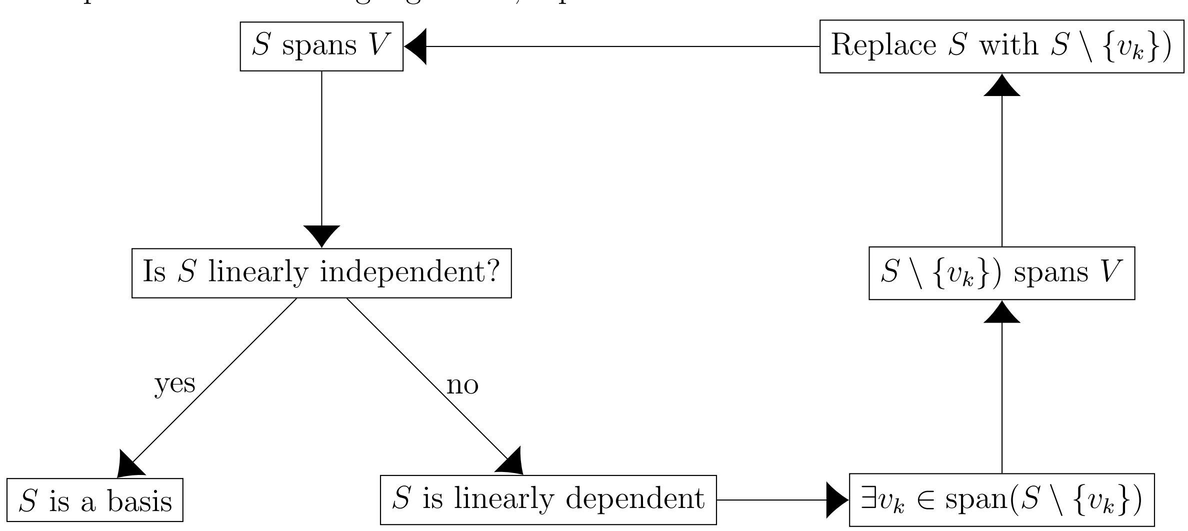
Now if $v \in V$, then there are $\mu_1, \ldots, \mu_n$ such that

$$\begin{aligned} v &= \sum_{i=1}^{n} \mu_i v_i \\ &= \mu_k v_k + \sum_{i=\neq k}^{n} \mu_i v_i \\ &= \mu_k \sum_{i=\neq k}^{n} \lambda_i v_i + \sum_{i=\neq k}^{n} \mu_i v_i \\ &= \sum_{i=\neq k}^{n} \mu_k \lambda_i v_i + \sum_{i=\neq k}^{n} \mu_i v_i \\ &= \sum_{i=\neq k}^{n} (\mu_k \lambda_i + \mu_i) v_i \in \operatorname{span}(S \setminus \{v_k\}). \end{aligned}$$

Hence $S \setminus \{v_k\}$ spans V. ☐

Theorem 8 (Contraction)**.** *Let* $S = \{v_1, \ldots, v_n\} \subset V$ *be a finite subset such that* S *spans* V*. Then there is a subset* $B \subset S$ *such that* B *is a basis of* V*.*

Proof. The proof is the following algorithm, represented as a flow chart:



Since S spans V, if S is also linearly independent, then S is a basis. If S is not linearly independent, then S is linearly dependent. Then there is a $v_k \in S$ such that v_k is a linear combination of the other elements of S. By Lemma 7, the set $S \setminus \{v_k\}$ still spans V, so we may replace S with $S \setminus \{v_k\}$ and repeat the above procedure with a new finite set with strictly fewer elements. This algorithm must eventually terminate, since the empty set is linearly independent by definition. ☐

Corollary 9 (Existence of Bases)**.** *Let V be a finite dimensional vector space. Then V has a basis.*

Proof. Since V is finite dimensional, there is a finite spanning set $S \subset V$. The Contraction Theorem 8 ensures there is a basis $B \subset S$. □

Lemma 10 (Extension Lemma)**.** *Suppose $S = \{v_1, \ldots, v_n\}$ is a linearly independent subset of V, and suppose $v \notin \operatorname{span}(S)$. Then $S \cup \{v\}$ is linearly independent.*

Proof. Since S is linearly independent, we know that

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \qquad \Longrightarrow \qquad \lambda_i = 0 \text{ for all } i. \tag{1}$$

Now suppose

$$\mu v + \sum_{i=1}^{n} \lambda_i v_i = 0. \tag{2}$$

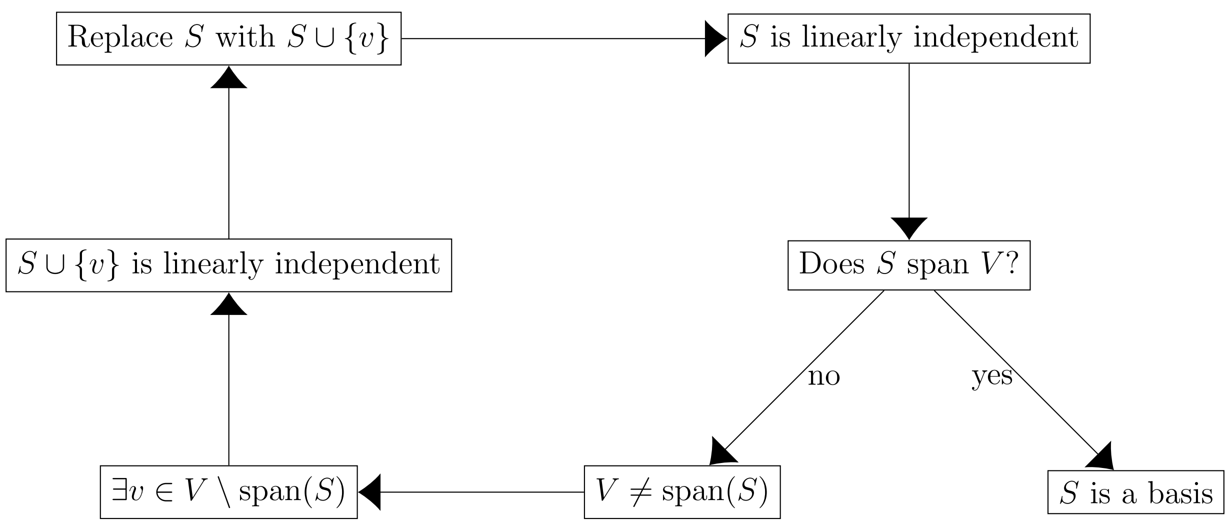
If $\mu \neq 0$, then

$$v = \frac{-1}{\mu} \sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n} \frac{-\lambda_i}{\mu} v_i \in \operatorname{span}(S),$$

a contradiction. Hence $\mu = 0$, and plugging this in to (2), we can conclude $\lambda_i = 0$ for all i by (1). Hence all the scalars in equation (2) must be zero, so $S \cup \{v\}$ is linearly independent. □

Theorem 11 (Extension)**.** *Let $S \subset V$ be a linearly independent subset. Then there is a finite set $B \supset S$ such that B is a basis for V.*

Proof. The proof is the following algorithm, represented as a flow chart:



Since S is linearly independent, if S spans V, then S is a basis for V. If S does not span V, then there is a $v \in V \setminus \operatorname{span}(S)$, and $S \cup \{v\}$ is linearly independent by Lemma 10. Thus we may replace S with $S \cup \{v\}$ and repeat the above procedure with a new linearly independent set with strictly more elements. This algorithm must eventually terminate, as once S has $\dim(V)$ many elements, it must span V. □

Corollary 12. *Let W be a subspace of V, and suppose S is a basis of W. There is a basis B for V with $S \subset B$.*