Homework 3
Math 2568   Due: January 30, 2019

Problem 1

§3.1, Exercise 7. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Denote the columns of the matrix A by

$$A_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \quad \cdots \quad A_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Show that the matrix vector product Ax can be written as

$$Ax = x_1A_1 + x_2A_2 + \cdots + x_nA_n,$$

where x_jA_j denotes scalar multiplication (see Chapter 1).

Compute Ax directly:

$$Ax = \begin{pmatrix} x_1a_{11} + x_2a_{12} + \cdots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + \cdots + x_na_{2n} \\ \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + \cdots + x_na_{mn} \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

So, it is indeed true that $Ax = x_1A_1 + x_2A_2 + \cdots + x_nA_n$.

Problem 2

§3.1, Exercise 8. Let

$$C = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find a 2-vector z such that $Cz = b$.

Answer: The equation $Cz = b$ is valid for $z = (\frac{2}{3}, \frac{1}{3})^t$.

Solution: Let $z = (z_1, z_2)^t$. Then $Cz = b$ implies

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which can be multiplied out, yielding the linear system

$$\begin{array}{rcrcl} z_1 & + & z_2 & = & 1 \\ 2z_1 & - & z_2 & = & 1. \end{array}$$

This system can be solved by substitution to obtain $z_1 = \frac{2}{3}$ and $z_2 = \frac{1}{3}$.

Problem 3 (MATLAB)

§3.1, Exercise 16.(MATLAB) Let

$$A = \begin{pmatrix} 2 & 4 & -1 \\ 1 & 3 & 2 \\ -1 & -2 & 5 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}. \tag{1}$$

Find a 3-vector x such that $Ax = b$.

Using MATLAB:

```
A\b =
    7.1111
   -2.7778
    1.1111
```

Problem 4

Find a nonzero vector that is mapped to the origin by the given matrix.

§3.2, Exercise 1. $A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$.

Answer: If $x = (x_1, 0)^t$, where x_1 is any real scalar, then $Ax = 0$.

Solution: Let $x = (x_1, x_2)^t$ and solve the system

$$Ax = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing A to obtain

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus, $Ax = 0$ when $x_2 = 0$.

Problem 5

§3.2, Exercise 5. What 2×2 matrix rotates the plane clockwise by $45°$?

Answer:

$$R_{(-45°)} = \begin{pmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Problem 6 (MATLAB)

Use `map` to help describe the planar motions of the associated linear mappings for the given 2×2 matrix.

§3.2, Exercise 21.(MATLAB) $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$

A rotates the plane $30°$ clockwise.

Problem 7

§3.3, Exercise 11. The *cross product* of two 3-vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is the 3-vector

$$x \times y = (x_2y_3 - x_3y_2, -(x_1y_3 - x_3y_1), x_1y_2 - x_2y_1).$$

Let $K = (2, 1, -1)$. Show that the mapping $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L(x) = x \times K$$

is a linear mapping. Find the 3×3 matrix A such that

$$L(x) = Ax,$$

that is, $L = L_A$.

Answer: The matrix of linear mapping L is

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}.$$

Solution: Let $X = (x_1, x_2, x_3)$ and let $Y = (y_1, y_2, y_3)$. Since $K = (2, 1, -1)$,

$$L(X) = (x_1, x_2, x_3) \times K = (-x_2 - x_3, x_1 + 2x_3, x_1 - 2x_2).$$

To show that $L(X)$ is a linear mapping, first demonstrate that (3.3.1) is valid:

$$\begin{aligned} L(X+Y) &= L(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (-(x_2 + y_2) - (x_3 + y_3), (x_1 + y_1) + 2(x_3 + y_3), (x_1 + y_1) - 2(x_2 + y_2)) \\ &= (-x_2 - x_3, x_1 + 2x_3, x_1 - 2x_2) + (-y_2 - y_3, y_1 + 2y_3, y_1 - 2y_2) \\ &= L(X) + L(Y), \end{aligned}$$

then show that (3.3.2) is valid:

$$\begin{aligned} cL(X) &= cL(x_1, x_2, x_3) \\ &= c(-x_2 - x_3, x_1 + 2x_3, x_1 - 2x_2) \\ &= (-cx_2 - cx_3, cx_1 + 2cx_3, cx_1 - 2cx_2) \\ &= L(cx_1, cx_2, cx_3) \\ &= L(cX). \end{aligned}$$

Find A by noting that $L(e_j) = Ae_j$ is the j^{th} column of A, and computing

$$\begin{aligned} L(e_1) = L(1, 0, 0) = (0, 1, 1) \\ L(e_2) = L(0, 1, 0) = (-1, 0, -2) \\ L(e_3) = L(0, 0, 1) = (-1, 2, 0). \end{aligned}$$

Problem 8

Determine whether the given transformation is linear.

§3.3, Exercise 8. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2 - 1)$.

The transformation $T(x, y) = (x + y, x - y - 1)$ is not linear because $T(0, 0) = (0, -1) \neq 0$, contradicting Theorem 3.3.5.

Problem 9 (MATLAB)

Use MATLAB to verify (3.3.1) and (3.3.2).

§3.3, Exercise 18.(MATLAB)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 4 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ -5 \\ 10 \end{pmatrix}, \quad c = 21; \tag{2}$$

Verify (3.3.1) by typing `A*(x + y)` to obtain:

```
ans =
    24
   -21
    21
```

Then, type `A*x + A*y`, which yields the same answer. Verify (3.3.2) by typing `c*(A*x)`, which gives the same answer as `A*(c*x)`, namely:

```
ans =
    84
    84
   231
```

Problem 10

§3.4, Exercise 3.

(a) Find all solutions to the homogeneous equation $Ax = 0$ where

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

(b) Find a single solution to the inhomogeneous equation

$$Ax = \begin{pmatrix} 6 \\ 6 \end{pmatrix}. \tag{3}$$

(c) Use your answers in (a) and (b) to find all solutions to (3).

(a) **Answer:** All solutions to the homogeneous equation are of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} -11 \\ 7 \\ 1 \end{pmatrix}.$$

Solution: Row reduce the matrix of the homogeneous system $Ax = 0$ to obtain:

$$\begin{pmatrix} 1 & 0 & 11 \\ 0 & 1 & -7 \end{pmatrix}.$$

So $x_1 = -11s$, $x_2 = 7s$ and $x_3 = s$.

(b) **Answer:** One possible solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution: Assign a value to x_3, then substitute into the two equations of the inhomogeneous system to obtain values for x_1 and x_2.

(c) All solutions to (3) can be found by adding a single solution of the inhomogeneous system to all solutions of the homogeneous system, so:

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -11 \\ 7 \\ 1 \end{pmatrix}.$$