Homework 4
Math 2568

Problem 1

§3.4, Exercise 2. Write all solutions to the homogeneous system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_4 - x_5 &= 0 \\ x_3 - 2x_4 + x_5 &= 0 \end{aligned}$$

as the general superposition of three vectors.

Answer: Every solution can be written as a superposition of the vectors

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Solution: Write the matrix of the homogeneous system:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

This matrix cannot be row reduced further. Every solution has the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_5 - x_4 - 2x_2 \\ x_2 \\ -x_5 + 2x_4 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 2

§3.4, Exercise 3.

(a) Find all solutions to the homogeneous equation $Ax = 0$ where

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

(b) Find a single solution to the inhomogeneous equation

$$Ax = \begin{pmatrix} 6 \\ 6 \end{pmatrix}. \tag{1}$$

(c) Use your answers in (a) and (b) to find all solutions to (1).

(a) **Answer:** All solutions to the homogeneous equation are of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} -11 \\ 7 \\ 1 \end{pmatrix}.$$

Solution: Row reduce the matrix of the homogeneous system $Ax = 0$ to obtain:

$$\begin{pmatrix} 1 & 0 & 11 \\ 0 & 1 & -7 \end{pmatrix}.$$

So $x_1 = -11s$, $x_2 = 7s$ and $x_3 = s$.

(b) **Answer:** One possible solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution: Assign a value to x_3, then substitute into the two equations of the inhomogeneous system to obtain values for x_1 and x_2.

(c) All solutions to (1) can be found by adding a single solution of the inhomogeneous system to all solutions of the homogeneous system, so:

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -11 \\ 7 \\ 1 \end{pmatrix}.$$

Problem 3

§3.5, Exercise 11. Let

$$A = \begin{pmatrix} 1 & 0 & -3 \\ -2 & 1 & 1 \\ 0 & 1 & -5 \end{pmatrix}.$$

Let A^t is the transpose of the matrix A, as defined in Section 1.3. Compute AA^t.

$$AA^t = \begin{pmatrix} 1 & 0 & -3 \\ -2 & 1 & 1 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -3 & 1 & -5 \end{pmatrix} = \begin{pmatrix} 10 & -5 & 15 \\ -5 & 6 & -4 \\ 15 & -4 & 26 \end{pmatrix}.$$

Problem 4

§3.5, Exercise 10. Let

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & 3 \\ b & 2 \end{pmatrix}.$$

For which values of a and b does $AB = BA$?

Answer:

$$B = \begin{pmatrix} \frac{4}{5} & 3 \\ \frac{3}{5} & 2 \end{pmatrix}.$$

Solution: Compute

$$\begin{aligned} AB &= BA \\ \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a & 3 \\ b & 2 \end{pmatrix} &= \begin{pmatrix} a & 3 \\ b & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} \\ \begin{pmatrix} 2a+5b & 16 \\ a+4b & 11 \end{pmatrix} &= \begin{pmatrix} 2a+3 & 5a+12 \\ 2b+2 & 5b+8 \end{pmatrix}. \end{aligned}$$

This equation can be rewritten as the system

$$\begin{aligned} 2a+5b &= 2a+3 \\ 16 &= 5a+12 \\ a+4b &= 2b+2 \\ 11 &= 5b+8 \end{aligned}$$

which yields the solution $a = 4/5$ and $b = 3/5$.

Problem 5 (MATLAB)

Decide for the given pair of matrices A and B whether or not the products AB or BA are defined and compute the products when possible.

§3.5, Exercise 14.(MATLAB)

$$A = \begin{pmatrix} -2 & -2 & 4 & 5 \\ 0 & -3 & -4 & 3 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & 0 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 3 & -4 & 5 \\ 4 & -3 & 0 & -2 \\ -3 & -4 & -4 & -3 \\ -2 & -2 & 3 & -1 \end{pmatrix} \tag{2}$$

Both AB and BA are defined and can be computed using MATLAB:

```
A*B
ans =
   -34   -26     7   -23
    -6    19    25    15
   -15     6    -5     7
    -4   -11    12    -6
```

```
B*A
ans =
    -8     4    -8    35
    -8    -1    28     3
     2    27     0   -43
     7     0     3   -17
```

Problem 6

§3.6, Exercise 4. Let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that $J^2 = -I$.

(b) Evaluate $(aI + bJ)(cI + dJ)$ in terms of I and J.

(a) Verify $J^2 = -I$ by computation:

$$J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

(b) **Answer:** $(aI + bJ)(cI + dJ) = (ac - bd)I + (ad + bc)J$.

Solution: Evaluate $(aI + bJ)(cI + dJ)$, yielding $acI^2 + adIJ + bcJI + bdJ^2$. Then, use the identities $IJ = JI = J$, $I^2 = I$, and $J^2 = -I$ to rewrite the expression in terms of I and J.

Problem 7 (MATLAB)

§3.6, Exercise 9.(MATLAB) Use the `rand(3,3)` command in MATLAB to choose five pairs of 3×3 matrices A and B at random. Compute AB and BA using MATLAB to see that in general these matrix products are unequal.

Computer experiment.

Problem 8

Use row reduction to find the inverse of the given matrix.

§3.7, Exercise 5. $\begin{pmatrix} 1 & 4 & 5 \\ 0 & 1 & -1 \\ -2 & 0 & -8 \end{pmatrix}$.

Answer: $A^{-1} = \dfrac{1}{10}\begin{pmatrix} -8 & 32 & -9 \\ 2 & 2 & 1 \\ 2 & -8 & 1 \end{pmatrix}$.

Solution: Let

$$M = (A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -2 & 0 & -8 & 0 & 0 & 1 \end{array}\right).$$

Then, row reduce M to obtain the augmented matrix $(I_3|A^{-1})$.

Problem 9 (MATLAB)

§3.7, Exercise 12.(MATLAB) Try to compute the inverse of the matrix

$$C = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 0 & 2 & 1 \end{pmatrix} \tag{3}$$

in MATLAB using the command `inv`. What happens — can you explain the outcome?

Now compute the inverse of the matrix

$$\begin{pmatrix} 1 & \epsilon & 3 \\ -1 & 2 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$

for some nonzero numbers ϵ of your choice. What can be observed in the inverse if ϵ is very small? What happens when ϵ tends to zero?

Typing `inv(C)` in MATLAB yields the response

```
Warning: Matrix is singular to working precision.
ans =
   Inf    Inf    Inf
   Inf    Inf    Inf
   Inf    Inf    Inf
```

Matrix C cannot be inverted because it is not row equivalent to I_3. We can type `rref(C)` to confirm that

```
ans =
    1.0000         0    3.0000
         0    1.0000    0.5000
         0         0         0
```

When `C(1,2)` is nonzero, C is invertible. As $\epsilon \to 0$ the entries of C^{-1} approach infinity. For example, if $\epsilon = 0.01$, then `inv(C)` yields

```
ans =
  600.0000  599.0000 -602.0000
  100.0000  100.0000 -100.0000
 -200.0000 -200.0000  201.0000
```

At $\epsilon = 0$, C^{-1} does not exist.

Problem 10

§3.7, Exercise 14. True or False: Determine whether the following statements are true or false, and explain your answer.

(a) The only 3×2 matrix A so that $Ax = 0$ for all $x \in \mathbb{R}^2$ is $A = 0$.

(b) A system of 5 equations in 3 unknowns with the solution $x_1 = 0, x_2 = -3, x_3 = 1$ must have infinitely many solutions.

(c) If A is a 2×2 matrix and $A^2 = 0$, then $A = 0$.

(d) If $u, v \in \mathbb{R}^3$ are perpendicular, then $\|u + v\| = \|u\| + \|v\|$.

(a) True:

$$A = \begin{pmatrix} Ae_1 & Ae_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

(b) False, it may have a unique solution. For example, the system of equations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

has $x_1 = 0, x_2 = -3, x_3 = 1$ as a unique solution.

(c) False: if

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then $A^2 = 0$.

(d) False:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2} \neq 2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

However, by the Pythagorean theorem,

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$