Homework 5
Math 2568

Problem 1

§3.3, Exercise 13. Let σ permute coordinates cyclically in $\mathbb{R}^3$; that is,

$$\sigma(x_1, x_2, x_3) = (x_2, x_3, x_1).$$

Find a 3×3 matrix A such that $\sigma = L_A$.

Answer: The matrix of linear mapping L_A is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Solution: Note that if $\sigma = L_A$, then $\sigma(e_j) = Ae_j$ is the j^{th} column of matrix A. Thus A is determined by

$$\begin{aligned} \sigma(e_1) &= \sigma(1,0,0) = (0,0,1) \\ \sigma(e_2) &= \sigma(0,1,0) = (1,0,0) \\ \sigma(e_3) &= \sigma(0,0,1) = (0,1,0). \end{aligned}$$

Problem 2

§3.7, Exercise 13. Let A and B be 3×3 invertible matrices so that

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Without computing A or B, determine the following:

(a) $\text{rank}(A)$

(b) The solution to

$$Bx = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) $(2BA)^{-1}$

(d) The matrix C so that $ACB + 3I_3 = 0$.

(a) A is an invertible 3×3 matrix, so $\text{rank}(A) = 3$.

(b) The solution is

$$x = B^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

(c)

$$(2BA)^{-1} = \frac{1}{2}A^{-1}B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ -2 & -2 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

(d) Recall that multiplication on the left by a matrix is not the same as multiplication on the right. We have that

$$\begin{aligned} ACB = -3I_3 &\implies A^{-1}ACB = -3A^{-1}I_3 && \text{multiplying on the left by } A^{-1} \\ &\implies CB = -3A^{-1} \\ &\implies CBB^{-1} = -3A^{-1}B^{-1} && \text{multiplying on the right by } B^{-1} \\ &\implies C = -3A^{-1}B^{-1} \\ &\implies C = -3\begin{pmatrix} 0 & 1 & 1 \\ -2 & -2 & -1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Problem 3

§3.8, Exercise 3. Show that the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is row equivalent to I_2 if and only if $ad - bc \neq 0$. **Hint:** Prove this result separately in the two cases $a \neq 0$ and $a = 0$.

Case: $a \neq 0$. A can be row reduced as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \to \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix}.$$

If $ad - bc \neq 0$, then the matrix can be row reduced to I_2, whereas if $ad - bc = 0$, the row reduced matrix is:

$$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 0 \end{pmatrix}$$

which cannot be reduced further and is not row equivalent to I_2.

Case: $a = 0$. If either $c = 0$ or $b = 0$, then the resulting matrices,

$$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

respectively, are not row equivalent to I_2, and $ad - bc = 0 - 0 = 0$. If $c \neq 0$ and $b \neq 0$, then the matrix can be row reduced:

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \to \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} \to \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & b \end{pmatrix}$$

which is row equivalent to I_2. So A is indeed row equivalent to I_2 if and only if $ad - bc \neq 0$.

Problem 4

§3.8, Exercise 4. Let A be a 2×2 matrix having integer entries. Find a condition on the entries of A that guarantees that A^{-1} has integer entries.

Answer: The matrix A^{-1} has integer entries when $|ad - bc| = 1$.

Solution: By (3.8.1),

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So, in order for A^{-1} to have integer entries, $\frac{1}{ad-bc}$ must be an integer. Since a, b, c, and d are integers, $\frac{1}{ad-bc}$ is an integer only if $|ad - bc| = 1$.

Problem 5

Use Cramer's rule (3.8.5) to solve the given system of linear equations.

§3.8, Exercise 8. Solve $\begin{aligned} 2x + 3y &= 2 \\ 3x - 5y &= 1 \end{aligned}$ for x.

Answer: $x = \frac{13}{19}$.

Solution: By Cramer's rule (see (3.8.5)),

$$x = \det \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \Big/ \det \begin{pmatrix} 2 & 3 \\ 3 & -5 \end{pmatrix} = \frac{-13}{-19}.$$

Problem 6 (MATLAB)

§3.8, Exercise 10. (MATLAB) Use MATLAB to choose five 2×2 matrices at random and compute their inverses. Do you get the impression that 'typically' 2×2 matrices are invertible? Try to find a reason for this fact using the determinant of 2×2 matrices.

A randomly selected 2×2 matrix is almost always invertible. A matrix will fail to be invertible only if the determinant of the matrix is 0, which is seldom the case.

Problem 7

Determine whether or not each of the given functions $x_1(t)$ and $x_2(t)$ is a solution to the given differential equation.

§4.1, Exercise 1. ODE: $\frac{dx}{dt} = \frac{t}{x-1}$.

Functions: $x_1(t) = t + 1$ and $x_2(t) = \frac{1+\sqrt{4t^2+1}}{2}$.

Answer: The function $x_1(t)$ is a solution to the differential equation; the function $x_2(t)$ is not a solution.

Solution: Compute

$$\frac{d}{dt}(x_1) = \frac{d}{dt}(t+1) = 1, \quad \text{and} \quad \frac{dx_1}{dt} = \frac{t}{x_1 - 1} = \frac{t}{(t+1)-1} = 1.$$

Thus, $x_1(t)$ is a solution to the differential equation. Then compute

$$\frac{d}{dt}(x_2) = \frac{d}{dt}\left(\frac{1+\sqrt{4t^2+1}}{2}\right) = \frac{4t}{\sqrt{4t^2+1}}, \quad \text{and} \quad \frac{dx_2}{dt} = \frac{t}{x_2 - 1} = \frac{2t}{\sqrt{4t^2+1}-1}.$$

Thus, $\frac{d}{dt}(x_2) \neq \frac{dx_2}{dt}$, so $x_2(t)$ is not a solution to the differential equation.

Problem 8

§4.1, Exercise 6. Solve the differential equation

$$\frac{dx}{dt} = -3x.$$

At what time t_1 will $x(t_1)$ be half of $x(0)$?

Answer: Using the initial value problem, we find that $\frac{dx}{dt} = -3x$ implies $x(t) = x_0e^{-3t}$. Given this equation, $x(t_1)$ will be half of $x(0)$ at time $t_1 = -\frac{1}{3}\ln(0.5)$.

Solution: Find this value of t_1 by substituting into the formula for x. That is, use:

$$x_0e^{-3t_1} = x(t_1) = \frac{1}{2}x_0$$

which implies

$$e^{-3t_1} = \frac{1}{2}.$$

Then solve for t_1.

Problem 9

Consider the uncoupled system of differential equations (4.3.2). For each choice of a and d, determine whether the origin is a saddle, source, or sink.

§4.3, Exercise 4. $a = -0.01$ and $d = -2.4$.

Answer: The origin is a sink.

Solution: For this uncoupled system, $A = -0.01 < 0$ and $D = -2.4 < 0$.

Problem 10

§4.3, Exercise 6. Let $(x(t), y(t))$ be the solution (4.3.3) of (4.3.2) with initial condition $(x(0), y(0)) = (x_0, y_0)$, where $x_0 \neq 0 \neq y_0$.

(a) Show that the points $(x(t), y(t))$ lie on the curve whose equation is:

$$y_0^a x^d - x_0^d y^a = 0.$$

(b) Verify that if $a = 1$ and $d = 2$, then the solution lies on a parabola tangent to the x-axis.

The solutions $x(t)$ and $y(t)$ are:

$$\begin{array}{rcl} x(t) & = & x_0e^{At} \\ y(t) & = & y_0e^{Dt} \end{array}.$$

We show that the point $(x(t), y(t))$ lies on the curve $y_0^A x^D - x_0^D y^A = 0$ as follows. Substitute the formulas for $x(t)$ and $y(t)$ into the equation to obtain

$$y_0^A\left(x_0e^{At}\right)^D - x_0^D\left(y_0e^{Dt}\right)^A = x_0^D y_0^A e^{ADt} - x_o^D y_0^A e^{ADt} = 0.$$

If $A = 1$ and $D = 2$, then the solutions lie on the curve $0 = y_0 x^2 - x_0^2 y$, which can be rewritten as $y = \frac{y_0}{x_0^2} x^2$. Since x_0 and y_0 are constants, this curve is a parabola tangent to the x-axis.