Homework 6
Math 2568

Problem 1 (MATLAB)

§4.4, Exercise 3.(MATLAB) Choose the linear system in pplane9 and set $a = d$ and $b = c$. Verify that for these systems of differential equations:

(a) When $|a| < b$ typical trajectories approach the line $y = x$ as $t \to \infty$ and the line $y = -x$ as $t \to -\infty$.

(b) Assume that b is positive, a is negative, and $b < -a$. With these assumptions show that the origin is a sink and that typical trajectories approach the origin tangent to the line $y = x$.

Graphs made in `pplane5` using the `axis('equal')` command verify these statements regarding linear systems where $A = D$ and $B = C$. Figure 3a, uses $A = D = -1$ and $B = C = 2$, and shows four sample trajectories which approach the line $y = x$ as $t \to \infty$. Figure 3b graphs the linear system with $A = D = -3$ and $B = C = 2$. It shows four sample trajectories, three of which approach the origin tangent to $y = x$. The fourth trajectory has an initial point $0 < x_0 = -y_0$ and approaches the origin on the straight line $y = -x$, which is orthogonal to $y = x$.

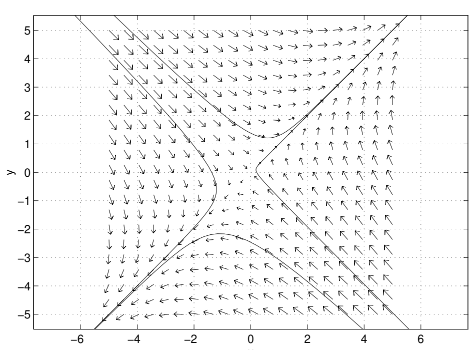

Figure 3a

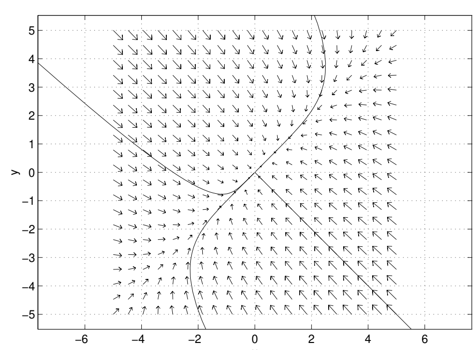

Figure 3b

Problem 2

Determine which of the function pairs $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions to the given system of ordinary differential equations.

§4.4, Exercise 6. The ODE is:

$$\begin{aligned} \dot{x} &= 2x - 3y \\ \dot{y} &= x - 2y. \end{aligned}$$

The pairs of functions are:

$$(x_1(t), y_1(t)) = e^t(3, 1) \quad \text{and} \quad (x_2(t), y_2(t)) = (e^{-t}, e^{-t}).$$

Answer: Both function pairs are solutions to the given system.

Solution: To determine whether $(x_1(t), y_1(t)) = (3e^t, e^t)$ is a solution to the system, compute the left hand sides of the equations:

$$\frac{dx_1}{dt}(t) = \frac{d}{dt}(3e^t) = 3e^t \quad \text{and} \quad \frac{dy_1}{dt}(t) = \frac{d}{dt}(e^t) = e^t.$$

Then compute the right hand sides of the equations:

$$2x_1(t) - 3y_1(t) = 2(3e^t) - 3e^t = 3e^t \quad \text{and} \quad x_1(t) - 2y_1(t) = 3e^t - 2e^t = e^t.$$

Since the left hand side of each equation equals the right hand side, the equations are consistent, and the pair of functions is a solution.

Similarly, to determine whether $(x_2(t), y_2(t)) = (e^{-t}, e^{-t})$ is a solution to the system, compute the left hand sides of the equations:

$$\frac{dx_2}{dt}(t) = \frac{d}{dt}(e^{-t}) = -e^{-t} \quad \text{and} \quad \frac{dy_2}{dt}(t) = \frac{d}{dt}(e^{-t}) = -e^{-t}.$$

Then compute the right hand sides of the equations:

$$2x_2(t) - 3y_2(t) = 2e^{-t} - 3e^{-t} = -e^{-t} \quad \text{and} \quad x_2(t) - 2y_2(t) = e^{-t} - 2e^{-t} = -e^{-t}.$$

Since the left hand side of each equation equals the right hand side, the equations are consistent, and the pair of functions is a solution.

Problem 3

§4.5, Exercise 4. Find a solution to

$$\dot{X}(t) = CX(t)$$

where

$$C = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and

$$X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Hint: Observe that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are eigenvectors of C.

Answer:

$$X(t) = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution: Note that if $Cv = \lambda v$, then $X(t) = e^{\lambda t}v$ is a solution to $\dot{X}(t) = CX(t)$. Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvalues corresponding to v_1 and v_2 are $\lambda_1 = 0$ and $\lambda_2 = 2$. This can be verified by calculating $Cv_1 = 0$ and $Cv_2 = 2v_2$. So,

$$X(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad X(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are both solutions to $\dot{X}(t) = CX$. By the principle of superposition,

$$X(t) = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is also a solution. Substitute the given the initial condition into the equation to obtain

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = X(0) = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now, solve the linear system

$$\begin{array}{rcrcr} 2 & = & \alpha & + & \beta \\ 1 & = & \alpha & - & \beta \end{array}$$

to find that $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$.

Problem 4 (MATLAB)

§4.5, Exercise 11.(MATLAB) Use MATLAB to verify that solutions to the system of linear differential equations

$$\begin{aligned}\frac{dx}{dt} &= 2x + y \\ \frac{dy}{dt} &= y\end{aligned}$$

are linear combinations of the two solutions

$$U(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V(t) = e^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

More concretely, proceed as follows:

(a) By superposition, the general solution to the differential equation has the form $X(t) = \alpha U(t) + \beta V(t)$. Find constants α and β such that $\alpha U(0) + \beta V(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) Graph the second component $y(t)$ of this solution using the MATLAB `plot` command.

(c) Use pplane9 to compute a solution via the Keyboard input starting at $(x(0), y(0)) = (0, 1)$ and then use the `y vs t` command in pplane9 to graph this solution.

(d) Compare the results of the two plots.

(e) Repeat steps (a)–(d) using the initial vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(a) **Answer:** If $\alpha = 1$ and $\beta = 1$, then

$$\alpha U(0) + \beta V(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Solution: Solve the linear system

$$\begin{aligned}\alpha - \beta &= 0 \\ \beta &= 1.\end{aligned}$$

(b) Figure 11a shows y as a function of t. The figure was created by the MATLAB commands:

```
t = linspace(-8,2);
y = exp(t);
plot(t,y)
```

(c) Figure 11b shows the `pplane5` graph of the system, and Figure 11c shows the `y vs. t` graph.

(d) The two plots are identical, since the `pplane5` command `y vs. t` graphs the y component of the solution, which is precisely what we did by hand in (b).

(e) In this case, $\alpha = 2$ and $\beta = 1$. Since the y component of $U(t)$ is zero, the graphs of $y(t)$ are identical to those in (b) and (c).

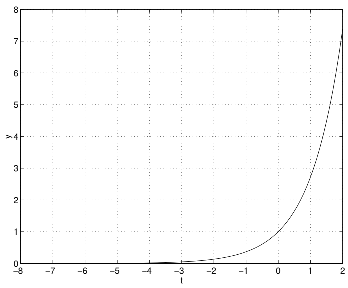

Figure 11a

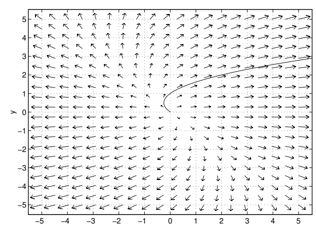

Figure 11b

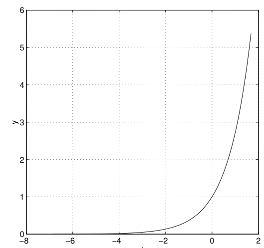

Figure 11c

Problem 5

§4.5, Exercise 6. Let

$$C = \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix}.$$

Show that C has no real eigenvectors.

A vector (x, y) is an eigenvector of C if

$$C \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

that is, if

$$(C - \lambda I_2) \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

In this case,

$$\begin{pmatrix} 1-\lambda & 2 \\ -3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

This equation will have a nonzero solution (x, y) only if

$$\begin{pmatrix} 1-\lambda & 2 \\ -3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is not row equivalent to the identity matrix. Row reducing the matrix yields

$$\begin{pmatrix} 1 & \frac{2}{1-\lambda} \\ 0 & -1-\lambda+\frac{6}{1-\lambda} \end{pmatrix}$$

so C has an eigenvector when

$$-1-\lambda+\frac{6}{1-\lambda}=0,$$

that is, when $\lambda^2=-5$. Therefore, C has no real eigenvectors.

Problem 6

§4.6, Exercise 1. For which values of λ is the matrix

$$\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$$

not invertible? **Note:** These values of λ are just the eigenvalues of the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Answer: The matrix is not invertible when $\lambda = 5$ or $\lambda = -1$.

Solution: Corollary 3.8.3 states that a matrix is not invertible if and only if the determinant is zero; in this case, if

$$(1-\lambda)(3-\lambda)-(2)(4)=\lambda^2-4\lambda-5=0.$$

Problem 7

Compute the determinant, trace, and characteristic polynomials for the given 2×2 matrix.

§4.6, Exercise 2. $\begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}$.

The determinant of the matrix is -1, the trace is 0, and the characteristic polynomial is $p(\lambda)=\lambda^2-1$.

Problem 8

Compute the eigenvalues for the given 2×2 matrix.

§4.6, Exercise 6. $\begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix}$.

Answer: The eigenvalues of the matrix are $\lambda_1 = -5$ and $\lambda_2 = 1$.

Solution: For any 2×2 matrix A, the eigenvalues are the roots of the characteristic polynomials, which can be found by solving the equation $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$. In this case, the characteristic polynomial of the matrix is $\lambda^2 + 4\lambda - 5$.

Problem 9

Compute the eigenvalues for the given 2×2 matrix.

§4.6, Exercise 8. $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$.

Answer: The matrix has a double eigenvalue at $\lambda = 1$.

Solution: The characteristic polynomial of the matrix is $\lambda^2 - 2\lambda + 1$. The eigenvalues are the roots of this polynomial.

Problem 10 (MATLAB)

§4.6, Exercise 15.(MATLAB) The MATLAB command `eig` computes the eigenvalues of matrices. Use `eig` to compute the eigenvalues of $A = \begin{pmatrix} 2.34 & -1.43 \\ \pi & e \end{pmatrix}$.

Answer: The eigenvalues of A are $\lambda \approx 2.5291 \pm 2.1111i$.

Solution: Enter the matrix A into MATLAB and find its eigenvalues by typing

```
A = [2.34 -1.43; pi exp(1)];
eig(A)
```