Homework 7
Math 2568

Problem 1

Find the solution to the system of differential equations $\dot{X} = CX$ satisfying $X(0) = X_0$.

§4.7, Exercise 2. $C = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$ and $X_0 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Answer: The solution to $\dot{X} = CX$ satisfying this initial condition is

$$X(t) = 3e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3e^{2t} - 2e^{-t} \\ -2e^{-t} \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 - \mathrm{tr}(C)\lambda + \det(C) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

So the eigenvalues are: $\lambda_1 = 2$ and $\lambda_2 = -1$. To find the eigenvector associated to each eigenvalue, solve the equation $(C - \lambda_j I_2)v_j = 0$ for $j = 1$ and $j = 2$. Solve

$$\left(\begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) v_1 = \begin{pmatrix} 0 & -3 \\ 0 & -3 \end{pmatrix} v_1 = 0$$

to obtain $v_1 = (1, 0)^t$ and solve

$$\left(\begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) v_2 = \begin{pmatrix} 3 & -3 \\ 0 & 0 \end{pmatrix} v_2 = 0$$

to obtain $v_2 = (1, 1)^t$. We can then write the general solution

$$X(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 = \alpha_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From this formula, find α_1 and α_2 by solving

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = X(0) = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2 \\ \alpha_2 \end{pmatrix}.$$

Solving the linear system

$$\begin{array}{ccccc} \alpha_1 & + & \alpha_2 & = & 1 \\ & & \alpha_2 & = & -2 \end{array}$$

we obtain $\alpha_1 = 3$ and $\alpha_2 = -2$ and find the solution to the differential equation.

Problem 2

§4.7, Exercise 5. Solve the initial value problem $\dot{X} = CX$ where $X_0 = e_1$ given that

(a) $X(t) = e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a solution,

(b) $\mathrm{tr}(C) = 3$, and

(c) C is a symmetric matrix.

Answer: The solution to the differential equation $\dot{X} = CX$ with the given restrictions is

$$X(t) = \frac{1}{5}e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{5}e^{4t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} e^{-t} + 4e^{4t} \\ 2e^{-t} - 2e^{4t} \end{pmatrix}.$$

Solution: First, find the matrix C using the given information: First, since C is symmetric, we can write

$$C = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

Then, we are given $\mathrm{tr}(C) = a + d = 3$, so we can rewrite C as

$$C = \begin{pmatrix} a & b \\ b & 3-a \end{pmatrix}.$$

Since $X(t) = e^{-t}(1,2)^t$ is a solution, $\lambda_1 = -1$ must be an eigenvalue of C with associated eigenvector $v_1 = (1,2)^t$. Thus $Cv_1 = \lambda_1 v_1$, or

$$\begin{pmatrix} a & b \\ b & 3-a \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a + 2b \\ b + 2(3-a) \end{pmatrix} = \begin{pmatrix} a + 2b \\ -2a + b + 6 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

This equation yields the linear system

$$\begin{array}{rcrcr} a & + & 2b & = & -1 \\ -2a & + & b & = & -8 \end{array}$$

which we can solve to obtain $a = 3$ and $b = -2$. So

$$C = \begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix}.$$

Now, find λ_2, the other root of

$$p_C(\lambda) = \lambda^2 - \mathrm{tr}(C) + \det(C) = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4).$$

Thus, the second eigenvalue of C is $\lambda_2 = 4$, and we can solve

$$(C - \lambda_2 I_2)v_2 = \left(\begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right)v_2 = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}v_2 = 0$$

to obtain $v_2 = (2, -1)^t$, the eigenvector associated to λ_2. The general solution to $\dot{X} = CX$ is

$$X(t) = \alpha_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 e^{4t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Find α_1 and α_2 by substituting the initial condition $X(0) = X_0$ into this formula:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = X(0) = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + 2\alpha_2 \\ 2\alpha_1 - \alpha_2 \end{pmatrix}.$$

Thus, $\alpha_1 = \frac{1}{5}$ and $\alpha_2 = \frac{2}{5}$, so we find the general solution.

Problem 3 (MATLAB)

With MATLAB assistance, find the solution to the system of differential equations $\dot{X} = CX$ satisfying $X(0) = X_0$.

§4.7, Exercise 6.(MATLAB) $C = \begin{pmatrix} 1.76 & 4.65 \\ 0.23 & 1.11 \end{pmatrix}$ and $X_0 = \begin{pmatrix} 0.34 \\ -0.50 \end{pmatrix}$.

Answer: The solution to the differential equation $\dot{X} = CX$ with the given initial condition is

$$X(t) \approx 0.8627 e^{2.5190t} \begin{pmatrix} -0.9869 \\ -0.1611 \end{pmatrix} - 1.2449 e^{0.3510t} \begin{pmatrix} -0.9570 \\ 0.2900 \end{pmatrix}.$$

Solution: In MATLAB, enter the matrix `C` and the vector `X0`. Then, type

```
lambda = eig(C)
```

to obtain the eigenvalues of C, which are $\lambda_1 \approx 2.5190$ and $\lambda_2 \approx 0.3510$. Find the eigenvectors v_1 and v_2 associated to λ_1 and λ_2 by typing

```
v1 = null(C - lambda(1)*eye(2))
v2 = null(C - lambda(2)*eye(2))
```

Thus, the general solution is

$$X(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 \approx \alpha_1 e^{2.5190t} \begin{pmatrix} -0.9869 \\ -0.1611 \end{pmatrix} + \alpha_2 e^{0.3510t} \begin{pmatrix} -0.9570 \\ 0.2900 \end{pmatrix}.$$

The initial condition is

$$X_0 = X(0) = \alpha_1 v_1 + \alpha_2 v_2.$$

We can solve this linear system by creating the matrix $A = (v_1 | v_2)$, and computing $A^{-1} X_0$. In MATLAB, type

```
A = [v1 v2]
alpha = inv(A)*X0
```

obtaining $\alpha_1 \approx 0.8627$ and $\alpha_2 \approx -1.2449$.

Problem 4 (MATLAB)

Find the solution to $\dot{X} = CX$ satisfying $X(0) = X_0$ in two different ways, as follows.

(a) Use pplane9 to find $X(0.5)$. **Hint**: Use the Specify a computation interval option in the PPLANE9 Keyboard input window to compute the solution to $t = 0.5$. Then use the zoom in square feature to determine an answer to three decimal places.

(b) Next use MATLAB to find the eigenvalues and eigenvectors of C and to find a closed form solution $X(t)$. Use this formula to evaluate $X(0.5)$ to three decimal places.

(c) Do the two answers agree?

§4.7, Exercise 8.(MATLAB) $C = \begin{pmatrix} 2.65 & -2.34 \\ -1.5 & -1.2 \end{pmatrix}$ and $X_0 = \begin{pmatrix} 0.5 \\ 0.1 \end{pmatrix}$.

Answer: $X(0.5) = (0.155, 0.386)^t$ and the two methods agree to three decimal places.

Solution: (a) The result of the pplane5 integration is given in Figure 8a. After zooming several times we arrive at Figure 8b. By inspection $X(0.5) = (0.155, 0.386)$.

(b) Enter the matrix C into MATLAB by typing

```
C = [2.65 -2.34; -1.5 -1.2];
```

Find the eigenvalues and eigenvectors of this matrix by typing `[V,D] = eig(C)` and obtaining

```
V =
    0.9510    0.4525
   -0.3093    0.8918
D =
    3.4112         0
         0   -1.9612
```

Therefore the general solution to this differential equation is:

$$X(t) = \alpha e^{3.4112t} \begin{pmatrix} 0.9510 \\ -0.3093 \end{pmatrix} + \beta e^{-1.9612t} \begin{pmatrix} 0.4525 \\ 0.8918 \end{pmatrix}.$$

It follows that

$$X(0) = V \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = V^{-1} X_0 = \begin{pmatrix} 0.9510 & 0.4525 \\ -0.3093 & 0.8918 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1.0 \end{pmatrix} = \begin{pmatrix} -0.0067 \\ 1.1191 \end{pmatrix}$$

The last calculation is done by typing `coeff = inv(V)*[0.5;1.0]`. Therefore, the solution to the initial value problem is:

$$X(t) = -0.0067 e^{3.4112t} \begin{pmatrix} 0.9510 \\ -0.3093 \end{pmatrix} + 1.1191 e^{-1.9612t} \begin{pmatrix} 0.4525 \\ 0.8918 \end{pmatrix}.$$

We can evaluate $X(0.5)$ in MATLAB by typing

```
X5 = coeff(1)*exp(D(1,1)*0.5)*V(:,1) + coeff(2)*exp(D(2,2)*0.5)*V(:,2)
```

and obtaining

```
X5 =
    0.1547
    0.3858
```

(c) The two answers agree to three decimal places.

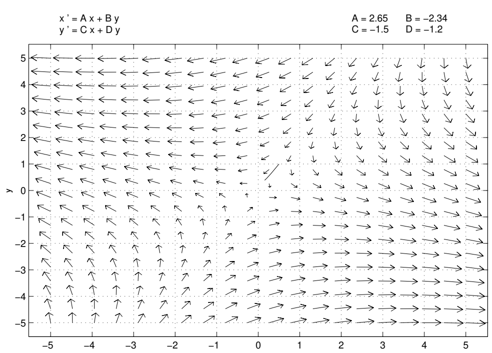

Figure 8a

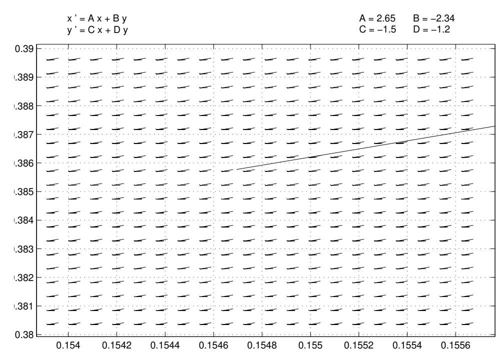

Figure 8b

Problem 5

§5.1, Exercise 3. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Let V_3 be the set of vectors $x \in \mathbb{R}^3$ such that $Ax = 0$. Verify that V_3 is a subspace of $\mathbb{R}^3$. Compare V_1 with V_3.

The set V_3 is a subspace of R^3 since the solution set to any equation $Ax = 0$ is a space. This is demonstrated by the principle of superposition introduced in Section 3.4. Also, $V_3 = V_1$.

We can show that $V_3 = V_1$ by row reducing to find the solutions to $Ax = 0$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}.$$

So all vectors in V_3 are of the form $x = s(-\frac{1}{2}, \frac{1}{2}, 1)$, where $s \in \mathbb{R}$. The vector x is an element of V_1 for each s.