Homework 8
Math 2568

Problem 1

A single equation in three variables is given. For each equation write the subspace of solutions in $\mathbb{R}^3$ as the span of two vectors in $\mathbb{R}^3$.

§5.2, Exercise 1. $4x - 2y + z = 0$.

Answer: The subspace of solutions can be spanned by the vectors $(1, 0, -4)^t$ and $(0, 1, 2)^t$.

Solution: All solutions to $4x - 2y + z = 0$ can be written in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 2y - 4x \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Problem 2

§5.2, Exercise 9. Write a system of two linear equations of the form $Ax = 0$ where A is a 2×4 matrix whose subspace of solutions in $\mathbb{R}^4$ is the span of the two vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Answer: The matrix A whose subspace of solutions in $\mathbb{R}^4$ is the span of v_1 and v_2 is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Solution: Note that all vectors x in the spanning set of v_1 and v_2 are of the form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ -a \\ b \\ -b \end{pmatrix}.$$

Therefore, $x_1 = -x_2$ and $x_3 = -x_4$. So,

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & & & & & = & 0 \\ & & & & x_3 & + & x_4 & = & 0. \end{array}$$

The matrix of this system is A.

Problem 3

§5.2, Exercise 20. Let $Ax = b$ be a system of m linear equations in n unknowns, and let $r = \text{rank}(A)$ and $s = \text{rank}(A|b)$. Suppose that this system has a unique solution. What can you say about the relative magnitudes of m, n, r, s?

Answer: The relationship of the constants is $m \geq n = r = s$.

Solution: The rank of matrix A cannot be greater than the rank of matrix $(A|b)$, since $(A|b)$ consists of A plus one column. The rank of A is the number of pivots in the row reduced matrix. $(A|b)$ can be row reduced through the same operations, and will have either the same number of pivots as A or, if there is a pivot in the last column, one more pivot than A. Since the system has a unique solution, it is consistent, and therefore $(A|b)$ cannot have a pivot in the $(n+1)^{st}$ column, so $r = \text{rank}(A) = \text{rank}(A|b) = s$.

The set of solutions is parameterized by $n - r$ parameters, where n is the number of columns of A. Since there is a unique solution, the set of solutions is parameterized by 0 parameters, so $n = r$.

The number m of rows of the matrix must be greater than or equal to n in order for the system to have a unique solution, since there must be n pivots, and each pivot must be in a separate row.

Problem 4 (MATLAB)

§5.3, Exercise 5.(MATLAB) Use row reduction to find the solutions to $Ax = 0$ where A is given in (**??**). Does your answer agree with the MATLAB answer using `null`? If not, explain why.

Answer: The solution set of $Bx = 0$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 + \frac{3}{4}x_4 \\ -3x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{3}{4} \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Solution: Row reduce B:

$$\begin{pmatrix} -4 & 0 & 4 & 3 \\ -4 & 1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & -\frac{3}{4} \\ 0 & 1 & 3 & -2 \end{pmatrix}.$$

The solution obtained by row reduction is not the same as the one obtained using `null`, but the solution vectors are linear combinations of the MATLAB solution vectors, so the answers are equivalent. By row reducing the matrix `[null(B) x]`, where $x = (-1, -3, 1, 0)$, we find that

$$\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} = -3.1009 \begin{pmatrix} 0.3225 \\ 0.8931 \\ -0.0992 \\ 0.2977 \end{pmatrix} + 1.1767 \begin{pmatrix} 0 \\ -0.1961 \\ 0.5883 \\ 0.7845 \end{pmatrix}.$$

By row reducing the matrix `[null(B) y]` where $y = (\frac{3}{4}, 2, 0, 1)$ we find that:

$$\begin{pmatrix} \frac{3}{4} \\ 2 \\ 0 \\ 1 \end{pmatrix} = 2.3257 \begin{pmatrix} 0.3225 \\ 0.8931 \\ -0.0992 \\ 0.2977 \end{pmatrix} + 0.3922 \begin{pmatrix} 0 \\ -0.1961 \\ 0.5883 \\ 0.7845 \end{pmatrix}.$$

Problem 5

§5.4, Exercise 1. Let w be a vector in the vector space V. Show that the sets of vectors $\{w, 0\}$ and $\{w, -w\}$ are linearly dependent.

To show that the set of vectors $\{w_1, w_2\}$ is linearly dependent, show that there exist nonzero a and b such that $aw_1 + bw_2 = 0$. For the set $\{w, 0\}$, if $a = 0$ and $b = 1$, then $0w + 1(0) = 0$, so the set is linearly dependent. For the set $\{w, -w\}$, if $a = 1$ and $b = 1$, then $w - w = 0$, so the set is linearly dependent.

Problem 6

§5.4, Exercise 3. Let

$$u_1 = (1, -1, 1) \quad u_2 = (2, 1, -2) \quad u_3 = (10, 2, -6).$$

Is the set $\{u_1, u_2, u_3\}$ linearly dependent or linearly independent?

Answer: The set is linearly dependent.

Solution: Let A be the matrix whose columns are u_1, u_2, and u_3. The set $\{u_1, u_2, u_3\}$ is linearly dependent if there exists a nonzero vector $r = (r_1, r_2, r_3)$ such that $r_1u_1 + r_2u_2 + r_3u_3 = 0$, that is, if the homogeneous system $Ar = 0$ has a nonzero solution. Row reduce:

$$\begin{pmatrix} 1 & 2 & 10 \\ -1 & 1 & 2 \\ 1 & -2 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, $Ar = 0$ when $r = r_3(-2, -4, 1)$. The value of r is nonzero for $r_3 \neq 0$, so the set is indeed linearly dependent. As an example, let $r_3 = 1$. Then,

$$-4u_1 - 2u_2 + u_3 = -2(1, -1, 1) - 4(2, 1, -2) + (10, 2, -6) = (0, 0, 0) = 0.$$

Problem 7

§5.4, Exercise 7. Suppose that the three vectors $u_1, u_2, u_3 \in \mathbb{R}^n$ are linearly independent. Show that the set

$$\{u_1 + u_2, u_2 + u_3, u_3 + u_1\}$$

is also linearly independent.

To show that the vectors $u_1 + u_2$, $u_2 + u_3$ and $u_3 + u_1$ are linearly independent, we assume that there exist scalars r_1, r_2, r_3 such that

$$r_1(u_1 + u_2) + r_2(u_2 + u_3) + r_3(u_3 + u_1) = 0.$$

We then prove that $r_1 = r_2 = r_3 = 0$, as follows. Use distribution to obtain

$$(r_1 + r_3)u_1 + (r_1 + r_2)u_2 + (r_2 + r_3)u_3 = 0.$$

Since the set $\{u_1, u_2, u_3\}$ is linearly independent,

$$\begin{array}{ccccccc} r_1 & & & + & r_3 & = & 0 \\ r_1 & + & r_2 & & & = & 0 \\ & & r_2 & + & r_3 & = & 0. \end{array}$$

Solving this system yields $r_1 = r_2 = r_3 = 0$, so the set $\{u_1 + u_2, u_2 + u_3, u_3 + u_1\}$ is linearly independent.

Problem 8 (MATLAB)

§5.4, Exercise 11.(MATLAB) Perform the following experiments.

(a) Use MATLAB to choose randomly three column vectors in $\mathbb{R}^3$. The MATLAB commands to choose these vectors are:

```
y1 = rand(3,1)
y2 = rand(3,1)
y3 = rand(3,1)
```

Use the methods of this section to determine whether these vectors are linearly independent or linearly dependent.

(b) Now perform this exercise five times and record the number of times a linearly independent set of vectors is chosen and the number of times a linearly dependent set is chosen.

(c) Repeat the experiment in (b) — but this time randomly choose four vectors in $\mathbb{R}^3$ to be in your set.

(a) The set of commands to perform this experiment is:

```
y1 = rand(3,1);
y2 = rand(3,1);
y3 = rand(3,1);
A = [y1 y2 y3];
rref(A)
```

If the resulting matrix is I_3, then the set is linearly independent.

(b) The most likely outcome is that all five trials result in linearly independent sets.

(c) Every trial yields a linearly dependent set of vectors.

Problem 9

§5.5, Exercise 1. Show that $\mathcal{U} = \{u_1, u_2, u_3\}$ where

$$u_1 = (1,1,0) \quad u_2 = (0,1,0) \quad u_3 = (-1,0,1)$$

is a basis for $\mathbb{R}^3$.

By Theorem 5.5.3, $\mathcal{U}$ is a basis for $\mathbb{R}^3$ if the vectors of $\mathcal{U}$ are linearly independent and span $\mathbb{R}^3$. By Lemma 5.5.4, the dimension of $\mathcal{U}$ is equal to the rank of the matrix whose rows are u_1, u_2, and u_3. Row reduce this matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\dim(\mathcal{U}) = 3 = \dim(\mathbb{R}^3)$, and we need now only show that u_1, u_2, and u_3 are linearly independent, which we can do by row reducing the matrix whose columns are the vectors of $\mathcal{U}$ as follows:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, there is no nonzero solution to the equation $\mathcal{U}r = 0$, so the vectors of $\mathcal{U}$ are linearly independent and $\mathcal{U}$ is a basis for $\mathbb{R}^3$.

Problem 10

§5.5, Exercise 2. Let $S = \text{span}\{v_1, v_2, v_3\}$ where

$$v_1 = (1, 0, -1, 0) \quad v_2 = (0, 1, 1, 1) \quad v_3 = (5, 4, -1, 4).$$

Find the dimension of S and find a basis for S.

Answer: The dimension of S is 2, and vectors v_1 and v_2 form a basis for S.

Solution: Row reduce the matrix A whose rows are v_1, v_2, and v_3. By Lemma 5.6.4, the number of nonzero rows in the reduced matrix is the dimension of S and these rows form a basis for S. So:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 5 & 4 & -1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$