Homework 7
Math 2568 Mar 20, 2019

Problem 1

You are given a pair of vectors v_1, v_2 spanning a subspace of $\mathbb{R}^3$. Decide whether that subspace is a line or a plane through the origin. If it is a plane, then compute a vector N that is perpendicular to that plane.

§5.6, Exercise 3. $v_1 = (0, 1, 0)$ and $v_2 = (4, 1, 0)$.

Answer: The span of v_1 and v_2 is a plane with normal vector $N = n_3(0, 0, 1)$, where n_3 is a nonzero scalar.

Solution: There is no scalar α such that $(0, 1, 0) = \alpha(4, 1, 0)$. Let $N = (n_1, n_2, n_3)$ be the vector perpendicular to the plane. Then:

$$\begin{array}{rcrcl} & & n_2 & = & 0 \\ 4n_1 & + & n_2 & = & 0 \end{array}$$

Solve for N by substitution to find that $n_1 = n_2 = 0$, and n_3 can be any nonzero real scalar.

Problem 2

§5.6, Exercise 4. The pairs of vectors

$$v_1 = (-1, 1, 0) \quad \text{and} \quad v_2 = (1, 0, 1)$$

span a plane P in $\mathbb{R}^3$. The pairs of vectors

$$w_1 = (0, 1, 0) \quad \text{and} \quad w_2 = (1, 1, 0)$$

span a plane Q in $\mathbb{R}^3$. Show that P and Q are different and compute the subspace of $\mathbb{R}^3$ that is given by the intersection $P \cap Q$.

Answer: The intersection of the planes is $P \cap Q = s(1, -1, 0)$ for any real scalar s.

Solution: The planes P and Q are not equal if the normal vectors P_N and Q_N point in different directions. Solving by row reduction yields $P_N = (-1, -1, 1)$ and $Q_N = (0, 0, 1)$, so $P \neq Q$.

Since P and Q are not the same plane and also are not parallel, they intersect in a line. The intersection $P \cap Q$ is the simultaneous solutions to the equations for planes P and Q, that is:

$$\begin{array}{rcrcrcl} -x & - & y & + & z & = & 0 \\ & & & & z & = & 0. \end{array}$$

Solve by row reduction or substitution to obtain $x = -y$ and $z = 0$.

Problem 3

§5.6, Exercise 6. Let

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 1 & 5 & 7 \\ 3 & 4 & 4 & 11 \end{pmatrix}.$$

(a) Find a basis for the subspace $\mathcal{C} \subset \mathbb{R}^3$ spanned by the columns of A.

(b) Find a basis for the subspace $\mathcal{R} \subset \mathbb{R}^4$ spanned by the rows of A.

(c) What is the relationship between $\dim \mathcal{C}$ and $\dim \mathcal{R}$?

Answer: (a) The vectors $(1, 2, 3)$ and $(3, 1, 4)$ form a basis for the subspace $\mathcal{C}$ of $\mathbb{R}^3$ spanned by the columns of A.

(b) The vectors $(1, 3, -1, 4)$ and $(2, 1, 5, 7)$ form a basis for the subspace $\mathbb{R}$ of $\mathbb{R}^4$ spanned by the rowss of A.

(c) $\dim \mathcal{C} = \dim \mathbb{R}$.

Solution: (a) Note that

$$\begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} = \frac{16}{5} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{7}{5} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$
$$\begin{pmatrix} 4 \\ 7 \\ 11 \end{pmatrix} = \frac{17}{5} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

So the two vectors $(1, 2, 3)$ and $(3, 1, 4)$ span $\mathcal{C}$. Since they are linearly independent, these vectors are a basis for $\mathcal{C}$ and $\dim = \mathcal{C} = 2$.

(b) Note that

$$(3, 4, 4, 11) = (1, 3, -1, 4) + (2, 1, 5, 7).$$

Therefore, $\{(1, 3, -1, 4), (2, 1, 5, 7)\}$ is a basis for $\mathcal{R}$ and $\dim \mathcal{R} = 2$

Problem 4

§5.6, Exercise 8. Let W be an infinite dimensional subspace of the vector space V. Show that V is infinite dimensional.

Let W be an infinite dimensional subspace of the vector space V. We want to show that V is infinite dimensional. Suppose that V is finite dimensional with $\dim V = n$. Then Corollary 5.6.3 states that any set of linear independent vectors in V has at most n vectors. Since W is infinite dimensional, there exists a linearly independent set of vectors in $W \subset V$ with more than n vectors. This is a contradiction and V must be infinite dimensional.

Problem 5

§5.6, Exercise 13. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix.

(a) Show that null space$(B) \subseteq$ null space(AB).

(b) Show that nullity$(B) \leq$ nullity(AB).

Solution: Note that if $x \in$ null space(B) then

$$ABx = A(Bx) = A0 = 0$$

so $x \in$ null space(AB). It follows that null space$(AB) \subseteq$ null space(B). The desired result immediately follows from Corollary 5.6.6. Alternatively, suppose by way of contradiction that dim null space$(AB) >$ dim null space(B), and let $\{v_1, \ldots, v_d\}$ be a basis for null space(AB), where $d =$ dim null space(AB). Then $\{v_1, \ldots, v_d\}$ is a set of $d >$ dim null space(B) linearly independent vectors in null space(B), which contradicts Corollary 5.6.3.

Problem 6

In Exercises 15-20 decide whether the statement is true or false, and explain your answer.

§5.6, Exercise 15. Every set of three vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$. **Answer:** False

Solution: The vectors could be linearly independent. For example $\{e_1, e_2, e_1 + e_2\}$ is not a basis for $\mathbb{R}^3$.

Problem 7

In Exercises 15-20 decide whether the statement is true or false, and explain your answer.

§5.6, Exercise 17. If $\{v_1, v_2\}$ is a basis for the plane $z = 0$ in $\mathbb{R}^3$, then $\{v_1, v_2, e_3\}$ is a basis for $\mathbb{R}^3$. **Answer:** True **Solution:** The vector e_3 is not contained in the plane $z = 0$, so Lemma 5.6.4 implies that $\{v_1, v_2, e_3\}$ is linearly independent. Therefore, $\{v_1, v_2, e_3\}$ is a basis for $\mathbb{R}^3$, because any set of three linearly independent vectors in a vector space of dimension three is a basis for that vector space.

Alternatively, $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$, so if $w \in \mathbb{R}^3$, then $w = a_1e_1 + a_2e_2 + a_3e_3$ for some scalars a_1 and a_2. It follows that $a_1e_1+a_2e_2$ is a vector in the plane $z = 0$, so there exist scalars b_1 and b_2 so $a_1e_1 + a_2e_2 = b_1v_1 + b_2v_2$. Therefore, $w = b_1v_1 + b_2v_2 + a_3e_3$, so $w \in \text{span}\{v_1, v_2, e_3\}$. Therefore, $\{v_1, v_2, e_3\}$ spans $\mathbb{R}^3$, and is a basis for $\mathbb{R}^3$ because any set of three linearly independent vectors in a vector space of dimension three is a basis for that vector space.

Problem 8

In Exercises 15-20 decide whether the statement is true or false, and explain your answer.

§5.6, Exercise 18. If $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$, the only subspaces of R^3 of dimension one are $\text{span}\{v_1\}, \text{span}\{v_2\}$, and $\text{span}\{v_3\}$. **Answer:** False **Solution:** For example, $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$ and $\text{span}\{e_1 + e_2\}$ does not equal the x, y, or z-axes.

Problem 9

In Exercises 15-20 decide whether the statement is true or false, and explain your answer.

§5.6, Exercise 19. The only subspace of $\mathbb{R}^3$ that contains finitely many vectors is $\{0\}$. **Answer:** True **Solution:** If a subspace of $\mathbb{R}^3$ contains a nonzero vector, it must contain all scalar multiples of that vector.

Problem 10

Compute the general solution for the given system of differential equations.

§6.2, Exercise 4. $\dfrac{dX}{dt} = \begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix} X.$

Answer: The general solution to the differential equation is

$$X(t) = \alpha_1 \begin{pmatrix} 2e^t\cos(2t) \\ e^t(\sin(2t)-\cos(2t)) \end{pmatrix} + \alpha_2 \begin{pmatrix} 2e^t\sin(2t) \\ -e^t(\sin(2t)+\cos(2t)) \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 - 2\lambda + 5.$$

The eigenvalues are $\lambda_1 = 1+2i$ and $\lambda_2 = 1-2i$. Then, find the eigenvector associated to λ_1 by solving the equation

$$(C-\lambda_1 I_2)v_1 = \left(\begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1+2i & 0 \\ 0 & 1+2i \end{pmatrix}\right)v_1 = \begin{pmatrix} -2-2i & -4 \\ 2 & 2-2i \end{pmatrix} v_1 = 0.$$

Solve this equation to find that

$$v_1 = \begin{pmatrix} 2 \\ -1-i \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

is an eigenvector of C. Since the eigenvalues of C are complex, we can find the general solution using (6.2.3) and (6.2.4). In this case, since $\lambda_1 = 1+2i$ is an eigenvalue, let $\sigma = 1$ and let $\tau = 2$. Then $v_1 = v + iw$, where $v = (2,-1)^t$ and $w = (0,-1)^t$. By (6.2.3),

$$X_1(t) = e^{\sigma t}(\cos(\tau t)v - \sin(\tau t)w) \quad \text{and} \quad X_2(t) = e^{\sigma t}(\sin(\tau t)v + \cos(\tau t)w)$$

are solutions to the differential equation. In this case,

$$\begin{aligned} X_1(t) &= e^t\left(\cos(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin(2t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = e^t\begin{pmatrix} 2\cos(2t) \\ \sin(2t)-\cos(2t) \end{pmatrix}. \\ X_2(t) &= e^t\left(\sin(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \cos(2t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = e^t\begin{pmatrix} 2\sin(2t) \\ -\sin(2t)-\cos(2t) \end{pmatrix}. \end{aligned}$$

The general solution consists of all linear combinations $X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t)$.

Problem 11

Compute the general solution for the given system of differential equations.

§6.2, Exercise 5. $\dfrac{dX}{dt} = \begin{pmatrix} 8 & -15 \\ 3 & -4 \end{pmatrix} X.$

Answer: The general solution to the differential equation is

$$X(t) = \alpha_1 \begin{pmatrix} 5e^{2t}\cos(3t) \\ e^{2t}(2\cos(3t)+\sin(3t)) \end{pmatrix} + \alpha_2 \begin{pmatrix} 5e^{2t}\sin(3t) \\ e^{2t}(2\sin(3t)-\cos(3t)) \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 - 4\lambda + 13.$$

The eigenvalues are $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$. Then, find the eigenvector associated to λ_1 by solving the equation

$$(C - \lambda_1 I_2)v_1 = \left(\begin{pmatrix} 8 & -15 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} 2+3i & 0 \\ 0 & 2+3i \end{pmatrix} \right) v_1 = \begin{pmatrix} 6-3i & -15 \\ 3 & -6-3i \end{pmatrix} v_1 = 0.$$

Solve this equation to find that

$$v_1 = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

is an eigenvector of C. Since the eigenvalues of C are complex, we can find the general solution using (6.2.3) and (6.2.4). In this case, since $\lambda_1 = 2 + 3i$ is an eigenvalue, let $\sigma = 2$ and let $\tau = 3$. Then $v_1 = v + iw$, where $v = (5, 2)^t$ and $w = (0, -1)^t$. By (6.2.3),

$$X_1(t) = e^{\sigma t}(\cos(\tau t)v - \sin(\tau t)w) \quad \text{and} \quad X_2(t) = e^{\sigma t}(\sin(\tau t)v + \cos(\tau t)w)$$

are solutions to the differential equation. In this case,

$$\begin{aligned} X_1(t) &= e^{2t}\left(\cos(3t)\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \sin(3t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = e^{2t}\begin{pmatrix} 5\cos(3t) \\ \sin(3t) + 2\cos(3t) \end{pmatrix}. \\ X_2(t) &= e^{2t}\left(\sin(3t)\begin{pmatrix} 5 \\ 2 \end{pmatrix} + \cos(3t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = e^{2t}\begin{pmatrix} 5\sin(3t) \\ 2\sin(3t) - \cos(3t) \end{pmatrix}. \end{aligned}$$

The general solution consists of all linear combinations $X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t)$.