Problem 1. True or false? Explain your reasoning.

(1)
$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

(2) The only solution to the differential equation
$$x'(t) = 3x(t)$$
is $x(t) = e^{3t}$.

(3) The inverse of the matrix
$$\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$
is
$$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

(4) A 2×2 matrix A is invertible if and only if its determinant is a nonzero number.

(5) The volume of the parallelogram given by the vectors $(0,0), (1,2), (4,5), (5,7)$ is 3.

Problem 2. Find a real number a so that the two vectors
$$x = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 1 \\ a \end{pmatrix}$$
meet at an angle of 90 degrees.

Problem 3. How many solutions does the equation
$$A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$
have, for the following choices of A? Explain your reasoning.

(1)
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(2)
$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3)
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Problem 4. First, determine which of the following maps are linear maps. Then for these linear maps that you have determined, writ down the matrix associated to the linear map. Explain all your reasoning.

(1)

$$L_1 : \mathbb{R}^2 \to \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \sin x \\ x + y \\ 2y + 1 \end{pmatrix}$$

(2)

$$L_2 : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + 3 \\ 2y + 1 \end{pmatrix}$$

(3)

$$L_3 : \mathbb{R}^2 \to \mathbb{R}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + y$$

Problem 5. You are asked to solve the equation $Ax = b$ for a 3×3 matrix A which has two pivots. You know that

$$x_1 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are two solutions to the equation $Ax = b$. Write down the full set of solutions for $Ax = b$.