Math 2568 Homework 10

Math 2568 Due: Wednesday, November 13, 2019

Problem 1

§6.3, Exercise 2. Use (4.6.13) in Chapter 3 to verify that the traces of similar matrices are equal.

Let A and B be similar matrices such that $A = P^{-1}BP$ for some matrix P. Then, using (4.6.13),

$$\mathrm{tr}(A) = \mathrm{tr}(P^{-1}BP) = \mathrm{tr}(BP^{-1}P) = \mathrm{tr}(B).$$

Problem 2

Determine whether or not the given matrices are similar, and why.

§6.3, Exercise 3. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ -3 & 8 \end{pmatrix}$.

Answer: Matrices A and B are not similar.

Solution: When two matrices are similar, the traces are equal. In this case, $\mathrm{tr}(A) = 5$ and $\mathrm{tr}(B) = 10$, so the matrices are not similar.

Problem 3

§6.3, Exercise 5. Let $B = P^{-1}AP$ so that A and B are similar matrices. Suppose that v is an eigenvector of B with eigenvalue λ. Show that Pv is an eigenvector of A with eigenvalue λ.

Since, A and B are similar matrices, if $Bv = \lambda v$, then

$$A(Pv) = PP^{-1}APv = PBv = \lambda(Pv).$$

Thus, Pv is an eigenvector of A with eigenvalue λ.

Problem 4

Determine whether or not the equilibrium at the origin in the system of differential equations $\dot{X} = CX$ is asymptotically stable.

§6.4, Exercise 1. $C = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$.

Answer: The origin is not asymptotically stable.

Solution: Theorem 6.4.1 states that the origin is a stable equilibrium only if all eigenvectors have negative real part. The characteristic polynomial of C is $p_C(\lambda) = \lambda^2 - 2\lambda - 5$. Thus, the eigenvalues are $\lambda_1 = 1 + \sqrt{6}$ and $\lambda_2 = 1 - \sqrt{6}$. Since $\lambda_1 > 0$, the origin is not stable.

Problem 5

Determine whether the equilibrium at the origin in the system of differential equations $\dot{X} = CX$ is a sink, a saddle or a source.

§6.4, Exercise 5. $C = \begin{pmatrix} 3 & 5 \\ 0 & -2 \end{pmatrix}$.

Answer: The origin of the system $\dot{X} = CX$ is a saddle.

Solution: The characteristic polynomial of C is $p_C(\lambda) = \lambda^2 - \lambda - 6$. So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$. Since one eigenvalue is negative and one is positive, the origin is a saddle.

Problem 6

compute the determinants of the given matrix.

§7.1, Exercise 1. $A = \begin{pmatrix} -2 & 1 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

Answer: The determinant of the matrix is -28.

Solution: Expand along the third column, obtaining:

$$\det \begin{pmatrix} -2 & 1 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 2 \end{pmatrix} = 2 \det \begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix} = 2(-14) = -28.$$

Problem 7

compute the determinants of the given matrix.

§7.1, Exercise 3. $C = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 0 \\ -3 & 2 & -2 & 0 & 0 \\ 1 & 1 & -1 & 2 & 4 \\ 0 & 2 & 3 & -1 & -3 \end{pmatrix}$.

Answer: The determinant of the matrix is 14.

Solution: Using Lemma 7.1.9, compute

$$\begin{aligned} \det(A) &= \det\begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -3 & 2 & -2 \end{pmatrix} \det\begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix} \\ &= \big(2(-2)(-2) + 1 \cdot 3(-3) + (-1)1 \cdot 2 - (-1)(-2)(-3) - 1 \cdot 1(-2) \\ &\quad -2 \cdot 3 \cdot 2\big)(-2) \\ &= (-7)(-2) \\ &= 14. \end{aligned}$$

Problem 8

Use row reduction to compute the determinant of the given matrix.

§7.1, Exercise 6. $A = \begin{pmatrix} -1 & -2 & 1 \\ 3 & 1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$.

Answer: The determinant is $\det(A) = 18$.

Solution: Compute by row reduction as follows:

$$\begin{aligned} \det\begin{pmatrix} -1 & -2 & 1 \\ 3 & 1 & 3 \\ -1 & 1 & 1 \end{pmatrix} &= -\det\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 6 \\ 0 & 3 & 0 \end{pmatrix} \\ &= 3\det\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 6 \end{pmatrix} \\ &= 3\det\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 18. \end{aligned}$$

Problem 9

Determine the characteristic polynomial and the eigenvalues of the given matrices.

§7.2, Exercise 2. $B = \begin{pmatrix} 2 & 1 & -5 & 2 \\ 1 & 2 & 13 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

Answer: The characteristic polynomial of B is $p_B(\lambda) = \lambda^4 - 8\lambda^3 + 23\lambda^2 - 28\lambda + 12$. The matrix B has single eigenvalues at $\lambda = 1$ and $\lambda = 3$ and a double eigenvalue at $\lambda = 2$.

Solution: Using Lemma 7.1.9, compute:

$$\begin{aligned} p_B(\lambda) &= \det(B - \lambda I_3) \\ &= \det \begin{pmatrix} 2-\lambda & 1 & -5 & 2 \\ 1 & 2-\lambda & 13 & 2 \\ 0 & 0 & 3-\lambda & -1 \\ 0 & 0 & 1 & 1-\lambda \end{pmatrix} \\ &= \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \det \begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} \\ &= ((2-\lambda)^2 - 1)((3-\lambda)(1-\lambda) + 1) \\ &= (\lambda - 3)(\lambda - 1)(\lambda - 2)^2. \end{aligned}$$

Problem 10

§7.2, Exercise 3. Find a basis for the eigenspace of

$$A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

corresponding to the eigenvalue $\lambda = 2$.

Answer: A basis for the eigenspace of A corresponding to the eigenvalue $\lambda = 2$ is:

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Solution: First, find all eigenvectors of A with eigenvalue $\lambda = 2$, that is, all vectors $v = (v_1, v_2, v_3)$ such that $(A - 2I_3)v = 0$. Solve the system

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

All solutions v to this system satisfy $v_1 = v_3 - v_2$. Thus:

$$v = \begin{pmatrix} v_3 - v_2 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the vectors $(-1,1,0)^t$ and $(1,0,1)^t$ form a basis for this eigenspace.

Problem 11

§7.2, Exercise 4. Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(a) Verify that the characteristic polynomial of A is $p_\lambda(A) = (\lambda-1)(\lambda+2)^2$.

(b) Show that $(1,1,1)$ is an eigenvector of A corresponding to $\lambda = 1$.

(c) Show that $(1,1,1)$ is orthogonal to every eigenvector of A corresponding to the eigenvalue $\lambda = -2$.

(a) Find the characteristic polynomial by solving

$$\begin{aligned}
p_A(\lambda) &= \det(A - \lambda I_3) \\
&= \begin{pmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{pmatrix} \\
&= (-1-\lambda)\det\begin{pmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} - \det\begin{pmatrix} 1 & 1 \\ 1 & -1-\lambda \end{pmatrix} + \\
&\quad \det\begin{pmatrix} 1 & 1 \\ -1-\lambda & 1 \end{pmatrix} \\
&= -(\lambda^3 + 3\lambda - 4) \\
&= -(\lambda-1)(\lambda+2)^2.
\end{aligned}$$

(b) Verify by computation:

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(c) Find the space of eigenvectors $v = (v_1, v_2, v_3)$ corresponding to $\lambda = -2$ by solving $(A - \lambda I_3)v = 0$ for $\lambda = -2$. That is, solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

to obtain

$$v = v_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Then compute

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0.$$

Since $(-1,1,0)^t$ and $(-1,0,1)^t$ form a basis for the space of eigenvectors of A corresponding to $\lambda = -2$ and since $(1,1,1)^t$ is orthogonal to these vectors, $(1,1,1)^t$ is orthogonal to every eigenvector of A corresponding to $\lambda = -2$.

Problem 12

§7.2, Exercise 5. Let

$$A = \begin{pmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{pmatrix}$$

One of the eigenvalues of A is -1. Find the other eigenvalues of A.

Answer: The other two eigenvalues of A are $-1 \pm \sqrt{2}$.

Solution: The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I_3) &= \begin{pmatrix} -\lambda & -3 & -2 \\ 1 & -4-\lambda & -2 \\ -3 & 4 & 1-\lambda \end{pmatrix} \\ &= -\lambda^3 - 3\lambda^2 - \lambda + 1 \end{aligned}$$

Since -1 is an eigenvalue of A, $\lambda + 1$ is a factor of the characteristic polynomial. We can solve the equation

$$\begin{aligned} -\lambda^3 - 3\lambda^2 - \lambda + 1 &= -(\lambda+1)(\lambda^2 + a\lambda + b) \\ &= -\lambda^3 - (a+1)\lambda^2 - (a+b)\lambda + b \end{aligned}$$

for

$$-\lambda^3 - 3\lambda^2 - \lambda + 1 = -(\lambda+1)(\lambda^2 + 2\lambda - 1).$$

Using the quadratic formula, the roots of $\lambda^2 + 2\lambda - 1$ are $-1 \pm \sqrt{2}$. Therefore, the eigenvalues are $-1, -1+\sqrt{2}, -1-\sqrt{2}$.

Problem 13

§7.2, Exercise 7. Find the characteristic polynomial and the eigenvalues of

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}.$$

Find eigenvectors corresponding to each of the three eigenvalues.

Answer: The characteristic polynomial of A is $p_A(\lambda) = -(\lambda^3 + 5\lambda^2 + 6\lambda) = -\lambda(\lambda + 2)(\lambda + 3)$. The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -2$, and $\lambda_3 = -3$, with eigenvectors $v_1 = (0, 1, -1)^t$, $v_2 = (2, -1, 0)^t$, and $v_3 = (1, 0, -1)^t$, respectively.

Solution: The eigenvalues are the roots of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I_3)$. The eigenvectors are vectors v such that $Av = \lambda v$, where λ is an eigenvalue of A. Find them by solving the system $(A - \lambda I_3)v = 0$.