Math 2568 Homework 11

Math 2568 Due: Monday, November 25, 2019

Problem 1

§7.2, Exercise 6. Consider the matrix $A = \begin{pmatrix} 8 & 5 \\ -10 & -7 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Show that the eigenvectors found in (a) form a basis for $\mathbb{R}^2$.

(c) Find the coordinates of the vector (x_1, x_2) relative to the basis in part (b).

(a) **Answer:** The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -2$, with corresponding eigenvectors $v_1 = (1, -1)^t$ and $v_2 = (1, -2)^t$, respectively.

Solution: The characteristic polynomial is $p_A(\lambda) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$. Then, solve $Av = \lambda v$ for each eigenvalue to find the corresponding eigenvectors.

(b) Two linearly independent vectors in $\mathbb{R}^2$ form a basis for $\mathbb{R}^2$. Note that $v_1 \neq \alpha v_2$ for any scalar α. Therefore, v_1 and v_2 form a basis for $\mathbb{R}^2$.

(c) **Answer:** The coordinates of (x_1, x_2) in the basis $\{v_1, v_2\}$ are $(2x_1 + x_2, -x_1 - x_2)$.

Solution: Find α_1 and α_2 such that $\alpha_1 v_1 + \alpha_2 v_2 = (x_1, x_2)^t$. That is, solve:

$$\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

to obtain $\alpha_1 = 2x_1 + x_2$ and $\alpha_2 = -x_1 - x_2$.

Problem 2

§7.2, Exercise 8. Let A be an $n \times n$ matrix. Suppose that

$$A^2 + A + I_n = 0.$$

Prove that A is invertible.

We are given $A^2 + A + I_n = 0$. Therefore, $I_n = -A^2 - A = A(-A - I_n)$. Thus, $A^{-1} = -A - I_n$ exists.

Problem 3

§7.2, Exercise 12. When n is odd show that every real $n \times n$ matrix has a real eigenvalue.

By Theorem 7.2.4, every $n \times n$ matrix has exactly n eigenvalues, which are either real or complex conjugate pairs. Since complex eigenvalues are paired, the number of complex eigenvalues must be even. Since n is odd, there can be no more than $n - 1$ complex eigenvalues; so the matrix has at least one real eigenvalue.

Problem 4

§7.3, Exercise 2. The eigenvalues of

$$A = \begin{pmatrix} -1 & 2 & -1 \\ 3 & 0 & 1 \\ -3 & -2 & -3 \end{pmatrix}$$

are $2, -2, -4$. Find the eigenvectors of A for each of these eigenvalues and find a 3×3 invertible matrix S so that $S^{-1}AS$ is diagonal.

The eigenvectors of A are $v_1 = (1, 1, -1)^t$ associated to eigenvalue $\lambda_1 = 2$; $v_2 = (1, -1, -1)^t$ associated to eigenvalue $\lambda_2 = -2$; and $v_3 = (1, -1, 1)^t$ associated to eigenvalue $\lambda_3 = -4$. Find these vectors by solving $(A - \lambda I_3)v = 0$ for each eigenvalue λ. The matrix S such that $S^{-1}AS$ is diagonal is

$$S = (v_1|v_2|v_3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Problem 5

§7.3, Exercise 3. Let

$$A = \begin{pmatrix} -1 & 4 & -2 \\ 0 & 3 & -2 \\ 0 & 4 & -3 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A, and find an invertible matrix S so that $S^{-1}AS$ is diagonal.

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. The eigenvector associated to λ_1 is $v_1 = (1, 1, 1)^t$. There are two eigenvectors associated to λ_2: $v_2 = (1, 0, 0)^t$

and $v_3 = (0,1,2)^t$.

$$S = (v_1|v_2|v_3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

Problem 6

§7.3, Exercise 4. Let A and B be similar $n \times n$ matrices.

(a) Show that if A is invertible, then B is invertible.

(b) Show that $A + A^{-1}$ is similar to $B + B^{-1}$.

(a) Let $B = P^{-1}AP$ be a matrix similar to some invertible matrix A. Then

$$B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P.$$

Since A^{-1} exists, B^{-1} exists also.
(b) If $B = P^{-1}AP$, then $B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}P$. Therefore,

$$B + B^{-1} = P^{-1}AP + P^{-1}A^{-1}P = P^{-1}(A + A^{-1})P$$

since matrix multiplication is associative. Therefore, $A + A^{-1}$ is similar to $B + B^{-1}$.

Problem 7

§7.3, Exercise 6. Let A be an $n \times n$ real diagonalizable matrix. Show that $A + \alpha I_n$ is also real diagonalizable.

Let S be a matrix such that $D = S^{-1}AS$ is a diagonal matrix. Then

$$S^{-1}(A + \alpha I_n)S = S^{-1}AS + S^{-1}(\alpha I_n)S = D + \alpha I_n.$$

The matrices D and αI_n are both diagonal; so $D + \alpha I_n$ is also diagonal. Therefore, $A + \alpha I_n$ is diagonalizable.

Problem 8

§7.3, Exercise 9. Let A be an $n \times n$ matrix all of whose eigenvalues equal ± 1. Show that if A is diagonalizable, the $A^2 = I_n$.

Since A is diagonalizable, there is an invertible matrix S such that $S^{-1}AS$ is diagonal. The diagonal entries of $S^{-1}AS$ are the eigenvalues of A; that is, the diagonal entries equal ± 1. Therefore, $(S^{-1}AS)^2 = I_n$. But $(S^{-1}AS)^2 = S^{-1}A^2S$. Therefore, $S^{-1}A^2S = I_n$ which implies that $A^2 = I_n$.

Problem 9

§8.1, Exercise 1. Use Theorem 8.1.2 and (8.1.3) to construct matrix of a linear mapping L from $\mathbb{R}^3$ to $\mathbb{R}^2$ with $L(v_i) = w_i$, $i = 1, 2, 3$, where

$$v_1 = (1, 0, 2) \quad v_2 = (2, -1, 1) \quad v_3 = (-2, 1, 0)$$

and

$$w_1 = (-1, 0) \quad w_2 = (0, 1) \quad w_3 = (3, 1).$$

Solution: Compute A, the matrix of L, using Equation (8.1.3):

$$A = (w_1^t|w_2^t|w_3^t)(v_1^t|v_2^t|v_3^t)^{-1} = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -7 & -11 & 3 \\ -4 & -7 & 2 \end{pmatrix}.$$

Problem 10

§8.1, Exercise 2. Let $\mathcal{P}_n$ be the vector space of polynomials $p(t)$ of degree less than or equal to n. Show that $\{1, t, t^2, \ldots, t^n\}$ is a basis for $\mathcal{P}_n$.

To show that the set $\{1, t, t^2, \ldots, t^n\}$ is a basis for $\mathcal{P}_n$, we must show that the $n+1$ polynomials are linearly independent and span $\mathcal{P}_n$. The polynomials are independent because the general polynomial of degree n:

$$\alpha_1 + \alpha_2 t + \alpha_3 t^2 + \cdots + \alpha_{n+1} t^n$$

is identically 0 for all values of t only when $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 0$. The polynomials span $\mathcal{P}_n$ because every polynomial $p(t)$ of degree n has the form

$$p(t) = \beta_1 + \beta_2 t + \cdots + \beta_{n+1} t^n$$

which is a linear combination of the polynomials $\{1, t, t^2, \ldots, t^n\}$ for any $p(t)$ in $\mathcal{P}_n$.

Problem 11

§8.1, Exercise 3. Show that

$$\frac{d}{dt} : \mathcal{P}_3 \to \mathcal{P}_2$$

is a linear mapping.

Let $\frac{d}{dt}$ be a transformation that maps $p(t) \mapsto \frac{d}{dt}p(t)$. For $p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$, then $\frac{d}{dt}p(t) = p_1 + 2p_2 t + 3p_3 t^2$, so $\frac{d}{dt}$ is a mapping $\mathcal{P}_3 \to \mathcal{P}_2$. From calculus, we know that, for any functions f and g:

$$\frac{d}{dt}(f+g)(t) = \frac{d}{dt}f(t) + \frac{d}{dt}g(t),$$

and that, for any scalar c:

$$\frac{d}{dt}(cf)(t) = c\frac{d}{dt}f(t).$$

Let f and g be elements of $\mathcal{P}_3$. Then $\frac{d}{dt} : \mathcal{P}_3 \to \mathcal{P}_2$ is a linear mapping.

Problem 12

§8.1, Exercise 4. Show that

$$L(p) = \int_0^t p(s)ds$$

is a linear mapping of $\mathcal{P}_2 \to \mathcal{P}_3$.

Let $p(t) = p_1 + p_2 t + p_3 t^2$. Then the transformation L maps $p(t) \mapsto L(p(t)) = p_1 t + \frac{1}{2}p_2 t^2 + \frac{1}{3}p_3 t^3$, so L is indeed a mapping $\mathcal{P}_2 \to \mathcal{P}_3$. We know from calculus that, for any functions f and g:

$$\int_0^t (f+g)(t) = \int_0^t f(t) + \int_0^t g(t)$$

And, for any scalar $c \in \mathbb{R}$,

$$\int_0^t (cf)(t) = c\int_0^t f(t).$$

Let f and g be elements of $\mathcal{P}_2$. Then L is a linear mapping.