Math 2568 Homework 12
Math 2568 Due: Monday, December 4, 2019

Problem 1

§8.2, Exercise 3. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 1 & 2 & -1 & 3 \end{pmatrix}.$$

(a) Find a basis for the row space of A and the row rank of A.

(b) Find a basis for the column space of A and the column rank of A.

(c) Find a basis for the null space of A and the nullity of A.

(d) Find a basis for the null space of A^t and the nullity of A^t.

(a) **Answer:** The vectors $(1, 0, 1, 0)$, $(0, 1, -1, 0)$ and $(0, 0, 0, 1)$ form a basis for the row space of A, and the row rank of A is 3.

Solution: Row reduce A:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 1 & 2 & -1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) **Answer:** The column rank of A is 3, and the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ form a basis for the column space of A.

Solution: Row reduce A^t:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) **Answer:** The vector $(-1, 1, 1, 0)$ is a basis for the null space. Since one vector forms the basis, the nullity of A is 1.

Solution: Solve $Ax = 0$ by row reducing A, which we have already done.

(d) **Answer:** The null space is trivial and the nullity of A^t is 0.

Solution: Find a basis by solving $A^t x = 0$ by row reduction. The row reduced matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

implies $x = (0, 0, 0)$.

Problem 2

§8.3, Exercise 2. Let $w_1 = (1, 2)$ and $w_2 = (0, 1)$ be a basis for $\mathbb{R}^2$. Let $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

in standard coordinates. Find the matrix $[L]_{\mathcal{W}}$.

Solution: From Section 8.3,

$$[L]_{\mathcal{W}} = (w_1^t | w_2^t)^{-1} L(w_1^t | w_2^t) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -9 & -2 \end{pmatrix}.$$

Problem 3

§8.3, Exercise 3. Let E_{ij} be the 2×3 matrix whose entry in the i^{th} row and j^{th} column is 1 and all of whose other entries are 0.

(a) Show that

$$\mathcal{V} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

is a basis for the vector space of 2×3 matrices.

(b) Compute $[A]_{\mathcal{V}}$ where

$$A = \begin{pmatrix} -1 & 0 & 2 \\ 3 & -2 & 4 \end{pmatrix}.$$

(a) By Theorem 5.5.3, the subset $\mathcal{V}$ is a basis for the vector space of 2×3 matrices if the vectors of $\mathcal{V}$ are linearly independent and span the vector space. Let

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}.$$

We show that B is in the span of $\mathcal{V}$ by noting that $B = b_{11}E_{11} + b_{12}E_{12} + b_{13}E_{13} + b_{21}E_{21} + b_{22}E_{22} + b_{23}E_{23}$. To show that the matrices E_{ij} are linearly independent, suppose $b_{11}E_{11} + b_{12}E_{12} + b_{13}E_{13} + b_{21}E_{21} + b_{22}E_{22} + b_{23}E_{23} = 0$. Then,

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = 0,$$

so $b_{ij} = 0$. Therefore, V is a basis for the given vector space.

(b) **Answer:** $[A]_{\mathcal{V}} = (-1, 0, 2, 3, -2, 4)$.

Solution: Compute $A = -E_{11} + 2E_{13} + 3E_{21} - 2E_{22} + 4E_{23}$.

Problem 4

§8.3, Exercise 4. Verify that $\mathcal{V} = \{p_1, p_2, p_3\}$ where

$$p_1(t) = 1 + 2t, \quad p_2(t) = t + 2t^2, \quad \text{and} \quad p_3(t) = 2 - t^2,$$

is a basis for the vector space of polynomials $\mathcal{P}_2$. Let $p(t) = t$ and find $[p]_{\mathcal{V}}$.

Answer: If $p(t) = t$, then $[p]_{\mathcal{V}} = (\frac{4}{7}, -\frac{1}{7}, -\frac{2}{7})$.

Solution: In order to verify that $\mathcal{V}$ is a basis for $\mathcal{P}_2$, first show that the set $\{1, t, t^2\}$ is a basis for $\mathcal{P}_2$. To prove this, note that any polynomial in $\mathcal{P}_2$ can be written as $p = \alpha_1 + \alpha_2 t + \alpha_3 t^2$, so the set spans $\mathcal{P}_2$. Also, $0 = \alpha_1 + \alpha_2 t + \alpha_3 t^2$ if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so the set is linearly independent.

The set $\{1, t, t^2\}$ has dimension 3 and is a basis for $\mathcal{P}_2$. Therefore, any linearly independent set of three vectors in $\mathcal{P}_2$ will span $\mathcal{P}_2$. So we need only show that $\mathcal{V}$ is a linearly independent set, which we do by solving:

$$\begin{aligned} 0 &= \alpha_1 p_1(t) + \alpha_2 p_2(t) + \alpha_3 p_3(t) \\ &= \alpha_1(1 + 2t) + \alpha_2(t + 2t^2) + \alpha_3(2 - t^2) \\ &= (\alpha_1 + 2\alpha_3) + (2\alpha_1 + \alpha_2)t + (2\alpha_2 - \alpha_3). \end{aligned}$$

This equation is identically 0 if

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0.$$

The only solution to this system is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so the elements are linearly independent and $\mathcal{V}$ is a basis for $\mathcal{P}_2$.

Let $p(t) = t$. Then find this vector $[p]_{\mathcal{V}}$ by solving $p(t) = \alpha_1 p_1(t) + \alpha_2 p_2(t) + \alpha_3 p_3(t)$. That is,

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Solve by substitution to obtain $\alpha_1 = \frac{4}{7}$, $\alpha_2 = -\frac{1}{7}$, and $\alpha_3 = -\frac{2}{7}$.

Problem 5

§9.1, Exercise 1. Find an orthonormal basis for the solutions to the linear equation

$$2x_1 - x_2 + x_3 = 0.$$

Answer: The vectors $w_1 = \frac{1}{\sqrt{3}}(1,1,-1)$ and $w_2 = \frac{1}{\sqrt{2}}(0,1,1)$ form an orthonormal basis for the solution set.

Solution: Find one vector which is a solution to the equation, for example $(1,1,-1)$. Then, divide the vector by its length, obtaining the unit vector w_1. By inspection, find a vector v_2 which satisfies both the given equation and $w_1 \cdot v_2 = 0$. Then set $w_2 = \frac{1}{||v_2||}v_2$.

Problem 6

§9.1, Exercise 2.

(a) Find the coordinates of the vector $v = (1,4)$ in the orthonormal basis $\mathcal{V}$

$$v_1 = \frac{1}{\sqrt{5}}(1,2) \quad \text{and} \quad v_2 = \frac{1}{\sqrt{5}}(2,-1).$$

(b) Let $A = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$. Find $[A]_{\mathcal{V}}$.

(a) By Theorem 9.1.3:

$$[v]_{\mathcal{V}} = (v \cdot v_1, v \cdot v_2) = \frac{1}{\sqrt{5}}(9,-2).$$

(b) By (9.1.1):

$$[A]_{\mathcal{V}} = \begin{pmatrix} Av_1 \cdot v_1 & Av_2 \cdot v_1 \\ Av_1 \cdot v_2 & Av_2 \cdot v_2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}.$$

Problem 7

§9.4, Exercise 1. Let

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

be the general real 2×2 symmetric matrix.

(a) Prove directly using the discriminant of the characteristic polynomial that A has real eigenvalues.

(b) Show that A has equal eigenvalues only if A is a scalar multiple of I_2.

(a) We can calculate the discriminant D of matrix A using (4.6.10):

$D = \mathrm{tr}(A)^2 - 4\det(A) = (a+d)^2 - 4(ad-b^2) = a^2+2ad+b^2-4ad+4b^2 = (a-d)^2+4b^2.$

Therefore, $D \geq 0$ for all real symmetric matrices A. The eigenvalues of A are

$$\lambda_1 = \frac{(a+d)+\sqrt{D}}{2} \quad \text{and} \quad \lambda_2 = \frac{(a+d)-\sqrt{D}}{2}.$$

Thus, λ_1 and λ_2 are real since D is non-negative.

(b) Matrix A has equal eigenvalues only if $D = 0$. According to the computation in (a) of this problem, $D = 0$ only if $a = d$ and $b = 0$. Therefore, if $\lambda_1 = \lambda_2$, then A is a multiple of I_2.

Problem 8

§9.4, Exercise 2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A and verify that the eigenvectors are orthogonal.

Answer: The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -3$, with respective eigenvectors $v_1 = (2, 1)$ and $v_2 = (1, -2)$.

Solution: Indeed, $v_1 \cdot v_2 = (2, 1) \cdot (1, -2) = 0$, so the eigenvectors are orthogonal.

Problem 9

Decide whether or not the given matrix is orthogonal.

§9.4, Exercise 5. $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$

The matrix is orthogonal, since

$$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} = I_3.$$

Problem 10

Decide whether or not the given matrix is orthogonal.

§9.4, Exercise 6. $\begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}$.

The matrix is orthogonal, since

$$\begin{pmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{pmatrix} \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix} = I_2.$$

Problem 11

Decide whether or not the given matrix is orthogonal.

§9.4, Exercise 7. $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix}$.

The matrix is not orthogonal, since all orthogonal matrices are square.

Problem 12

§9.4, Exercise 8. Let Q be an orthogonal $n \times n$ matrix. Show that Q preserves the length of vectors, that is

$$\|Qv\| = \|v\| \quad \text{for all } v \in \mathbb{R}^n.$$

For this proof, we use the fact that, if C is a complex matrix, then $(Cv) \cdot w = v \cdot (\overline{C}^t w)$. This was shown in the discussion of Hermitian inner products in Section 9.4. In particular, since Q is a real matrix, $(Qv) \cdot w = v \cdot (Q^t w)$. Therefore, since Q is orthogonal:

$$||Qv||^2 = (Qv) \cdot (Qv) = (Q^t Qv) \cdot v = (I_n v) \cdot v = v \cdot v = ||v||^2.$$