Math 2568 Homework 3
Math 2568 Due: Monday, September 9, 2019

Problem 1

Row reduce the given matrix to reduced echelon form by hand and determine its rank.

§2.4, Exercise 1. $A = \begin{pmatrix} 1 & 2 & 1 & 6 \\ 3 & 6 & 1 & 14 \\ 1 & 2 & 2 & 8 \end{pmatrix}$

The reduced echelon form of the matrix is:

$$A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The rank of A is two, since the reduced echelon matrix has two nonzero rows.

Problem 2

§2.4, Exercise 3.

How many solutions does the equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

have for the following choices of A. Explain your reasoning.

(a) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Answer: (a) no solutions; (b) 1 solution; (c) infinitely many solutions

Solution:

(a) The third equation in this system is $0 = 1$ and that is inconsistent.

(b) A is invertible; so there is 1 solution

(c) Reduce the augmented matrix to echelon form. The rank of A is 2 as is the rank of the augmented matrix. Therefore, there exists a one-parameter set of solutions.

Problem 3

§2.4, Exercise 6.

Consider the system of equations

$$\begin{aligned} x_1 + 3x_3 &= 1 \\ -x_1 + 2x_2 - 3x_3 &= 1 \\ 2x_2 + ax_3 &= b \end{aligned}$$

For which real numbers a and b does the system have no solutions, a unique solution, or infinitely many solutions? Your answer should subdivide the ab-plane into three disjoint sets.

Answer: Unique solutions occur when $a \neq 0$; no solution occurs when $a = 0$ and $b \neq 2$; and infinitely many solutions exist when $a = 0$ and $b = 2$.

Solution: Use row reduction on the augmented matrix to obtain

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ -1 & 2 & -3 & 1 \\ 0 & 2 & a & b \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & a & b \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & a & b-2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a & b-2 \end{pmatrix}$$

If $a \neq 0$ the system has a unique solution. If $a = 0$ we obtain the echelon form matrix

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & b-2 \end{pmatrix}$$

There are no solutions if $b \neq 2$ and infinitely many solutions if $b = 2$.

Problem 4

§2.4, Exercise 14. Prove that the rank of an $m \times n$ matrix A is less than or equal to the minimum of m and n.

Suppose A is row equivalent to the $m \times n$ reduced row echelon matrix E. The rank of A equals the number of pivots in E. Since there is at most 1 pivot in each column, the number of pivots is less than or equal to the number of columns n of E. Similarly, since each row of E contains at most one pivot, the number of pivots in E is at most the number m of rows of E. It follows that the rank of A is less than or equal to both m and n and hence the minimum of m and n.

Problem 5

§3.1, Exercise 1. Let

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Compute Ax.

$$Ax = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 6-2 \\ -3-8 \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \end{pmatrix}$$

Problem 6

§3.1, Exercise 7. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Denote the columns of the matrix A by

$$A_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \quad \cdots \quad A_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Show that the matrix vector product Ax can be written as

$$Ax = x_1A_1 + x_2A_2 + \cdots + x_nA_n,$$

where x_jA_j denotes scalar multiplication (see Chapter 1).

Compute Ax directly:

$$Ax = \begin{pmatrix} x_1a_{11} + x_2a_{12} + \cdots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + \cdots + x_na_{2n} \\ \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + \cdots + x_na_{mn} \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

So, it is indeed true that $Ax = x_1A_1 + x_2A_2 + \cdots + x_nA_n$.

Problem 7

§3.1, Exercise 9. Write the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 4 \\ 6x_1 - 5x_3 &= 1 \end{aligned}$$

in the matrix form $Ax = b$.

$$\begin{pmatrix} 2 & 3 & -2 \\ 6 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Problem 8

§3.1, Exercise 10. Find all solutions to

$$\begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 1 & 5 & 7 \\ 3 & 4 & 4 & 11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 14 \\ 17 \\ 31 \end{pmatrix}.$$

Answer: All solutions are of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{37}{5} - \frac{16}{5}x_3 - \frac{17}{5}x_4 \\ \frac{11}{5} + \frac{7}{5}x_3 - \frac{1}{5}x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

where x_3 and x_4 are free parameters.

Solution: Create the augmented matrix

$$\left(\begin{array}{rrrr|r} 1 & 3 & -1 & 4 & 14 \\ 2 & 1 & 5 & 7 & 17 \\ 3 & 4 & 4 & 11 & 31 \end{array}\right)$$

which can be row reduced to

$$\left(\begin{array}{rrrr|r} 1 & 0 & \frac{16}{5} & \frac{17}{5} & \frac{37}{5} \\ 0 & 1 & -\frac{7}{5} & \frac{1}{5} & \frac{11}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

yielding the desired solution.

Problem 9

§3.1, Exercise 13. Is there an upper triangular 2×2 matrix A such that

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}? \tag{1}$$

Is there a symmetric 2×2 matrix A satisfying (1)?

Answer: There is no 2×2 upper triangular matrix A that satisfies equation (1), but any symmetric matrix A of the form

$$A = \begin{pmatrix} 1 & 2 \\ 2 & a_{22} \end{pmatrix},$$

where a_{22} is a real number, satisfies (1).

Solution: Let A be the upper triangular matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}.$$

The resulting matrix equation

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

yields the linear equations

$$\begin{aligned} a_{11} &= 1 \\ 0 &= 2. \end{aligned}$$

The second equation is inconsistent, so there is no solution.

Then let A be the symmetric matrix

$$\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array} \right).$$

Write the matrix equation

$$\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array} \right) \left(\begin{array}{c} 1 \\ 0 \end{array} \right) = \left(\begin{array}{c} 1 \\ 2 \end{array} \right),$$

from which we obtain the consistent linear system

$$\begin{array}{rcl} a_{11} & = & 1 \\ a_{12} & = & 2. \end{array}$$