Math 2568 Homework 4
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Problem 1

§2.4, Exercise 15. Consider the augmented matrix

$$A = \left(\begin{array}{cc|c} 1 & -r & 1 \\ r & -1 & 1 \end{array} \right)$$

where r is a real parameter.

1 Find all r so that $\text{rank}(A) = 2$.

2 Find all r for which the corresponding linear system has

(a) no solution,

(b) one solution, and

(c) infinitely many solutions.

Solution: Subtracting r times the first row of A from the second row of that matrix yields

$$\left(\begin{array}{cc|c} 1 & -r & 1 \\ 0 & r^2-1 & 1-r \end{array} \right) = \left(\begin{array}{cc|c} 1 & -r & 1 \\ 0 & (r+1)(r-1) & 1-r \end{array} \right)$$

So the reduced row echelon form of A is

$$\text{RREF}(A) = \begin{cases} \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{1+r} \\ 0 & 1 & -\frac{1}{1+r} \end{array} \right) & r \neq \pm 1 \\ \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array} \right) & r = 1 \\ \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) & r = -1 \end{cases}$$

1 $\text{rank}(A) = 2$ if $r \neq 1$.

2 The linear system corresponding to the augmented matrix A has

(a) no solution if $r = -1$,

(b) one solution if $r \neq \pm 1$, and

(c) infinitely many solutions if $r = 1$.

Problem 2

§3.1, Exercise 12. Let A be a 2×2 matrix. Find A so that

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
$$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Answer: The equations are valid when

$$A = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}.$$

Solution: Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3. \end{pmatrix}$$

These matrix equations yield the linear system

$$\begin{array}{ccccccccc} a_{11} & + & a_{12} & & & & & = & 2 \\ & & & & a_{21} & + & a_{22} & = & -1 \\ a_{11} & - & a_{12} & & & & & = & 4 \\ & & & & a_{21} & - & a_{22} & = & 3, \end{array}$$

which can be written as an augmented matrix and row-reduced to yield the values a_{ij}:

$$\left(\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right).$$

Problem 3

Determine whether the given transformation is linear.

§3.3, Exercise 7. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_1x_2, 2x_2)$.

Answer: The transformation $T(x, y) = (x + xy, 2y)$ is not linear.

Solution: If T is a linear transformation, then

$$T(x_1 + x_2, y_1 + y_2) = T(x_1, y_1) + T(x_2, y_2)$$

for any real numbers x_1,x_2,y_1,y_2. However,

$$\begin{aligned} T(1,1) &= (2,2) \\ T(1,0) + T(0,1) &= (1,0) + (0,2) = (1,2). \end{aligned}$$

Therefore $T(1,1) \neq T(1,0) + T(0,1)$ and T is not linear.

Problem 4

Determine whether the given transformation is linear.

§3.3, Exercise 9. $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (1, x_1 + x_2, 2x_2)$

The transformation $T(x, y) = (1, x + y, 2y)$ is not linear because $T(0, 0) = (1, 0, 0) \neq 0$.

Problem 5

§3.3, Exercise 14. Let $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ permute coordinates cyclically; that is,

$$\sigma(x_1, x_2, x_3) = (x_2, x_3, x_1).$$

Find the 3×3 matrix A such that $\sigma = L_A$.

Answer: The matrix A of the linear mapping L_A is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Solution: Note that if $\sigma = L_A$, then $\sigma(e_j) = Ae_j$ is the j^{th} column of matrix A. Thus A is determined by

$$\begin{aligned} \sigma(e_1) = \sigma(1, 0, 0) = (0, 0, 1) \\ \sigma(e_2) = \sigma(0, 1, 0) = (1, 0, 0) \\ \sigma(e_3) = \sigma(0, 0, 1) = (0, 1, 0). \end{aligned}$$

Problem 6

§3.3, Exercise 16. Let $P : \mathbb{R}^n \to \mathbb{R}^m$ and $Q : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings.

(a) Prove that $S : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$S(x) = P(x) + Q(x)$$

is also a linear mapping.

(b) Theorem 3.3.5 states that there are matrices A, B and C such that

$$P = L_A \quad \text{and} \quad Q = L_B \quad \text{and} \quad S = L_C.$$

What is the relationship between the matrices A, B, and C?

Solution: The mapping L is linear if $L(x + y) = L(x) + L(y)$ and if $cL(x) = L(cx)$.

(a) We can use the assumption that $P(x)$ and $Q(x)$ are linear mappings to show:

$$\begin{aligned} S(x + y) &= P(x + y) + Q(x + y) \\ &= P(x) + P(y) + Q(x) + Q(y) \\ &= [P(x) + Q(x)] + [P(y) + Q(y)] \\ &= S(x) + S(y) \end{aligned}$$

and

$$\begin{aligned} cS(x) &= cP(x) + cQ(x) \\ &= P(cx) + Q(cx) \\ &= S(cx). \end{aligned}$$

(b) Assume that $S = L_C$, $P = L_A$ and $Q = L_B$ for $m \times n$ matrices A, B, C. We claim that $A = B + C$. By definition, $A(e_j) = L_A(e_j) = L_B(e_j) + L_C(e_j) = (B + C)(e_j)$. Lemma 3.3.4 implies that the j^{th} column of C is the sum of the j^{th} column of A and the j^{th} column of B for all columns j, so $C = A + B$.

Problem 7

§3.4, Exercise 3.

(a) Find all solutions to the homogeneous equation $Ax = 0$ where

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

(b) Find a single solution to the inhomogeneous equation

$$Ax = \begin{pmatrix} 6 \\ 6 \end{pmatrix}. \tag{1}$$

(c) Use your answers in (a) and (b) to find all solutions to (1).

(a) **Answer:** All solutions to the homogeneous equation are of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} -11 \\ 7 \\ 1 \end{pmatrix}.$$

Solution: Row reduce the matrix of the homogeneous system $Ax = 0$ to obtain:

$$\begin{pmatrix} 1 & 0 & 11 \\ 0 & 1 & -7 \end{pmatrix}.$$

So $x_1 = -11s$, $x_2 = 7s$ and $x_3 = s$.

(b) **Answer:** One possible solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution: Assign a value to x_3, then substitute into the two equations of the inhomogeneous system to obtain values for x_1 and x_2.

(c) All solutions to (1) can be found by adding a single solution of the inhomogeneous system to all solutions of the homogeneous system, so:

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -11 \\ 7 \\ 1 \end{pmatrix}.$$

Problem 8

§3.4, Exercise 4. How many solutions can a homogeneous system of 4 linear equations in 7 unknowns have?

Answer: The system must have infinitely many solutions.

The system must have a solution because homogeneous systems are always consistent. The system cannot have a unique solution because the rank of the corresponding augmented matrix cannot exceed 4 which is less than the number of variables 7.

Problem 9

Compute the given matrix product.

§3.5, Exercise 8. $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 5 \\ 1 & 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ -2 & -1 \\ -5 & 3 \end{pmatrix}$.

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 5 \\ 1 & 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ -2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2+2-15 & 14+1+9 \\ 1-25 & 7+15 \\ 1-10+5 & 7-5-3 \end{pmatrix} = \begin{pmatrix} -11 & 24 \\ -24 & 22 \\ -4 & -1 \end{pmatrix}.$$

Problem 10

§3.6, Exercise 4. Let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that $J^2 = -I$.

(b) Evaluate $(aI + bJ)(cI + dJ)$ in terms of I and J.

(a) Verify $J^2 = -I$ by computation:

$$J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

(b) **Answer:** $(aI + bJ)(cI + dJ) = (ac - bd)I + (ad + bc)J$.

Solution: Evaluate $(aI + bJ)(cI + dJ)$, yielding $acI^2 + adIJ + bcJI + bdJ^2$. Then, use the identities $IJ = JI = J$, $I^2 = I$, and $J^2 = -I$ to rewrite the expression in terms of I and J.

Problem 11

§3.7, Exercise 1. Verify by matrix multiplication that the following matrices are inverses of each other:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 2 \\ 2 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix}.$$

If two matrices are inverses of each other, then their product is the identity matrix. So:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 2 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 12 (MATLAB)

§3.6, Exercise 10 (MATLAB).(MATLAB) Experimentally, find two symmetric 2×2 matrices A and B for which the matrix product AB is *not* symmetric.

Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

be symmetric matrices. Then

$$AB = \begin{pmatrix} 0 & 3 \\ 5 & -4 \end{pmatrix}$$

is not symmetric. In general, for

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix},$$

AB is symmetric if $a_{12}b_{11} + a_{22}b_{12} = a_{11}b_{12} + a_{12}b_{22}$.