Math 2568 Homework 6
Math 2568   Due: Monday, October 7, 2019

Problem 1

You are given a vector space V and a subset W. For each pair, decide whether or not W is a subspace of V, and explain why.

§5.1, Exercise 6. $V = \mathbb{R}^2$ and W consists of vectors in $\mathbb{R}^2$ for which the sum of the components is 1.

Answer: W is not a subspace of V.

Solution: The subset W is closed neither under addition nor under scalar multiplication. For example, let $w_1 = (3, -2)$ and $w_2 = (0, 1)$ be elements of W. Then,

$$w_1 + w_2 = (3, -2) + (0, 1) = (3, -3).$$

The sum of the elements $3 - 3 = 0 \neq 1$.

Problem 2

You are given a vector space V and a subset W. For each pair, decide whether or not W is a subspace of V, and explain why.

§5.1, Exercise 8. $V = \mathcal{C}^1$ and W consists of functions $x(t) \in \mathcal{C}^1$ satisfying $\int_{-2}^{4} x(t)dt = 0$.

W is a subspace of V, since W is closed under addition and scalar multiplication.

Problem 3

§5.1, Exercise 16. Let V be a vector space and let W_1 and W_2 be subspaces. Show that the intersection $W_1 \cap W_2$ is also a subspace of V.

The subset $W_1 \cap W_2$ is a subspace of V. To show that this subset is closed under addition and scalar multiplication, let x and y be vectors in $W_1 \cap W_2$. It follows that $x, y \in W_1$ and $x, y \in W_2$. Therefore, by the definition of a subspace, $x + y \in W_1$ and $x + y \in W_2$, so $x + y \in W_1 \cap W_2$. Also by definition, $rx \in W_1$ and $rx \in W_2$, for some scalar r, so $rx \in W_1 \cap W_2$.

Problem 4

§5.1, Exercise 18. For which scalars a, b, c, d do the solutions to the equation

$$ax + by + cz = d$$

form a subspace of $\mathbb{R}^3$?

Answer: By the same proof as in Exercise 17, the solutions to the equation $ax+by+cz = d$ form a subspace of $\mathbb{R}^3$ when $d = 0$, and do not form a subspace when $d \neq 0$.

Problem 5

A single equation in three variables is given. For each equation write the subspace of solutions in $\mathbb{R}^3$ as the span of two vectors in $\mathbb{R}^3$.

§5.2, Exercise 2. $x - y + 3z = 0$.

Answer: The subspace of solutions can be spanned by the vectors $(1, 1, 0)^t$ and $(-3, 0, 1)^t$.

Solution: All solutions to $x - y + 3z = 0$ can be written in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - 3z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 6

Each of the given matrices is in reduced echelon form. Write solutions of the corresponding homogeneous system of linear equations as a span of vectors.

§5.2, Exercise 8. $B = \begin{pmatrix} 1 & -1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$.

Answer: The subspace of solutions to $Bx = 0$ is spanned by the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -5 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Solution: Let $x = (x_1, \ldots, x_6)$ be a solution to $Bx = 0$. All solutions to this equation have the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x_2 - 5x_4 \\ x_2 \\ -2x_4 - 2x_6 \\ x_4 \\ -2x_6 \\ x_6 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Problem 7

§5.2, Exercise 9. Write a system of two linear equations of the form $Ax = 0$ where A is a 2×4 matrix whose subspace of solutions in $\mathbb{R}^4$ is the span of the two vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Answer: The matrix A whose subspace of solutions in $\mathbb{R}^4$ is the span of v_1 and v_2 is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Solution: Note that all vectors x in the spanning set of v_1 and v_2 are of the form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ -a \\ b \\ -b \end{pmatrix}.$$

Therefore, $x_1 = -x_2$ and $x_3 = -x_4$. So,

$$\begin{array}{ccccccccc} x_1 & + & x_2 & & & & & = & 0 \\ & & & & x_3 & + & x_4 & = & 0. \end{array}$$

The matrix of this system is A.

Problem 8

§5.2, Exercise 20. Let $W \subset \mathbb{R}^4$ be the subspace that is spanned by the vectors

$$w_1 = (-1, 2, 1, 5) \quad \text{and} \quad w_2 = (2, 1, 3, 0).$$

Find a linear system of two equations such that $W = \text{span}\{w_1, w_2\}$ is the set of solutions of this system.

Answer: The span of W is the set of solutions to the system

$$\begin{array}{ccccccccc} x_1 & + & x_2 & - & x_3 & & & = & 0 \\ & & 3x_2 & - & x_3 & - & x_4 & = & 0 \end{array}.$$

where $x = (x_1, x_2, x_3, x_4) \in W$. Row reduction of the associated matrix demonstrates that this system is a valid solution set.

Solution: Solve for x as a linear combination of w_1 and w_2 by creating the matrix whose columns are w_1 and w_2, then setting up the equation:

$$\begin{pmatrix} -1 & 2 \\ 2 & 1 \\ 1 & 3 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

where a and b are scalars. Then row reduce the associated augmented matrix:

$$\left(\begin{array}{cc|c} -1 & 2 & x_1 \\ 2 & 1 & x_2 \\ 1 & 3 & x_3 \\ 5 & 0 & x_4 \end{array}\right) \longrightarrow \left(\begin{array}{cc|l} 1 & 3 & x_3 \\ 0 & -5 & x_2 - 2x_3 \\ 0 & 0 & x_1 + x_2 - x_3 \\ 0 & 0 & -3x_2 + x_3 + x_4 \end{array}\right).$$

Extract from this solution the values that are independent of a and b to obtain the linear system above.

Problem 9

§5.2, Exercise 22. Let V be a vector space and let $v, w \in V$ be vectors. Show that

$$\text{span}\{v, w\} = \text{span}\{v, w, v + 3w\}.$$

Every vector $x \in \text{span}\{v, w\}$ is of the form

$$x = av + bw = av + bw + 0(v + 3w) \in \text{span}\{v, w, v + 3w\}.$$

Also, every vector $y \in \text{span}\{v, w, v + 3w\}$ is of the form

$$y = cv + dw + f(v + 3w) = (c + f)v + (d + 3f)w \in \text{span}\{v, w\}.$$

Therefore, $\text{span}\{v, w\} = \text{span}\{v, w, v + 3w\}$.

Problem 10

§5.2, Exercise 24. Let $Ax = b$ be a system of m linear equations in n unknowns, and let $r = \text{rank}(A)$ and $s = \text{rank}(A|b)$. Suppose that this system has a unique solution. What can you say about the relative magnitudes of m, n, r, s?

Answer: The relationship of the constants is $m \geq n = r = s$.

Solution: The rank of matrix A cannot be greater than the rank of matrix $(A|b)$, since $(A|b)$ consists of A plus one column. The rank of A is the number of pivots in the row reduced matrix. $(A|b)$ can be row reduced through the same operations, and will have either the same number of pivots as A or, if there is a pivot in the last column, one more pivot than A. Since the system has a unique solution, it is consistent, and therefore $(A|b)$ cannot have a pivot in the $(n+1)^{st}$ column, so $r = \text{rank}(A) = \text{rank}(A|b) = s$.

The set of solutions is parameterized by $n - r$ parameters, where n is the number of columns of A. Since there is a unique solution, the set of solutions is parameterized by 0 parameters, so $n = r$.

The number m of rows of the matrix must be greater than or equal to n in order for the system to have a unique solution, since there must be n pivots, and each pivot must be in a separate row.

Problem 11

§5.4, Exercise 1. Let w be a vector in the vector space V. Show that the sets of vectors $\{w, 0\}$ and $\{w, -w\}$ are linearly dependent.

To show that the set of vectors $\{w_1, w_2\}$ is linearly dependent, show that there exist nonzero a and b such that $aw_1 + bw_2 = 0$. For the set $\{w, 0\}$, if $a = 0$ and $b = 1$, then $0w + 1(0) = 0$, so the set is linearly dependent. For the set $\{w, -w\}$, if $a = 1$ and $b = 1$, then $w - w = 0$, so the set is linearly dependent.

Problem 12

§5.4, Exercise 3. Let

$$u_1 = (1, -1, 1) \quad u_2 = (2, 1, -2) \quad u_3 = (10, 2, -6).$$

Is the set $\{u_1, u_2, u_3\}$ linearly dependent or linearly independent?

Answer: The set is linearly dependent.

Solution: Let A be the matrix whose columns are u_1, u_2, and u_3. The set $\{u_1, u_2, u_3\}$ is linearly dependent if there exists a nonzero vector $r = (r_1, r_2, r_3)$

such that $r_1u_1 + r_2u_2 + r_3u_3 = 0$, that is, if the homogeneous system $Ar = 0$ has a nonzero solution. Row reduce:

$$\begin{pmatrix} 1 & 2 & 10 \\ -1 & 1 & 2 \\ 1 & -2 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, $Ar = 0$ when $r = r_3(-2, -4, 1)$. The value of r is nonzero for $r_3 \neq 0$, so the set is indeed linearly dependent. As an example, let $r_3 = 1$. Then,

$$-4u_1 - 2u_2 + u_3 = -2(1, -1, 1) - 4(2, 1, -2) + (10, 2, -6) = (0, 0, 0) = 0.$$

Problem 13

§5.4, Exercise 8. Suppose that the three vectors $u_1, u_2, u_3 \in \mathbb{R}^n$ are linearly independent. Show that the set

$$\{u_1 + u_2, u_2 + u_3, u_3 + u_1\}$$

is also linearly independent.

To show that the vectors $u_1 + u_2$, $u_2 + u_3$ and $u_3 + u_1$ are linearly independent, we assume that there exist scalars r_1, r_2, r_3 such that

$$r_1(u_1 + u_2) + r_2(u_2 + u_3) + r_3(u_3 + u_1) = 0.$$

We then prove that $r_1 = r_2 = r_3 = 0$, as follows. Use distribution to obtain

$$(r_1 + r_3)u_1 + (r_1 + r_2)u_2 + (r_2 + r_3)u_3 = 0.$$

Since the set $\{u_1, u_2, u_3\}$ is linearly independent,

$$\begin{array}{ccccccc} r_1 & & & + & r_3 & = & 0 \\ r_1 & + & r_2 & & & = & 0 \\ & & r_2 & + & r_3 & = & 0. \end{array}$$

Solving this system yields $r_1 = r_2 = r_3 = 0$, so the set $\{u_1 + u_2, u_2 + u_3, u_3 + u_1\}$ is linearly independent.