Math 2568 Homework 9
Math 2568 Due: Monday, October 28, 2019

Problem 1

Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 65x + 42y \\ \frac{dy}{dt} &= -99x - 64y. \end{aligned} \tag{1}$$

§6.1, Exercise 1. Verify that

$$v_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -7 \\ 11 \end{pmatrix}$$

are eigenvectors of the coefficient matrix of (2) and find the associated eigenvalues.

Answer: The vector $v_1 = (2, -3)^t$ is an eigenvector with associated eigenvalue $\lambda_1 = 2$. The vector $v_2 = (-7, 11)^t$ is an eigenvector with associated eigenvalue $\lambda_2 = -1$.

Solution: Calculate:

$$\begin{aligned} \begin{pmatrix} 65 & 42 \\ -99 & -64 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} &= \begin{pmatrix} 4 \\ -6 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -3 \end{pmatrix}. \\ \begin{pmatrix} 65 & 42 \\ -99 & -64 \end{pmatrix}\begin{pmatrix} -7 \\ 11 \end{pmatrix} &= \begin{pmatrix} 7 \\ -11 \end{pmatrix} = -1\begin{pmatrix} -7 \\ 11 \end{pmatrix}. \end{aligned}$$

Problem 2

Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 65x + 42y \\ \frac{dy}{dt} &= -99x - 64y. \end{aligned} \tag{2}$$

§6.1, Exercise 2. Find the solution to (2) satisfying initial conditions $X(0) = (-14, 22)^t$.

Answer: The solution to (2) with initial condition $X(0) = (-14, 22)^t$ is

$$X(t) = 2e^{-t}\begin{pmatrix} -7 \\ 11 \end{pmatrix}.$$

Solution: We are given two linearly independent initial conditions: v_1 and v_2. Therefore, by Theorem 6.1.1, the general solution to (2) with initial condition $X(0)$ is

$$X(t) = r_1 e^{2t} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + r_2 e^{-t} \begin{pmatrix} -7 \\ 11 \end{pmatrix}.$$

Find r_1 and r_2 by solving:

$$X(0) = r_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} + r_2 \begin{pmatrix} -7 \\ 11 \end{pmatrix}. \tag{3}$$

Problem 3

Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x - y \\ \frac{dy}{dt} &= -x + y. \end{aligned} \tag{4}$$

§6.1, Exercise 5. The eigenvalues of the coefficient matrix of (5) are 0 and 2. Find the associated eigenvectors.

Answer: The eigenvector associated to $\lambda_1 = 0$ is $v_1 = (1,1)^t$, and the eigenvector associated to $\lambda_2 = 2$ is $v_2 = (1,-1)^t$.

Solution: Solve the systems

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} v_1 = 0 \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} v_2 = 2v_2.$$

Problem 4

Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x - y \\ \frac{dy}{dt} &= -x + y. \end{aligned} \tag{5}$$

§6.1, Exercise 6. Find the solution to (5) satisfying initial conditions $X(0) = (2,-2)^t$.

Answer: The solution to (5) with initial condition $X(0) = (2,-2)^t$ is

$$X(t) = 2e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution: Note that initial conditions v_1 and v_2 are linearly independent. Therefore, by Theorem 6.1.1, the general solution to (5) with initial condition $X(0)$ is

$$X(t) = r_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + r_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find values for r_1 and r_2 by solving:

$$X(0) = r_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + r_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{6}$$

Problem 5

Consider the system of differential equations

$$\begin{aligned} \tfrac{dx}{dt} &= -2x + 7y \\ \tfrac{dy}{dt} &= 5y, \end{aligned} \tag{7}$$

§6.1, Exercise 13. Find a solution to (7) satisfying the initial condition $(x(0), y(0)) = (1, 0)$.

Answer: If $(x(0), y(0)) = (1, 0) = X_2(0)$, then

$$(x(t), y(t)) = e^{-2t}(1, 0).$$

Solution: The general solution to the system is

$$X(t) = r_1 e^{5t}(1, 1) + r_2 e^{-2t}(1, 0).$$

To obtain this solution, first rewrite the system of differential equations as

$$\frac{dX}{dt} = CX = \begin{pmatrix} -2 & 7 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

By inspection of C, $(1, 1)^t$ and $(1, 0)^t$ are eigenvectors with eigenvalues 5 and -2 respectively. Therefore:

$$X_1(t) = e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad X_2(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are solutions to the differential equation. The initial values $X_1(0) = (1, 1)$ and $X_2(0) = (1, 0)$ are linearly independent, so the general solution is valid. To find r_1 and r_2, evaluate

$$X(0) = (x(0), y(0)) = r_1(1, 1) + r_2(1, 0) = (r_1 + r_2, r_1).$$

Problem 6

In modern language De Moivre's formula states that

$$e^{ni\theta} = \left(e^{i\theta}\right)^n.$$

In Exercises 2 - 3 use De Moivre's formula coupled with Euler's formula (6.2.5) to determine trigonometric identities for the given quantity in terms of $\cos\theta$, $\sin\theta$, $\cos\varphi$, $\sin\varphi$.

§6.2, Exercise 2. $\cos(\theta+\varphi)$.

Answer: $\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$.

Solution: Using Euler's formula ((6.2.5)):

$$\begin{aligned}
\cos(\theta+\varphi) + i\sin(\theta+\varphi) &= e^{i(\theta+\varphi)} \\
&= e^{i\theta}e^{i\varphi} \\
&= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \\
&= \cos\theta\cos\varphi + i\sin\theta\cos\varphi + i\sin\varphi\cos\theta - \sin\theta\sin\varphi \\
&= (\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\sin\theta\cos\varphi + \sin\varphi\cos\theta)
\end{aligned}$$

The real part of this formula is equal to $\cos(\theta+\varphi)$.

Problem 7

Compute the general solution for the given system of differential equations.

§6.2, Exercise 4. $\dfrac{dX}{dt} = \begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix} X$.

Answer: The general solution to the differential equation is

$$X(t) = \alpha_1 \begin{pmatrix} 2e^t\cos(2t) \\ e^t(\sin(2t) - \cos(2t)) \end{pmatrix} + \alpha_2 \begin{pmatrix} 2e^t\sin(2t) \\ -e^t(\sin(2t) + \cos(2t)) \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 - 2\lambda + 5.$$

The eigenvalues are $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Then, find the eigenvector associated to λ_1 by solving the equation

$$(C-\lambda_1 I_2)v_1 = \left(\begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1+2i & 0 \\ 0 & 1+2i \end{pmatrix}\right) v_1 = \begin{pmatrix} -2-2i & -4 \\ 2 & 2-2i \end{pmatrix} v_1 = 0.$$

Solve this equation to find that

$$v_1 = \begin{pmatrix} 2 \\ -1-i \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

is an eigenvector of C. Since the eigenvalues of C are complex, we can find the general solution using (6.2.3) and (6.2.4). In this case, since $\lambda_1 = 1 + 2i$ is an eigenvalue, let $\sigma = 1$ and let $\tau = 2$. Then $v_1 = v + iw$, where $v = (2, -1)^t$ and $w = (0, -1)^t$. By (6.2.3),

$$X_1(t) = e^{\sigma t}(\cos(\tau t)v - \sin(\tau t)w) \quad \text{and} \quad X_2(t) = e^{\sigma t}(\sin(\tau t)v + \cos(\tau t)w)$$

are solutions to the differential equation. In this case,

$$\begin{aligned} X_1(t) &= e^t\left(\cos(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin(2t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = e^t\begin{pmatrix} 2\cos(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix}. \\ X_2(t) &= e^t\left(\sin(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \cos(2t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = e^t\begin{pmatrix} 2\sin(2t) \\ -\sin(2t) - \cos(2t) \end{pmatrix}. \end{aligned}$$

The general solution consists of all linear combinations $X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t)$.

Problem 8

Compute the general solution for the given system of differential equations.

§6.2, Exercise 5. $\dfrac{dX}{dt} = \begin{pmatrix} 8 & -15 \\ 3 & -4 \end{pmatrix} X$.

Answer: The general solution to the differential equation is

$$X(t) = \alpha_1 \begin{pmatrix} 5e^{2t}\cos(3t) \\ e^{2t}(2\cos(3t) + \sin(3t)) \end{pmatrix} + \alpha_2 \begin{pmatrix} 5e^{2t}\sin(3t) \\ e^{2t}(2\sin(3t) - \cos(3t)) \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 - 4\lambda + 13.$$

The eigenvalues are $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$. Then, find the eigenvector associated to λ_1 by solving the equation

$$(C - \lambda_1 I_2)v_1 = \left(\begin{pmatrix} 8 & -15 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} 2+3i & 0 \\ 0 & 2+3i \end{pmatrix}\right) v_1 = \begin{pmatrix} 6-3i & -15 \\ 3 & -6-3i \end{pmatrix} v_1 = 0.$$

Solve this equation to find that

$$v_1 = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

is an eigenvector of C. Since the eigenvalues of C are complex, we can find the general solution using (6.2.3) and (6.2.4). In this case, since $\lambda_1 = 2 + 3i$ is an eigenvalue, let $\sigma = 2$ and let $\tau = 3$. Then $v_1 = v + iw$, where $v = (5, 2)^t$ and $w = (0, -1)^t$. By (6.2.3),

$$X_1(t) = e^{\sigma t}(\cos(\tau t)v - \sin(\tau t)w) \quad \text{and} \quad X_2(t) = e^{\sigma t}(\sin(\tau t)v + \cos(\tau t)w)$$

are solutions to the differential equation. In this case,

$$\begin{aligned} X_1(t) &= e^{2t}\left(\cos(3t)\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \sin(3t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = e^{2t}\begin{pmatrix} 5\cos(3t) \\ \sin(3t) + 2\cos(3t) \end{pmatrix}. \\ X_2(t) &= e^{2t}\left(\sin(3t)\begin{pmatrix} 5 \\ 2 \end{pmatrix} + \cos(3t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = e^{2t}\begin{pmatrix} 5\sin(3t) \\ 2\sin(3t) - \cos(3t) \end{pmatrix}. \end{aligned}$$

The general solution consists of all linear combinations $X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t)$.

Problem 9

Compute the general solution for the given system of differential equations.

§6.2, Exercise 6. $\dfrac{dX}{dt} = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} X.$

Answer: The general solution to the differential equation is

$$X(t) = \alpha e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{4t}\begin{pmatrix} t + \frac{1}{2} \\ t - \frac{1}{2} \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.$$

Thus, C has a double eigenvalue at $\lambda_1 = 4$. Since C is not a multiple of I_2, C has only one linearly independent eigenvector. Find this eigenvector by solving the equation

$$(C - \lambda_1 I_2)v_1 = \left(\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right)v_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} v_1 = 0,$$

obtaining $v_1 = (1, 1)^t$. Find the generalized eigenvector w_1 by solving the equation $(C - \lambda_1 I_2)w_1 = v_1$, that is

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So $w_1 = (\frac{1}{2}, -\frac{1}{2})^t$ is the generalized eigenvector. Now, by (6.2.16), we know that the general solution to $\dot{X} = CX$ when C has equal eigenvalues and only one independent eigenvector is

$$X(t) = e^{\lambda_1 t}(\alpha v_1 + \beta(w_1 + t v_1)).$$

In this case,

$$X(t) = e^{4t}\left(\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta\left(\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\right).$$

Problem 10

Compute the general solution for the given system of differential equations.

§6.2, Exercise 7. $\dfrac{dX}{dt} = \begin{pmatrix} -4 & 4 \\ -1 & 0 \end{pmatrix} X.$

Answer: The general solution to the differential equation is

$$X(t) = \alpha e^{-2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 2t+1 \\ t+1 \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2.$$

Thus, C has a double eigenvalue at $\lambda_1 = -2$. Since C is not a multiple of I_2, C has only one linearly independent eigenvector. Find this eigenvector by solving the equation

$$(C - \lambda_1 I_2)v_1 = \left(\begin{pmatrix} -4 & 4 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)v_1 = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}v_1 = 0,$$

obtaining $v_1 = (2,1)^t$. Find the generalized eigenvector w_1 by solving the equation $(C - \lambda_1 I_2)w_1 = v_1$, that is,

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} w_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

So $w_1 = (1,1)^t$ is the generalized eigenvector. Now, by (6.2.16), we know that the general solution to $\dot{X} = CX$ when C has equal eigenvalues and only one independent eigenvector is

$$X(t) = e^{\lambda_1 t}(\alpha v_1 + \beta(w_1 + tv_1)).$$

In this case,

$$X(t) = e^{-2t}\left(\alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)\right).$$