MATH 2568 EXAM 1 SOLUTIONS

Problem 1.

(a) **FALSE**: Every square matrix is invertible.

(b) Suppose $L : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and $L\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix}$ and $L\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix}$.

Find the 3×2 matrix A such that $L = L_A$. $A = \begin{pmatrix}2 & 3\\2 & 3\\2 & 3\end{pmatrix}$.

(c) **FALSE**: If A is an $m \times n$ matrix and the linear system $A\vec{x} = \vec{b}$ is consistent, then $m \leq n$.

(d) (2 points) Compute $\det\begin{pmatrix}1 & 2\\3 & 5\end{pmatrix} = -1$.

(e) **FALSE**: Suppose A is an $m \times n$ matrix in reduced row echelon form, and $A\vec{x} = \vec{0}$ has a unique solution. Then A has a pivot in every row.

Problem 2.

(a) **FALSE**: The dot product between two vectors in $\mathbb{R}^n$ is an element in $\mathbb{R}^n$.

(b) Compute $\begin{pmatrix}1 & 2 & 1\\1 & 1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$.

(c) **FALSE**: The matrix $\begin{pmatrix}1 & 2 & 1\\1 & 1 & 0\end{pmatrix}$ from (2b) above is invertible.

(d) Compute the angle between the vectors $\begin{pmatrix}2\\1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$. **Since** $\cos(\theta) = \frac{3}{\sqrt{6}\sqrt{3}} = \frac{1}{\sqrt{2}}$**,**

$\theta = \pi/4$**.**

Problem 3. Find all solutions to the linear system associated to the augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 0 & 2 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

Express your answer in the form of a particular solution plus a superposition (a.k.a. linear combination) of solutions to the homogeneous system.

Solution: The matrix is already in RREF. There are 2 columns without pivots which give 2 free parameters. So the solutions are all of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2-3r-s \\ 1-r+2s \\ r \\ s \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 3 \end{pmatrix} + r \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 4. Consider the system of linear equations below, where $c, d \in \mathbb{R}$:

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + cx_3 &= 1 \\ 3x_1 + 2x_2 + x_3 &= d. \end{aligned}$$

(a) Find the augmented matrix $\left(A \mid \vec{b} \right)$ corresponding to the system $A\vec{x} = \vec{b}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & c & 1 \\ 3 & 2 & 1 & d \end{array}\right)$$

(b) (*4 points*) Perform Gaussian elimination to get a matrix *as close as possible* to row echelon form (you may have an expression in a variable instead of a pivot in one entry. Your answer need not be in reduced row echelon form). You cannot divide by zero!

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & c & 1 \\ 3 & 2 & 1 & d \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & c & 1 \\ 0 & 2 & -2 & d-3 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & c & 1 \\ 0 & 0 & -2-2c & d-5 \end{array}\right)$$

(c) (*4 points*) Find $c, d \in \mathbb{R}$ such that this system has infinitely many solutions. Explain your reasoning.

Solution: There are infinitely many solutions if the system is consistent and there is a column without a pivot to the left of the augmentation line. There will be no pivot in the third column if and only if $c = -1$. In this case, in order for the system to be consistent, we must have $d = 5$.

Problem 5. (*10 points*) Suppose A is a 3×2 matrix and $\vec{b} \in \mathbb{R}^3$. Prove that if $A\vec{x} = \vec{b}$ has a solution $\vec{x} \in \mathbb{R}^2$, then $\vec{b}$ is a superposition (a.k.a. linear combination) of the columns of A.

Solution 1: Suppose $\vec{x} \in \mathbb{R}^2$ is a solution to $A\vec{x} = \vec{b}$. Then $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$ for some $x_1, x_2 \in \mathbb{R}$. Since left multiplication by A is linear, we have

$$\vec{b} = A\vec{x} = A(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1 \underbrace{A\vec{e}_1}_{A_1} + x_2 \underbrace{A\vec{e}_2}_{A_2} = x_1 A_1 + x_2 A_2$$

where A_1, A_2 are the first and second columns of A respectively.

Solution 2: Since A is 3×2, there are $a, b, c, d, e, f \in \mathbb{R}$ such that

$$A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}.$$

Multiplying by $\vec{x} \in \mathbb{R}^2$, we have:

$$\vec{b} = A\vec{x} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \\ ex_1 + fx_2 \end{pmatrix} = x_1 \begin{pmatrix} a \\ c \\ e \end{pmatrix} + x_2 \begin{pmatrix} b \\ d \\ f \end{pmatrix}.$$

This final expression is a linear combination of the columns of A.