Problem 1. (*10 points*) There is no partial credit on this problem.

(a) (2 points) Circle TRUE or FALSE: Every $n \times n$ matrix with real entries has at least one real eigenvalue.
Solution: False. The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is rotation by $\pi/2$ and has no real eigenvalues.

(b) (2 points) Suppose A is a 4×8 matrix whose nullity is 6. What is the dimension of the row space of A?
Solution: By the Rank-Nullity Theorem, the rank plus the nullity is equal to the number of columns. The rank is equal to the number of pivots in RREF, which is equal to the dimension of the row space. So $6 + \dim(RS(A)) = 8$, and $\dim(RS(A)) = 2$.

(c) (2 points) Circle TRUE or FALSE: Suppose A is a 2×2 matrix whose eigenvalues are 0 and 1. Then A is invertible.
Solution: False. Observe that when 0 is an eigenvalue, there is a non-zero $\vec{x}$ such that $A\vec{x} = \vec{0}$. Thus A is invertible if and only if 0 is not an eigenvalue of A by the Square Matrix Theorem.

(d) (2 points) Suppose A is an 2×2 matrix with an eigenvector $\vec{v} \in \mathbb{R}^2$ with eigenvalue 3. What is the eigenvalue for the eigenvector $\vec{v}$ for the 2×2 matrix $A + I$?
Solution: We calculate that $(A + I)\vec{v} = A\vec{v} + \vec{v} = 3\vec{v} + \vec{v} = 4\vec{v}$. Thus the eigenvalue for $\vec{v}$ for $A + I$ is 4.

(e) (2 points) Circle TRUE or FALSE: Suppose V is a vector space and $\{v_1, v_2, v_3\} \subset V$. If $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_1, v_3\}$ are linearly independent, then $\{v_1, v_2, v_3\}$ is linearly independent.
Solution: False. Pick $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Any two of these three vectors are linearly independent. But all three must be linearly dependent since $\dim(\mathbb{R}^2) = 2$.

Problem 2. (*10 points*) For the questions below, you must provide a type. Select the single answer which **best** answers the question, and write it on the blank provided. There is no partial credit on this problem. The possible answers are as follows. Answers can be used more than once or not at all.

{ number, vector, set of vectors, subspace, matrix, linear transformation }

(A) (2 points) Which type of thing has a dimension?
Solution: Subspace

(B) (2 points) Which type of thing has a domain?
Solution: Linear transformation

(C) (2 points) Which type of thing is span$\{v_1, \ldots, v_k\}$?
Solution: Subspace

(D) (2 points) Which type of thing can span $\mathbb{R}^n$?
Solution: Set of vectors

(E) (2 points) Which type of thing can be linearly independent?
Solution: Set of vectors

Problem 3. (*10 points*) Find bases for the null, row, and columns spaces of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 3 & 1 \\ 2 & 2 & -1 & 5 & 4 \\ 3 & 3 & -1 & 8 & 6 \end{pmatrix}.$$

Solution: We row reduce A to obtain

$$\begin{pmatrix} 1 & 1 & 0 & 3 & 1 \\ 2 & 2 & -1 & 5 & 4 \\ 3 & 3 & -1 & 8 & 6 \end{pmatrix} \xrightarrow{\text{R3}\leftarrow\text{R3-3R1}} \begin{pmatrix} 1 & 1 & 0 & 3 & 1 \\ 2 & 2 & -1 & 5 & 4 \\ 0 & 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow{\text{R2}\leftarrow\text{R2-2R1}} \begin{pmatrix} 1 & 1 & 0 & 3 & 1 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow{\text{R2}\leftarrow\text{-R2}} \begin{pmatrix} 1 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow{\text{R3}\leftarrow\text{R3+R2}} \begin{pmatrix} 1 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{R2}\leftarrow\text{R2+2R3}} \begin{pmatrix} 1 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{R1}\leftarrow\text{R1-R2}} \begin{pmatrix} 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since there are 2 columns of $RREF(A)$ without pivots, the null space has dimension 2. The solutions of the homogeneous system are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -r-3s \\ r \\ -s \\ s \\ 0 \end{pmatrix} = r\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad r, s \in \mathbb{R}.$$

So a basis for $NS(A)$ is given by

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Since all three rows have pivots, all three rows of A or all three rows of $RREF(A)$ give bases for $RS(A)$:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -1 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -1 \\ 8 \\ 6 \end{pmatrix} \right\}.$$

Since only the first, third, and fifth columns of $RREF(A)$ have pivots, the corresponding columns of A give a basis for $CS(A)$:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} \right\}.$$

Point breakdown: 2 points for correct RREF, 3 points for a correct basis of NS, 2 points each for a correct basis for RS and CS, and 1 point for correct notation for a basis as set of vectors. One point was subtracted in addition if your answer violated the Rank Nullity Theorem (rank + nullity = number of columns).

Problem 4. (*10 points*) Consider the matrix

$$A = \begin{pmatrix} -7 & 9 \\ -4 & 5 \end{pmatrix}$$

(a) (2 points) Compute the characteristic polynomial of A, and factorize your answer.
Solution: $p_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$.

(b) (4 points) Find a basis of $\mathbb{R}^2$ consisting of an eigenvector and a generalized eigenvector for A.
Solution: The only eigenvalue is -1. We see that

$$A + I = \begin{pmatrix} -6 & 9 \\ -4 & 6 \end{pmatrix},$$

so here are the most likely choices you might have picked for your eigenvector $\vec{v}$:

$$\vec{v} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}, \qquad \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

We now need to solve $(A+I)\vec{w} = \vec{v}$. Based on your $\vec{v}$, here are some likely choices for your $\vec{w}$ respectively:

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Remember that the choice of $\vec{w}$ is not at all unique, and may differ by a multiple of $\vec{v}$.

(c) (2 points) Find a basis for the solution space of the linear system $AX(t) = X'(t)$.
Solution: A basis is given by $\{e^{-t}\vec{v}, e^{-t}(\vec{w} + t\vec{v})\}$ where your $\vec{v}$ and $\vec{w}$ are taken from your answer in part (b).

(d) (2 points) Find the particular solution with initial condition $X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Solution: This will again depend on your $\vec{v}$ and $\vec{w}$ from part (b). One must solve

$$\alpha\vec{v} + \beta\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For the three choices listed above, the particular solutions are:

$$(\alpha, \beta) = \left(\frac{1}{9}, -\frac{2}{3}\right), \qquad (0, -1), \qquad \text{or} \qquad (1, -2)$$

Problem 5. (*10 points*) Suppose A is a 3×3 matrix, and $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$ are two eigenvectors for A with eigenvalues $\lambda_1 \neq \lambda_2$. Prove $\vec{v}_1, \vec{v}_2$ are linearly independent.
Solution 1: We prove the contrapositive. Suppose that $\vec{v}_1 = \alpha\vec{v}_2$. We apply A to $\vec{v}_1$ in two ways to obtain:

$$\begin{aligned} A\vec{v}_1 &= \lambda_1\vec{v}_1 \\ A\vec{v}_1 &= A(\alpha\vec{v}_2 = \alpha(A\vec{v}_2) = \alpha\lambda_2\vec{v}_2 = \lambda_2(\alpha\vec{v}_2) = \lambda_2\vec{v}_1. \end{aligned}$$

This means that $\lambda_1\vec{v}_1 = \lambda_2\vec{v}_2$, so $(\lambda_1 - \lambda_2)\vec{v}_1 = \vec{0}$. Since $\vec{v}_1 \neq \vec{0}$, $\lambda_1 - \lambda_2 = 0$, so $\lambda_1 = \lambda_2$.

Solution 2: Suppose

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \tag{1}$$

We need to show $c_1 = c_2 = 0$. We apply A to the equation to get

$$\vec{0} = A\vec{0} = A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1A\vec{v}_1 + c_2A\vec{v}_2 = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2. \tag{2}$$

We now subtract λ_1 times (1) from (2) to get

$$\vec{0} = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 - \lambda_1(c_1\vec{v}_1 + c_2\vec{v}_2) = c_2(\lambda_2 - \lambda_1)\vec{v}_2.$$

Since $\vec{v}_2 \neq \vec{0}$, $c_2(\lambda_2 - \lambda_1) = 0$. Since $\lambda_1 \neq \lambda_2$, $c_2 = 0$. Now plugging $c_2 = 0$ into (1), we get

$$c_1\vec{v}_1 = \vec{0}.$$

Since $\vec{v}_1 \neq \vec{0}$, we have $c_1 = 0$. So $c_1 = c_2 = 0$ as desired.