Problem 1. True or false? Explain your reasoning.

(1) If A and B are matrices such that $AB = I_n$, then $BA = I_n$.
(2) If A, B, C are matrices such that $AB = I_n$ and $BC = I_n$, then $A = C$.
(3) If the system of equations $Ax = b$ has a unique solution, then A is invertible.
(4) If $L : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation and A is a 3×3 matrix, then the map $x \mapsto A(Lx)$ is linear.
(5) If A is an $m \times n$ matrix and $Ax = b$ is inconsistent, then $m > n$.

Problem 2. Find all solutions to the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 2x_4 + x_5 &= 2 \\ x_1 + 2x_3 + x_5 &= -2 \\ x_2 - x_3 + x_4 &= 2 \end{aligned}$$

Express your answer as a particular solution plus a superposition of solutions to the homogeneous system.

Problem 3. Suppose $L : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix} \qquad \text{and} \qquad L\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}.$$

Find the 3×2 matrix A such that $L = L_A$.

Problem 4. Determine whether the following matrices are invertible. If they are, find their inverses. Explain all your reasoning.

$$\begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \qquad \begin{pmatrix}2 & 0 & -1\\1 & 1 & 2\\3 & 1 & 1\end{pmatrix} \qquad \begin{pmatrix}0 & -3 & -2\\1 & -4 & -2\\-3 & 4 & 1\end{pmatrix}$$

Problem 5. Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ x_2 + x_3 &= b \\ ax_1 - x_3 &= 1. \end{aligned}$$

For which values of $a, b \in \mathbb{R}$ does this system have no solutions, a unique solution, or infinitely many solutions? Your answer should split the possibilities for $a, b \in \mathbb{R}$ into 3 disjoint sets.

Problem 6. Suppose A is an invertible $n \times n$ matrix and B is an $m \times n$ matrix. Prove that B and AB are row equivalent.