Math 2568 Practice Final **November 19, 2019**

Problem 1. True or false? Explain your reasoning. If false, find a condition on A which makes it true.

(1) If the columns of A are linearly independent, then so are the rows.
(2) If A is invertible, then so is A^T.
(3) If $Ax = b$ has a unique solution for some $b \in \mathbb{R}^m$, then $\text{nullity}(A) = 0$.
(4) If the only eigenvalue of A is 1, then A is diagonalizable.
(5) If A maps orthogonal vectors to orthogonal vectors, then A is orthogonal.

Problem 2. Short answer. The answers to all questions below are a single digit between 0 and 9.

(1) Compute the determinant of $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.
(2) Suppose 1 is an eigenvalue of A. What number is then an eigenvalue of $A + I$?
(3) Suppose A is a 7×9 matrix whose rank is 4. What is the dimension of the row space of A?
(4) The matrix $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix}$ has only one eigenvalue. What is its geometric multiplicity?
(5) Suppose $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 5 & k \end{pmatrix}$ is not invertible. What is k?

Problem 3. Find the Jordan normal form of the following matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

Problem 4. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(1) What are the eigenvalues of A?
(2) What are the algebraic multiplicities of the eigenvalues of A?
(3) What are the geometric multiplicities of the eigenvalues of A?
(4) Is A diagonalizable? If so, find S and D such that $S^{-1}AS = D$ is diagonal.

Problem 5. Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(1) Find 3 linearly independent solutions $X_1(t), X_2(t), X_3(t)$ of the differential equation $X'(t) = AX(t)$.
(2) Find a particular solution of the form $a_1X_1(t) + a_2X_2(t) + a_3X_3(t)$ when $X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Problem 6. Find bases for the column space, row space, and null space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{pmatrix}.$$

Problem 7. Let V be the vector space of polynomials of degree less than or equal to 2, and let W be the vector space of polynomials of degree less than or equal to 3. Consider the map $T : V \to W$ given by

$$p(x) = ax^2 + bx + c \longmapsto \int_{t=0}^{x} p(t)\, dt = \frac{ax^3}{3} + \frac{bx^2}{2} + cx.$$

(1) Show that T is linear.
(2) Calculate coordinates $[T]_B^C$ for T with respect to the ordered bases $B = \{1, x, x^2\}$ for V and $C = \{1, x, x^2, x^3\}$ for W.

Problem 8. Suppose A, B are $n \times n$ matrices such that $AB = I_n$. Show that $BA = I_n$.