Practice proof problems

Problems 1-5 below are for Midterm 1, Problems 6-10 below are for Midterm 2, and all problems are for the final.

Practice proof problems for Midterm 1.

Problem 1. Suppose the $m \times n$ matrices A and B are row equivalent. Prove that $A\vec{x} = \vec{0}$ if and only if $B\vec{x} = \vec{0}$.

Problem 2. Suppose A is a 3×2 matrix and $\vec{b} \in \mathbb{R}^3$. Prove that if $A\vec{x} = \vec{b}$ has a solution $\vec{x} \in \mathbb{R}^2$, then $\vec{b}$ is a superposition (a.k.a. linear combination) of the columns of A.
Variant: Suppose A is a 3×2 matrix. Prove that $\vec{b} \in \mathbb{R}^m$ is a superposition (a.k.a. linear combination) of the columns of A if and only if there is an $\vec{x} \in \mathbb{R}^2$ such that $A\vec{x} = \vec{b}$.

Problem 3. Suppose A is an invertible $m \times m$ matrix and B is an $m \times n$ matrix. Prove that B and AB are row equivalent.
Variant: Show two $m \times n$ matrices A, B are row equivalent if and only if there is an invertible $m \times m$ matrix S such that $A = SB$.

Problem 4. Show directly using Gaussian Elimination that the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad a \neq 0$$

is row equivalent to I_2 if and only if $ad - bc \neq 0$.

Problem 5. Suppose A, B are 3×3 matrices such that $AB = I_3$. Prove A is invertible.
Variant: Suppose A, B are $n \times n$ matrices such that $AB = I_n$. Prove A is invertible.

Practice proof problems for Midterm 2.

Problem 6. Suppose $\vec{w}_1, \vec{w}_2, \vec{w}_3 \in \mathbb{R}^5$. Prove that

$$\operatorname{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\} := \left\{a_1\vec{w}_1 + a_2\vec{w}_2 + a_3\vec{w}_3 \middle| a_1, a_2, a_3 \in \mathbb{R}\right\}$$

is a subspace.

Problem 7. Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are vectors in $\mathbb{R}^5$ and A is a 4×5 matrix. Suppose $\{A\vec{v}_1, A\vec{v}_2, A\vec{v}_3\} \subset \mathbb{R}^4$ is linearly independent. Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset \mathbb{R}^5$ is linearly independent.

Problem 8. Suppose V is a vector space and $v_1, \ldots, v_n \in V$ are not linearly independent. Prove there is a *redundant vector*, i.e., for some $k \in \{1, \ldots, n\}$, $v_k \in \operatorname{span}\{v_1, \ldots, v_{k-1}\}$.

Problem 9. Suppose $\{\vec{v}_1, \vec{v}_2\} \subset \mathbb{R}^4$ is linearly independent. Suppose $\vec{v}_3 \in \mathbb{R}^4$, but $\vec{v} \notin \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$. Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.
Variant: Suppose V is a vector space and $\{v_1, v_2\} \subset V$ is linearly independent. Suppose $v_3 \in V$, but $v \notin \operatorname{span}\{v_1, v_2\}$. Show that $\{v_1, v_2, v_3\}$ is linearly independent.

Problem 10. State and prove the Rank Nullity Theorem for an $m \times n$ matrix A.

Additional practice proof problems for final.

Problem 11. Suppose A is a 3×3 matrix, and $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$ are two eigenvectors for A with eigenvalues $\lambda_1 \neq \lambda_2$. Prove $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.
Variant: Suppose A is a 4×4 matrix, and $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^4$ are three eigenvectors for A with distinct eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. Prove $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Problem 12. Show that a 3×3 matrix A is diagonalizable if and only if there is a basis of $\mathbb{R}^3$ consisting of eigenvectors for A.
Hint: Consider the equation $AS = SD$ where D is diagonal.

Problem 13. Show that two similar 3×3 matrices have the same eigenvalues.

Problem 14. Suppose A is a 3×3 matrix.

(1) Explain why the (complex) eigenvalues of A are exactly the roots of the characteristic polynomial of A.
Recall that λ is an eigenvalue of A if there is a non-zero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

(2) Prove that $\det(A)$ is the product of the (complex) eigenvalues of A.
Hint: Every polynomial factorizes over the complex numbers.

Variant: Suppose A is an $n \times n$ matrix. Prove that $\det(A)$ *is the product of the (complex) eigenvalues of A.*

Problem 15. Prove that if a 3×3 matrix A is orthogonally diagonalizable, then $A = A^T$.

Problem 16. Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^5$ are orthonormal. Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.