Homework 11
Math 2568   April 10, 2019

Problem 1

compute the determinants of the given matrix.

§7.1, Exercise 2. $B = \begin{pmatrix} 1 & 0 & 2 & 3 \\ -1 & -2 & 3 & 2 \\ 4 & -2 & 0 & 3 \\ 1 & 2 & 0 & -3 \end{pmatrix}$.

Answer: The determinant of the matrix is -110.

Solution: First row reduce:

$$\det\begin{pmatrix} 1 & 0 & 2 & 3 \\ -1 & -2 & 3 & 2 \\ 4 & -2 & 0 & 3 \\ 1 & 2 & 0 & -3 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & -2 & 5 & 5 \\ 0 & -2 & -8 & -9 \\ 0 & 2 & -2 & -6 \end{pmatrix}.$$

Then, use formula (7.1.9):

$$\begin{aligned} \det\begin{pmatrix} -2 & 5 & 5 \\ -2 & -8 & -9 \\ 2 & -2 & -6 \end{pmatrix} &= \begin{array}{l} (-2)(-8)(-6) + 5(-9)2 + 5(-2)(-2) - 5(-8)2 \\ -5(-2)(-6) - (-2)(-9)(-2) \end{array} \\ &= -96 - 90 + 20 + 80 - 60 + 36 \\ &= -110. \end{aligned}$$

Problem 2

compute the determinants of the given matrix.

§7.1, Exercise 3. $C = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 0 \\ -3 & 2 & -2 & 0 & 0 \\ 1 & 1 & -1 & 2 & 4 \\ 0 & 2 & 3 & -1 & -3 \end{pmatrix}$.

Answer: The determinant of the matrix is 14.

Solution: Using Lemma 7.1.9, compute

$$\begin{aligned}
\det(A) &= \det\begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -3 & 2 & -2 \end{pmatrix} \det\begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix} \\
&= \big(2(-2)(-2) + 1\cdot 3(-3) + (-1)1\cdot 2 - (-1)(-2)(-3) - 1\cdot 1(-2) \\
&\quad -2\cdot 3\cdot 2\big)(-2) \\
&= (-7)(-2) \\
&= 14.
\end{aligned}$$

Problem 3

§7.1, Exercise 4. Find $\det(A^{-1})$ where $A = \begin{pmatrix} -2 & -3 & 2 \\ 4 & 1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$.

Answer: The solution is $\det(A^{-1}) = \frac{1}{35}$.

Solution: By Definition 7.1.1(c), $\det(A)\det(A^{-1}) = \det(I_3) = 1$. Therefore,

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Now compute $\det(A)$ using (7.1.9):

$$\det(A) = -2\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + 3\begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix} + 2\begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix} = 35.$$

Problem 4

§7.1, Exercise 15. Suppose that two $n\times p$ matrices A and B are row equivalent. Show that there is an invertible $n \times n$ matrix P such that $B = PA$.

By Proposition 7.1.4, every elementary row operation on A can be represented by an invertible $n \times n$ matrix E. That is, the matrix EA is row equivalent to A. If A and B are row equivalent, then there exist matrices E_j such that $B = E_k \dots E_1 A$. The product of invertible $n \times n$ matrices is an invertible $n \times n$ matrix. Thus $P = E_k \dots E_1$ is an invertible $n \times n$ matrix such that $B = PA$.

Problem 5

§7.2, Exercise 4. Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(a) Verify that the characteristic polynomial of A is $p_\lambda(A) = (\lambda - 1)(\lambda + 2)^2$.

(b) Show that $(1, 1, 1)$ is an eigenvector of A corresponding to $\lambda = 1$.

(c) Show that $(1, 1, 1)$ is orthogonal to every eigenvector of A corresponding to the eigenvalue $\lambda = -2$.

(a) Find the characteristic polynomial by solving

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda I_3) \\ &= \begin{pmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{pmatrix} \\ &= (-1-\lambda)\det\begin{pmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} - \det\begin{pmatrix} 1 & 1 \\ 1 & -1-\lambda \end{pmatrix} + \\ &\quad \det\begin{pmatrix} 1 & 1 \\ -1-\lambda & 1 \end{pmatrix} \\ &= -(\lambda^3 + 3\lambda - 4) \\ &= -(\lambda - 1)(\lambda + 2)^2. \end{aligned}$$

(b) Verify by computation:

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(c) Find the space of eigenvectors $v = (v_1, v_2, v_3)$ corresponding to $\lambda = -2$ by solving $(A - \lambda I_3)v = 0$ for $\lambda = -2$. That is, solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

to obtain

$$v = v_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Then compute

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0.$$

Since $(-1,1,0)^t$ and $(-1,0,1)^t$ form a basis for the space of eigenvectors of A corresponding to $\lambda = -2$ and since $(1,1,1)^t$ is orthogonal to these vectors, $(1,1,1)^t$ is orthogonal to every eigenvector of A corresponding to $\lambda = -2$.

Problem 6

§7.2, Exercise 7. Let A be an $n \times n$ matrix. Suppose that

$$A^2 + A + I_n = 0.$$

Prove that A is invertible.

We are given $A^2 + A + I_n = 0$. Therefore, $I_n = -A^2 - A = A(-A - I_n)$. Thus, $A^{-1} = -A - I_n$ exists.

Problem 7

Decide whether the given statements are *true* or *false*. If the statements are false, give a counterexample; if the statements are true, give a proof.

§7.2, Exercise 8. If the eigenvalues of a 2×2 matrix are equal to 1, then the four entries of that matrix are each less than 500.

Answer: The statement is false.

Solution: A counterexample is the matrix $A = \begin{pmatrix} 1 & 500 \\ 0 & 1 \end{pmatrix}$.

Problem 8

Decide whether the given statements are *true* or *false*. If the statements are false, give a counterexample; if the statements are true, give a proof.

§7.2, Exercise 9. If A is a 4×4 matrix and $\det(A) > 0$, then $\det(-A) > 0$.

Answer: The statement is true.

Solution: Since A is a 4×4 matrix, $\det(-A) = (-1)^4 \det(A) = \det(A) > 0$.

Problem 9

Decide whether the given statements are *true* or *false*. If the statements are false, give a counterexample; if the statements are true, give a proof.

§7.2, Exercise 10. The trace of the product of two $n \times n$ matrices is the product of the traces.

Answer: The statement is false.

Solution: For example, let

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}.$$

Then $\mathrm{tr}(A)\mathrm{tr}(B) = 2(1) = 2$, and $\mathrm{tr}(AB) = -1$.

Problem 10

§10.1, Exercise 6. Let A be an $n \times n$ real diagonalizable matrix. Show that $A + \alpha I_n$ is also real diagonalizable.

Let S be a matrix such that $D = S^{-1}AS$ is a diagonal matrix. Then

$$S^{-1}(A + \alpha I_n)S = S^{-1}AS + S^{-1}(\alpha I_n)S = D + \alpha I_n.$$

The matrices D and αI_n are both diagonal; so $D + \alpha I_n$ is also diagonal. Therefore, $A + \alpha I_n$ is diagonalizable.

Problem 11

§10.1, Exercise 9. Let A be an $n \times n$ matrix all of whose eigenvalues equal ± 1. Show that if A is diagonalizable, the $A^2 = I_n$.

Since A is diagonalizable, there is an invertible matrix S such that $S^{-1}AS$ is diagonal. The diagonal entries of $S^{-1}AS$ are the eigenvalues of A; that is, the diagonal entries equal ± 1. Therefore, $(S^{-1}AS)^2 = I_n$. But $(S^{-1}AS)^2 = S^{-1}A^2S$. Therefore, $S^{-1}A^2S = I_n$ which implies that $A^2 = I_n$.