Problem 1. Below is a list of statements. Decide which are true and which are false. On the left of each, write "TRUE" or "FALSE" in capital letters. You must also write your answer ("TRUE" or "FALSE" in capital letters) on the front page of the exam.

There is no partial credit on this problem.

(A) (2 points) Every square matrix is invertible.
Solution: FALSE. A square zero matrix is not invertible.

(B) (2 points) Suppose A is an $m \times n$ matrix in reduced row echelon form, and $A\vec{x} = \vec{0}$ has a unique solution. Then A has a pivot in every row.
Solution: FALSE. $A\vec{x} = \vec{0}$ has a unique solution if and only if A has a pivot in every **column**. For example, $\begin{pmatrix} 1 & 0 \end{pmatrix} x = 0$ has infinitely many solutions.

(C) (2 points) If A is an $m \times n$ matrix and the linear system $A\vec{x} = \vec{b}$ is consistent, then $m \leq n$.
Solution: FALSE. For example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is consistent and has the solution $x = 0$.

(D) (2 points) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in reduced row echelon form and $a = 0$, then A is the zero matrix.
Solution: FALSE. Since $a = 0$, we must have $c = 0$. Then it could be the case that $b = 1$ and $d = 0$.

(E) (2 points) If $AB = 0_n$ (the $n \times n$ zero matrix), then $BA = 0_n$.
Solution: FALSE. The matrices A and B need not be square! Even if they are square, it's still false. For example, if we let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

then $AB = 0$ implies $a = c = 0$. But then

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

which is nonzero as long as $b \neq 0$.

Problem 2. (*10 points*) You must show all work to get partial credit.

(a) (5 points) Use Gaussian elimination to compute the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

Solution: Form the following augmented matrix and row reduce:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right) \xrightarrow{\text{R2}\leftarrow\text{ R2+(-3)R1}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right)$$
$$\xrightarrow{\text{R2}\leftarrow\text{ (-1)R2}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right)$$
$$\xrightarrow{\text{R1}\leftarrow\text{ R1+(-2)R2}} \left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right)$$

We conclude that

$$A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

One can double check this using the formula for a 2×2 inverse as $\det(A) = -1$, or by directly checking $AA^{-1} = I_2$.

(b) (5 points) Find a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$, but $BA \neq I_3$.
Hint: You can do this only using 0*'s and* 1*'s for the entries of* A *and* B.
Solution: There are many answers here, including:

$$A = \begin{pmatrix} I_2 & 0_{2\times1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} I_2 \\ 0_{1\times2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One checks directly that $AB = I_2$, but $BA \neq I_2$.

Problem 3. (*10 points*) Follow the steps below to find all solutions to the system

$$\begin{aligned} 2x_1 + 4x_3 &= 6 \\ 2x_1 + x_2 + 6x_3 + x_4 &= 16 \\ x_2 + 2x_3 + x_4 &= 10. \end{aligned}$$

You must show all work to get partial credit.

(a) (*2 points*) Write down the augmented matrix $\left(A \mid \vec{b} \right)$ corresponding to the system $A\vec{x} = \vec{b}$.

Solution: $\left(\begin{array}{cccc|c} 2 & 0 & 4 & 0 & 6 \\ 2 & 1 & 6 & 1 & 16 \\ 0 & 1 & 2 & 1 & 10 \end{array}\right)$

(b) (*4 points*) Perform Gaussian elimination to the above augmented matrix to get a matrix in reduced row echelon form. Circle the pivots.

Solution:

$$\left(\begin{array}{cccc|c} 2 & 0 & 4 & 0 & 6 \\ 2 & 1 & 6 & 1 & 16 \\ 0 & 1 & 2 & 1 & 10 \end{array}\right) \xrightarrow{\text{R2}\leftarrow\text{R2+(-1)R1}} \left(\begin{array}{cccc|c} 2 & 0 & 4 & 0 & 6 \\ 0 & 1 & 2 & 1 & 10 \\ 0 & 1 & 2 & 1 & 10 \end{array}\right)$$

$$\xrightarrow{\text{R3}\leftarrow\text{R3+(-1)R2}} \left(\begin{array}{cccc|c} 2 & 0 & 4 & 0 & 6 \\ 0 & 1 & 2 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{R1}\leftarrow\text{(1/2)R1}} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 2 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(c) (*2 points*) How many free parameters are there?

Solution: Two. They correspond to the columns without pivots.

(d) (*2 points*) Write down the parametrized solutions to $A\vec{x} = \vec{b}$ in the form of a particular solution plus a superposition of solutions to the homogeneous system.

Solution: By looking at the row reduced matrix in (b) above, solutions to $A\vec{x} = \vec{b}$ must be of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 - 2r \\ 10 - 2r - s \\ r \\ s \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 4. Suppose $L : \mathbb{R}^2 \to \mathbb{R}^5$ is a linear transformation such that

$$L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix} \qquad \text{and} \qquad L\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}4\\3\\2\\1\\0\end{pmatrix}.$$

(a) (*4 points*) Express e_1 and e_2 in $\mathbb{R}^2$ as linear combinations of $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$.

Solution: Observe that

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix} \qquad \text{and} \qquad e_2 = \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}.$$

(b) (*3 points*) Use your answer in part (a) to compute $L(e_1)$ and $L(e_2)$.

Solution: Since L is a linear transformation, we have

$$L(e_1) = \frac{1}{2}L\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}L\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}4\\3\\2\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\2\\2\\2\end{pmatrix}$$

$$L(e_2) = \frac{1}{2}L\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}L\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix} - \frac{1}{2}\begin{pmatrix}4\\3\\2\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\-1\\0\\1\\2\end{pmatrix}$$

(c) (*3 points*) Find the 5×2 matrix A such that $L = L_A$.

Solution: The matrix is given by

$$A = \big(\ L(e_1)\ \big|\ L(e_2)\ \big) = \begin{pmatrix}2 & -2\\2 & -1\\2 & 0\\2 & 1\\2 & 2\end{pmatrix}.$$

Problem 5. Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ x_2 + x_3 &= b \\ ax_1 - x_3 &= 1. \end{aligned}$$

Find $a, b \in \mathbb{R}$ such that this system has infinitely many solutions. You must show all your steps and explain your reasoning.

Solution1: We form the following augmented matrix and begin to row reduce:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & b \\ a & 0 & -1 & 1 \end{array}\right) \xrightarrow{\text{R3}\leftarrow\text{R3 + (-}a\text{)R1}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & b \\ 0 & -2a & -1-a & 1-3a \end{array}\right)$$

$$\xrightarrow{\text{R3}\leftarrow\text{R3 + (2}a\text{)R2}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & b \\ 0 & 0 & -1+a & 1-3a+2ab \end{array}\right)$$

We will get infinitely many solutions if and only if the last row is all zeros. This means we must have $-1 + a = 0$, so $a = 1$. Then we must have $-2 + 2b = 0$, so $b = 1$.

Solution2: We form the following augmented matrix and begin to row reduce:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & b \\ a & 0 & -1 & 1 \end{array}\right) \xrightarrow{\text{R1}\leftarrow\text{R1 + (-2)R2}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 3-2b \\ 0 & 1 & 1 & b \\ a & 0 & -1 & 1 \end{array}\right)$$

We will get infinitely many solutions if and only if the first row and the last row are proportional. This means there must be a single scalar s such that

$$s\begin{pmatrix} 1 & 0 & -1 & 3-2b \end{pmatrix} = \begin{pmatrix} a & 0 & -1 & 1 \end{pmatrix}$$

Looking at the third entry, we see that $s = 1$, so we must have $a = 1$ and $3 - 2b = 1$, so $b = 1$.