MATH 2568 EXAM 2 SAMPLE SOLUTIONS

Problem 1. Below is a list of statements. Decide which are true and which are false. On the left of each, write "TRUE" or "FALSE" in capital letters. You must also write your answer ("TRUE" or "FALSE" in capital letters) on the front page of the exam.

There is no partial credit on this problem.

(A) (2 points) The set of all non-invertible 2×2 matrices is a subspace of $M_{2\times 2}(\mathbb{R})$.
Solution: False. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are not invertible, but their sum I_2 is invertible.

(B) (2 points) Suppose A is an $n \times n$ matrix. The columns of A span $\mathbb{R}^n$ if and only if the rows of A span $\mathbb{R}^n$.
Solution: True. Both are equivalent to A has rank n.

(C) (2 points) Suppose A is a matrix such that $NS(A) = \{0\}$. Then A is invertible.
Solution: False. Consider $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The answer would be true if A were square.

(D) (2 points) Suppose A is an $n \times n$ matrix. If the only eigenvalue of A is zero, then A is the zero matrix.
Solution: False. Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(E) (2 points) Suppose A is a 2×2 matrix. Then A and A^T have the same eigenvalues.
Solution: True. We calculate

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc = \begin{vmatrix} \lambda - a & -c \\ -b & \lambda - d \end{vmatrix} = \det(\lambda I - A).$$

Problem 2. (*10 points*) Fill in the blanks from the word bank below to complete the proof of a theorem from class. Each word may be used more than once in the proof below.

Let A be an $m \times n$ matrix. The rest of the proof uses $E :=$RREF(A), which is obtained from A by elementary row operations. The number of columns of E *with* pivots is equal to the rank of A. The number of columns of E *without* rank is equal to the nullity of A. Hence the rank (or nullity) plus the nullity (or rank) is equal to n.

Word bank: A, RREF(A), rows, columns, pivots, dimension, basis, rank, nullity, m, n

Problem 3. (*10 points*) Find bases for the null, row, and columns spaces of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 5 & 4 \\ 3 & 6 & -1 & 8 & 5 \end{pmatrix}.$$

Solution: We row reduce:

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 5 & 4 \\ 3 & 6 & -1 & 8 & 5 \end{pmatrix} \xrightarrow{\text{R2} \leftarrow \text{R2 + (-2)R1}} \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & -1 & -1 & 2 \\ 3 & 6 & -1 & 8 & 5 \end{pmatrix}$$

$$\xrightarrow{\text{R3} \leftarrow \text{R3 + (-3)R1}} \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{\text{R2} \leftarrow \text{R2 + (-1)R3}} \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{R2} \leftarrow \text{(-1)R2}} \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We find a basis for the null space as follows. The free variables are $x_2 = r$, $x_4 = s$, and $x_5 = t$. We have

$$x_1 + 2r + 3s + t = 0 \qquad \text{and} \qquad x_3 + s - 2t = 0.$$

This means

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2r - 3s - t \\ r \\ -s + 2t \\ s \\ t \end{pmatrix} = r \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

is the general solution to $A\vec{x} = \vec{0}$. A basis for the null space is then given by

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We get a basis for the row space by taking the transpose of the nonzero rows in the reduced row echelon form above:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \end{pmatrix} \right\}.$$

Finally, we get a basis for the column space by looking at where the pivots lie in the reduced row echelon form, and taking the corresponding columns of A:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right\}.$$

Notice that the nullity (3) plus the rank (2) is equal to the number of columns (5).

Problem 4. (*10 points*) Consider the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$$

1. (3 points) Find a basis of $\mathbb{R}^2$ consisting of an eigenvector and a generalized eigenvector for A.
   Solution: First we calculate the characteristic polynomial:
   $$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 3 & 1 \\ -4 & \lambda + 1 \end{pmatrix} = (\lambda - 3)(\lambda + 1) + 4 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2,$$
   We see that the vector
   $$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
   lies in the null space of $I - A$, and is thus an eigenvector for A corresponding to $\lambda = 1$. Now we need to find a w such that $(A - I)w = v$. Guess and check easily finds a few choices:
   $$w = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2. (4 points) Find a basis for the solution space of the linear system $AX(t) = X'(t)$.
   Solution: A basis is given by $\{X_1(t), X_2(t)\}$ where
   $$X_1(t) = e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad X_1(t) = e^t \left(w + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$
   where w is any generalized eigenvector satisfying $(A - I)w = v$ as in part 1.

3. (3 points) Find the particular solution with initial condition $X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
   Solution: We need to find $\alpha_1, \alpha_2 \in \mathbb{R}$ such that
   $$X_0 = \alpha_1 X_1(0) + \alpha_2 X(0) \qquad \Longleftrightarrow \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 w$$
   The specific α_1 will depend on your choice of w, but α_2 and the final solution will not.
   $$w = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \Longrightarrow \quad \alpha_1 = 1, \alpha_2 = 2$$
   $$w = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad \alpha_1 = -1, \alpha_2 = 2$$
   In either case, the particular solution is
   $$X(t) = e^t \begin{pmatrix} 1 + 2t \\ 4t \end{pmatrix}.$$

Problem 5. Let V be a vector space containing vectors v_1, v_2, v_3. Suppose w_1, w_2, w_3 are vectors contained in $\text{span}\{v_1, v_2, v_3\}$. Show that $\text{span}\{w_1, w_2, w_3\} \subset \text{span}\{v_1, v_2, v_3\}$. That is, show that an arbitrary $w \in \text{span}\{w_1, w_2, w_3\}$ is an element of $\text{span}\{v_1, v_2, v_3\}$.

Solution: We know that $w_1, w_2, w_3 \in \text{span}\{v_1, v_2, v_3\}$. This means there are real numbers $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ such that

$$\begin{aligned} w_1 &= a_1 v_1 + a_2 v_2 + a_3 v_3 \\ w_2 &= b_1 v_1 + b_2 v_2 + b_3 v_3 \\ w_3 &= c_1 v_1 + c_2 v_2 + c_3 v_3 \end{aligned}$$

Since $w \in \text{span}\{w_1, w_2, w_3\}$, there are real numbers d_1, d_2, d_3 such that

$$\begin{aligned} w &= d_1 w_1 + d_2 w_2 + d_3 w_3 \\ &= d_1(a_1 v_1 + a_2 v_2 + a_3 v_3) + d_2(b_1 v_1 + b_2 v_2 + b_3 v_3) + d_3(c_1 v_1 + c_2 v_2 + c_3 v_3) \\ &= \underbrace{(d_1 a_1 + d_2 b_1 + d_3 c_1)}_{x} v_1 + \underbrace{(d_1 a_2 + d_2 b_2 + d_3 c_2)}_{y} v_2 + \underbrace{(d_1 a_3 + d_2 b_3 + d_3 c_3)}_{z} v_3 \\ &= x v_1 + y v_2 + z v_3 \end{aligned}$$

where x, y, z are real numbers. Thus w is in $\text{span}\{v_1, v_2, v_3\}$.