Problem 1. True or false? Explain your reasoning. If false, find a condition on A which makes it true.

(1) If the columns of A are linearly independent, then so are the rows.
(2) If A is invertible, then so is A^T.
(3) If $Ax = b$ has a unique solution for some $b \in \mathbb{R}^m$, then $\text{nullity}(A) = 0$.
(4) If the only eigenvalue of A is 1, then A is diagonalizable.
(5) If A maps orthogonal vectors to orthogonal vectors, then A is orthogonal.

Problem 2. Short answer. The answers to all questions below are a single digit between 0 and 9.

(1) Compute the determinant of $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.
(2) Suppose 1 is an eigenvalue of A. What number is then an eigenvalue of $A + I$?
(3) Suppose A is a 7×9 matrix whose rank is 4. What is the dimension of the row space of A?
(4) The matrix $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix}$ has only one eigenvalue. What is its geometric multiplicity?
(5) Suppose $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 5 & k \end{pmatrix}$ is not invertible. What is k?

Problem 3. Find the Jordan normal form of the following matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

Problem 4. Find an orthogonal matrix Q such that $Q^T A Q$ is diagonal for

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Problem 5. Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(1) Find 3 linearly independent solutions $X_1(t), X_2(t), X_3(t)$ of the differential equation $X'(t) = AX(t)$.
(2) Find a particular solution of the form $a_1X_1(t) + a_2X_2(t) + a_3X_3(t)$ when $X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Problem 6. Find bases for the column space, row space, and null space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{pmatrix}.$$

Problem 7. Let $v \in \mathbb{R}^3$ and let A be a 3×3 matrix. Suppose v, Av, A^2v are all nonzero and $A^3 = 0$.

(1) Show that $B = \{v, Av, A^1v\}$ is linearly independent. Deduce B is a basis for $\mathbb{R}^3$.
(2) Calculate $[L_A]_B$ where L_A is the linear map given by $L_Ax = Ax$.

Problem 8. Suppose A, B are $n \times n$ matrices such that $AB = I_n$. Show that $BA = I_n$.