Problem 1. True or false? Explain your reasoning. If false, find a condition on A which makes it true.

(1) The function $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a linear map, where $M_{n\times n}(\mathbb{R})$ denotes the vector spaces of $n \times n$ matrices with the entries in $\mathbb{R}$.
(2) If an $n \times n$ matrix A has rank n, then $\det(A) = 0$.
(3) For any $n \times n$ matrix A, $\det(A)$ is the product of its eigenvalues, counting multiplicities.
(4) Similar matrices always have the same eigenvalues.
(5) Similar matrices always have the same eigenvectors.
(6) Any linear map on $\mathbb{R}^n$ that has fewer than n distinct eigenvalues is not real diagonalizable.
(7) For any (not necessarily square) matrix A, the rank of A is equal to the rank of A^T.
(8) For any $n \times n$ matrix A, we have $\det(kA) = k^n \det(A)$.
(9) There is a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(1, 2, 3) = (0, 1)$ and $L(2, 4, 6) = (1, 1)$.

Problem 2. Evaluate the determinant of the matrix

$$A = \begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$$

in two ways:

(1) First, write $A = E_n E_{n-1} \cdots E_1 U$, where E_i are the elementary matrices and U an upper triangular matrix; then from this expression, evaluate $\det A$.
(2) Secondly, perform a cofactor expansion along the fourth row to evaluate $\det A$.

Problem 3. An $n \times n$ matrix A is called orthogonal if $AA^T = I_n$. Show that if A is orthogonal, then $\det(A) = \pm 1$.

Problem 4. Decide whether the following matrix is real diagonalizable. If so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}.$$

Problem 5. Let V be the vector space of polynomials of degree ≤ 2, and consider the following map of vector spaces:

$$L : V \to V$$
$$ax^2 + bx + c \mapsto (-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2.$$

(1) For the basis $B = \{x^2, x, 1\}$, write down the matrix that represents L with respect to B.
(2) For the basis $C = \{x - x^2, -1 + x^2, -1 - x + x^2\}$, what is the matrix that represents L with respect to C?

Problem 6. Find bases for the column space, row space, and null space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}.$$

Problem 7. Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\\2\\1\end{pmatrix}.$$

Use the Gram-Schmidt process to compute an orthonormal basis for V.

Problem 8. Let A be a 3×3 matrix with 3 distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Show that the eigenvectors corresponding to these eigenvalues are linearly independent.