Math 3345 Homework 5

Recall from last week's class that we defined the equivalence class of an integer modulo $m$. Fix $m \in \mathbb{N}$. We define the *equivalence class of a module* $m$ to be

$$\overline{a} = \{b \in \mathbb{Z} | a \equiv b \mod m\}.$$

Recall that we defined addition and multiplication for equivalence classes by

$$\overline{a} + \overline{b} = \overline{a+b} \text{ and}$$
$$\overline{a} \cdot \overline{b} = \overline{ab}.$$

To show these operations are *well-defined*, we must check that if $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$, then

$$\overline{a_1 + b_1} = \overline{a_2 + b_2} \text{ and}$$
$$\overline{a_1 b_1} = \overline{a_2 b_2}.$$

We did this in class assuming using the fact from the problem below.

**Problem 1.** Fix $m \in \mathbb{N}$. Suppose $a, b \in \mathbb{Z}$. Show that $\overline{a} = \overline{b}$ if and only if $a \equiv b \mod m$.

**Problem 2.** Fix $m \in \mathbb{N}$ such that $m \geq 2$. Show that the following two properties are equivalent.

(a) $m$ is prime.

(b) For all $a, b \in \mathbb{Z}$, $\overline{a} \cdot \overline{b} = \overline{0}$ implies $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$.

**Problem 3.** (Falkner Section 5 Exercise 1) Use the Principle of Mathematical Induction to prove that for all $n \in \mathbf{N}$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

**Problem 4.** (Edited from Falkner Section 5 Exercise 6) Prove that for each $n \in \mathbb{N}$, 6 divides $n^3 - n$.

**Problem 5.**

(a) Falkner Section 5 Exercise 11

(b) Falkner Section 5 Exercise 12
*Hint: Use part (a).*

**Optional problem 1.** Suppose you have cards with 3 sides. Each card has a number on side 1, a letter on side 2, and a color (red or blue) on side 3. You can only see one side at a time, and you know which way to flip the card to see the other two sides. You want to verify the following rule:

- If side 1 shows an even number then (if side 2 shows a vowel, then side 3 must show the color blue).

You see cards which display 1, 2, A, B, red, and blue. Which cards do you need to flip to verify the rule? Which ways do you need to flip them?