Math 3345 Exam 1 Solutions

Problem 1. Suppose there are four cards lying on a table. One shows the number 4, one shows the number 7, one is plain blue, and one is plain green. Which cards must you flip over to determine if the following rule is valid?

- A card has one side plain blue whenever the opposite side has an even number.

Fully explain your answer.

Solution. This rule can be restated as, "When one side is even, the other side must be plain blue." We can think of this rule as "For all cards x, $P(x)$ implies $Q(x)$ where $P(x)$ is the statement "one side of x is even" and $Q(x)$ is "the other side of x is blue".

1. We must flip over the card showing 4, since we must check if the other side is blue to verify the rule $P(x) \Rightarrow Q(x)$.
2. We do not need to flip over the card showing 7. Here, we have $\neg P(x)$, so $P(x) \Rightarrow Q(x)$ holds vacuously.
3. We do not need to flip over the blue card. Here, $Q(x)$ holds, so $P(x)$ always implies $Q(x)$ regardless of the truth value of $P(x)$.
4. We must flip over the green card, since we have $\neg Q(x)$, which must imply $\neg P(x)$. This is the contrapositive of $P(x) \Rightarrow Q(x)$, which is logically equivalent. That is, we must check that the other side of this card is not even, or else the rule would be violated. □

Problem 2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Consider the following statement, which will denote P:

- For every $\varepsilon > 0$, there is a $M > 0$ such that $x > M$ implies $|f(x) - L| < \varepsilon$.

Deduce the statement $\neg P$, and explain all your steps. Simplify your answer as much as possible.

Solution. Recall that the negation of $(\forall a \in A)P(a)$ is $(\exists a \in A)\neg P(a)$, and the negation of $(\exists b \in B)Q(b)$ is $(\forall b \in B)\neg Q(b)$ by De Morgan's laws. Applying this, we see that the negation starts with "There is an $\varepsilon > 0$ such that for every $M > 0$..."

The next step is a bit tricky. We want to negate $x > M$ implies $|f(x) - L| < \varepsilon$. However, x is not a free variable in this sentence. Here, x is any arbitrary number bigger than M, and any time $x > M$, we have $|f(x) - L| < \varepsilon$. So this part can also be written as "for all $x > M$, $|f(x) - L| < \varepsilon$." This means the original statement is also read as:

- For every $\varepsilon > 0$, there is a $M > 0$ such that for all $x > M$, $|f(x) - L| < \varepsilon$.

Thus there is a hidden quantifier lurking here using universal generalization. So again using De Morgan's laws, the negation reads: "There is an $\varepsilon > 0$ such that for every $M > 0$, there is an $x > M$ such that ..."

Finally, we need to negate the inner most statement, which is $|f(x) - L| < \varepsilon$. The negation here is $|f(x) - L| \geq \varepsilon$. Thus the final answer is:

- There is an $\varepsilon > 0$ such that for every $M > 0$, there is an $x > M$ such that $|f(x) - L| \geq \varepsilon$.

$\square$

Remark. Most people did not find this hidden quantifier, and instead they negated an implication $P \Rightarrow Q$ to obtain $P \wedge \neg Q$. So I only took 1 point off for this. Unfortunately, most people also did not translate the symbols back into English words, and so the most common answer was:

- $(\exists \varepsilon > 0)(\forall M > 0)[(x > M) \wedge |f(x) - L| \geq \varepsilon]$.

The problem with this is that when you reinterpret the symbolic sentence into English words, you get a statement that does not make any sense:

- There is a $\varepsilon > 0$ such that for all $M > 0$, $x > M$ and $|f(x) - L| \geq \varepsilon$.

It's not possible for an $x \in \mathbb{R}$ to be greater than M for every $M > 0$.

Problem 3. It is known that π is irrational. Using this, prove that for all $x > 0$,

$$\sqrt{\frac{\pi}{x}} \text{ is irrational or } \sqrt{x\pi} \text{ is irrational.}$$

You may use without proof facts from the class and from the homework about combining rational numbers with rational numbers, but not facts about combining irrational numbers with rational numbers.

Solution. We are asked to prove "For all $x > 0$,

$$\sqrt{\frac{\pi}{x}} \text{ is irrational or } \sqrt{x\pi} \text{ is irrational."}$$

We will give a proof by contradiction.

Assume for contradiction that the above statement is false, i.e., it's negation is true. The negation is "There is an $x > 0$ such that $\sqrt{\frac{\pi}{x}}$ and $\sqrt{\pi x}$ are rational. We know from class that the product of two rational numbers is rational, so

$$\sqrt{\frac{\pi}{x}} \cdot \sqrt{x\pi} = \sqrt{\frac{\pi^2 x}{x}} = \sqrt{\pi^2} = \pi$$

is rational. But this is a contradiction to the fact that π is irrational. This is a contradiction.

$\square$

Remark. Here, as announced during the test, you must prove the statement for any real number $x > 0$. The most common mistake was assuming x was an integer or assuming x was rational.

Problem 4. Suppose $a, b \in \mathbb{N}$. Prove that if a divides b and b divides a, then $a = b$.

We'll give two solutions. One will be from the basic definition of 'divides', and the other will use a proposition from class.

Solution 1. Suppose that $a, b \in \mathbb{N}$ such that a divides b and b divides a. Then there are $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that $a = kb$ and $b = na$. We then have that $a = kb = k(na) = (kn)a$, so $(kn)a - a = (kn - 1)a = 0$. Since $a \neq 0$, we know $kn - 1 = 0$. But k and n are integers, so either $k = n = 1$ or $k = n = -1$. Since a and b are both greater than zero, we must have $k = n = 1$, and thus $a = b$. □

Remark. We also showed in class that if $a, b \in \mathbb{N}$ such that a divides b, then there is a $k \in \mathbb{N}$ such that $b = ka$. That is, we may assume the k is actually in $\mathbb{N}$, not just in $\mathbb{Z}$. This shortens the above proof considerably as follows:

Suppose that $a, b \in \mathbb{N}$ such that a divides b and b divides a. Then there are $k \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $a = kb$ and $b = na$. We then have that $a = kb = k(na) = (kn)a$, so $(kn)a - a = (kn - 1)a = 0$. Since $a \neq 0$, we know $kn - 1 = 0$. But $k, n \in \mathbb{N}$, so $k = n = 1$. Thus $a = b$.

Solution 2. In class, we proved that if we have $d, n \in \mathbb{N}$ such that d divides n, then $d \leq n$. Since a divides b, $a \leq b$. Since b divides a, $b \leq a$. Thus $a \leq b$ and $b \leq a$. The only way this is possible is if $a = b$. □

Remark. The most common mistake here was dividing. You are not allowed to divide when doing 'a divides b' proofs.

Problem 5. Use the method of conditional proof to explain in words why the sentence

$$(P \Rightarrow Q) \Rightarrow [(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)]$$

is a tautology. Be careful not to skip any steps. Be explicit about discharging assumptions.

Solution.

<u>A1:</u> Assume $P \Rightarrow Q$.
We need to show $(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.

<u>A2:</u> Assume $(Q \Rightarrow R)$.
We need to show $(P \Rightarrow R)$.

<u>A3:</u> Assume P.
We need to show R. We have P by A3, and by A1, $P \Rightarrow Q$. By modus ponens, we have Q.
Now we have Q and by A2, we have $Q \Rightarrow R$. By modus ponens, we have R.

Discharging A3, we see that $P \Rightarrow R$ under A1 and A2 alone.

Discharging A2, we see that $(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$ under A1 alone.

Discharging A1, we see that $(P \Rightarrow Q) \Rightarrow [(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)]$ under no assumptions. This means it is always true, so it is a tautology. $\square$

Remark. Most mistakes here were in the discharging phase of the proof. The sentences below are similar to the most common mistakes.

- "Discharging A3, we see that $P \Rightarrow R$ under no assumptions."
  This is not true. You still need to know A1 and A2 to conclude $P \Rightarrow R$.

- "We see that R is true. Thus discharging A3, R is still true, so $P \Rightarrow R$ under A1 and A2."
  This is not true. When you discharge A3, you no longer can conclude R is true under A1 and A2, but you can conclude $P \Rightarrow R$ under A1 and A2. The difference here is that if you only have A1 and A2, then you only know $P \Rightarrow Q$ and $Q \Rightarrow R$. However, once you discharge A3, you no longer know P is true, so you cannot conclude that R is true.