## 1. Topology

**Problem 1.** Two metrics $\rho_1, \rho_2$ on $X$ are called *equivalent* if there is a $C > 0$ such that

$$C^{-1}\rho_1(x,y) \le \rho_2(x,y) \le C\rho_1(x,y) \qquad \forall x, y \in X.$$

Show that equivalent metrics induce the same topology on $X$. That is, show that $U \subset X$ is open with respect to $\rho_1$ if and only if $U$ is open with respect to $\rho_2$.

**Problem 2** (Sarason)**.** Let $(X, \rho)$ be a metric space.

(1) Let $\alpha : [0, \infty) \to [0, \infty)$ be a continuous non-decreasing function satisfying
- $\alpha(s) = 0$ if and only if $s = 0$, and
- $\alpha(s + t) \le \alpha(s) + \alpha(t)$ for all $s, t \ge 0$.

Define $\sigma(x, y) := \alpha(\rho(x, y))$. Show that $\sigma$ is a metric, and $\sigma$ induces the same topology on $X$ as $\rho$.

(2) Define $\rho_1, \rho_2 : X \times X \to [0, \infty)$ by

$$\rho_1(x,y) := \begin{cases} \rho(x,y) & \text{if } \rho(x,y) \le 1 \\ 1 & \text{otherwise.} \end{cases}$$

$$\rho_2(x,y) := \frac{\rho(x,y)}{1+\rho(x,y)}.$$

Use part (1) to show that $\rho_1$ and $\rho_2$ are metrics on $X$ which induce the same topology on $X$ as $\rho$.

**Problem 3.** A collection of subsets of $(F_i)_{i\in I}$ of $X$ has the *finite intersection property* if for any finite $J \subset I$, we have $\bigcap_{j\in J} F_j \neq \emptyset$. Prove that for a metric (or topological) space, the following are equivalent.

(1) Every open cover of $X$ has a finite subcover.
(2) For every collection of closed subsets $(F_i)_{i\in I}$ with the finite intersection property, $\bigcap_{i\in I} F_i \neq \emptyset$.

**Problem 4** (Adapted from Wikipedia `https://en.wikipedia.org/wiki/Locally_compact_space`)**.** Consider the following conditions:

(1) Every point of $X$ has a compact neighborhood.
(2) Every point of $X$ has a closed compact neighborhood.
(3) Every point of $X$ has a relatively compact neighborhood.
(4) Every point of $X$ has a local base of relatively compact neighborhoods.
(5) Every point of $X$ has a local base of compact neighborhoods.
(6) For every point $x$ of $X$, every neighborhood of $x$ contains a compact neighborhood of $x$.

Determine which conditions imply which other conditions. Then show all the above conditions are equivalent when $X$ is Hausdorff.

**Problem 5.** Suppose $(X, \tau)$ is a locally compact Hausdorff topological space and suppose $K \subset X$ is a non-empty compact set.

(1) Suppose $K \subset U$ is an open set. Show there is a continuous function $f : X \to [0, 1]$ with compact support such that $f|_K = 1$ and $f|_{U^c} = 0$.
(2) Suppose $f : K \to \mathbb{C}$ is continuous. Show there is a continuous function $F : X \to \mathbb{C}$ such that $F|_K = f$.

**Problem 6.** Suppose $(X, \tau)$ is a locally compact topological space and $(f_n)$ is a sequence of continuous $\mathbb{C}$-valued functions on $X$. Show that the following are equivalent:

(1) There is a continuous function $f : X \to \mathbb{C}$ such that $f_n|_K \to f|_K$ uniformly on every compact $K \subset X$.
(2) For every compact $K \subset X$, $(f_n|_K)$ is uniformly Cauchy.

**Problem 7.**

(1) Show that every open subset of $\mathbb{R}$ is a countable union of open intervals where both endpoints are rational.
(2) Suppose $U \subset \mathbb{R}$ is open and suppose $((a_j, b_j))_{j \in J}$ is a collection of open intervals which cover $U$:
$$U \subset \bigcup_{j \in J} (a_j, b_j).$$
Show there is a countable sub-cover, i.e., show that there is a countable subset $I \subset J$ such that
$$U \subset \bigcup_{i \in I} (a_i, b_i).$$
(3) Suppose $((a_j, b_j])_{j \in J}$ is a collection of half-open intervals which cover $(0, 1]$:
$$(0, 1] \subset \bigcup_{j \in J} (a_j, b_j].$$
Show there is a countable sub-cover, i.e., show that there is a countable subset $I \subset J$ such that
$$(0, 1] \subset \bigcup_{i \in I} (a_i, b_i].$$

**Problem 8.** Suppose $X$ is a locally compact Hausdorff space, $K \subset X$ is compact, and $\{U_1, \ldots, U_n\}$ is an open cover of $K$. Prove that there are $g_i \in C_c(X, [0, 1])$ for $i = 1, \ldots, n$ such that $g_i = 0$ on $U_i^c$ and $\sum_{i=1}^n g_i = 1$ everywhere on $K$.

**Problem 9** (Pedersen *Analysis Now*, E 1.3.4 and E 1.3.6)**.** A *filter* on a set $X$ is a collection $\mathcal{F}$ of non-empty subsets of $X$ satisfying

- $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, and
- $A \in \mathcal{F}$ and $A \subset B$ implies $B \in \mathcal{F}$.

Suppose $\tau$ is a topology on $X$. We say a filter $\mathcal{F}$ *converges* to $x \in X$ if every open neighborhood $U$ of $x$ lies in $\mathcal{F}$.

(1) Show that $A \subset X$ is open if and only if $A \in \mathcal{F}$ for every filter $\mathcal{F}$ that converges to a point in $A$.
(2) Show that if $\mathcal{F}$ and $\mathcal{G}$ are filters and $\mathcal{F} \subset \mathcal{G}$ ($\mathcal{G}$ is a *subfilter* of $\mathcal{F}$), then $\mathcal{G}$ converges to $x$ whenever $\mathcal{F}$ converges to $x$.
(3) Suppose $(x_\lambda)$ is a net in $X$. Let $\mathcal{F}$ be the collection of sets $A$ such that $(x_\lambda)$ is eventually in $A$. Show that $\mathcal{F}$ is a filter. Then show that $x_\lambda \to x$ if and only if $\mathcal{F}$ converges to $x$.

**Problem 10** (Pedersen *Analysis Now*, E 1.3.5)**.** A filter $\mathcal{F}$ on a set $X$ is called an *ultrafilter* if it is not properly contained in any other filter.

(1) Show that a filter $\mathcal{F}$ is an ultrafilter if and only if for every $A \subset X$, we have either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.
(2) Use Zorn's Lemma to prove that every filter is contained in an ultrafilter.

**Problem 11.** Let $(X, \tau)$ be a topological space. A net $(x_\lambda)_{\lambda \in \Lambda}$ is called *universal* if for every subset $Y \subset X$, $(x_\lambda)$ is either eventually in $Y$ or eventually in $Y^c$.

(1) Show that every net has a universal subnet.
(2) Show that $(X, \tau)$ is compact if and only if every universal net converges.
*Note: You may use part (1) to prove part (2) even if you choose not to prove part (1).*

*Hint for (1): Let $(x_\lambda)$ be a net in $X$. Define a* filter for $(x_\lambda)$ *to be a collection $\mathcal{F}$ of non-empty subsets of $X$ such that:*

- *$\mathcal{F}$ is closed under finite intersections,*
- *If $F \in \mathcal{F}$ and $F \subset G$, then $G \in \mathcal{F}$, and*
- *$(x_\lambda)$ is frequently in every $F \in \mathcal{F}$.*

*(1) Show that the set of filters for $(x_\lambda)$ is non-empty.*
*(2) Order the set of filters for $(x_\lambda)$ by inclusion. Show that if $(\mathcal{F}_j)$ is a totally ordered set of filters for $(x_\lambda)$, then $\cup \mathcal{F}_j$ is also a filter for $(x_\lambda)$.*
*(3) Use Zorn's Lemma to assert there is a maximal filter $\mathcal{F}$ for $(x_\lambda)$.*
*(4) Show that $\mathcal{F}$ is an ultrafilter.*
*(5) Find a subnet of $(x_\lambda)$ that is universal.*

**Problem 12.** Show the following collections of functions are uniformly dense in the appropriate algebras:

(1) For $a < b$ in $\mathbb{R}$, the polynomials $\mathbb{R}[t] \subset C([a, b], \mathbb{R})$.
(2) For $a < b$ in $\mathbb{R}$, the piece-wise linear functions $PWL \subset C([a, b], \mathbb{R})$.
(3) For $K \subset \mathbb{C}$ compact, the polynomials $\mathbb{C}[z, \overline{z}] \subset C(K)$.
(4) For $\mathbb{R}/\mathbb{Z}$, the trigonometric polynomials $\operatorname{span}\{\sin(2\pi nx), \cos(2\pi nx) | n \in \mathbb{N} \cup \{0\}\} \subset C(\mathbb{R}/\mathbb{Z}, \mathbb{R})$.

**Problem 13.** Let $X, Y$ be compact Hausdorff spaces. For $f \in C(X)$ and $g \in C(Y)$, define $(f \otimes g)(x, y) := f(x)g(y)$. Prove that $\operatorname{span}\{f \otimes g | f \in C(X) \text{ and } g \in C(Y)\}$ is uniformly dense in $C(X \times Y)$.

**Problem 14.** Suppose $X$ is locally compact Hausdorff and $A \subset C_0(X, \mathbb{C})$ is a subalgebra which separates points and is closed under complex conjugation. Show that either $\overline{A} = C_0(X, \mathbb{C})$ or there is an $x_0 \in X$ such that $\overline{A} = \{f \in C_0(X, \mathbb{C}) | f(x_0) = 0\}$.

**Problem 15** (Adapted from `http://u.cs.biu.ac.il/~tsaban/RT/Book/Chapter3.pdf`)**.** Let $\mathcal{U}\mathbb{N}$ be the set of ultrafilters on $\mathbb{N}$. For a subset $S \subset \mathbb{N}$, define $[S] := \{\mathcal{F} \in \mathcal{U}\mathbb{N} | S \in \mathcal{F}\}$. Show that the function $S \mapsto [S]$ satisfies the following properties:

(1) $[\emptyset] = \emptyset$ and $[\mathbb{N}] = \mathcal{U}\mathbb{N}$.
(2) For all $S, T \subset \mathbb{N}$,
  (a) $[S] \subset [T]$ if and only if $S \subset T$.
  (b) $[S] = [T]$ if and only if $S = T$.
  (c) $[S] \cup [T] = [S \cup T]$.
  (d) $[S] \cap [T] = [S \cap T]$.
  (e) $[S^c] = [S]^c$.
(3) Find a sequence of subsets $(S_n)$ of $\mathbb{N}$ such that $[\bigcup S_n] \neq \bigcup [S_n]$.
(4) Find a sequence of subsets $(S_n)$ of $\mathbb{N}$ such that $[\bigcap S_n] \neq \bigcap [S_n]$.

**Problem 16** (Adapted from `http://u.cs.biu.ac.il/~tsaban/RT/Book/Chapter3.pdf`)**.** Assume the notation of Problem 15.

(1) Show that $\{[S] | S \subset \mathbb{N}\}$ is a base for a topology on $\mathcal{U}\mathbb{N}$.
(2) Show that all the sets $[S]$ are both closed and open in $\mathcal{U}\mathbb{N}$.
(3) Show that $\mathcal{U}\mathbb{N}$ is compact.
(4) For $n \in \mathbb{N}$, let $\mathcal{F}_n = \{S \subset \mathbb{N} | n \in S\}$. Show $\mathcal{F}_n$ is an ultrafilter on $\mathbb{N}$.
*Note: Each $\mathcal{F}_n$ is called a* principal *ultrafilter on* $\mathbb{N}$.
(5) Show that $\{\mathcal{F}_n | n \in \mathbb{N}\}$ is dense in $\mathcal{U}\mathbb{N}$.

(6) Show that for every compact Hausdorff space $K$ and every function $f : \mathbb{N} \to K$, there is a continuous function $\widetilde{f} : \mathcal{U}\mathbb{N} \to K$ such that $\widetilde{f}(\mathcal{F}_n) = f(n)$ for every $n \in \mathbb{N}$. Deduce that $\mathcal{U}\mathbb{N}$ is homeomorphic to the Stone-Čech compactification $\beta\mathbb{N}$.
*Hint: Show that* $f^*(\mathcal{F}) := \left\{A \subset K \middle| f^{-1}(A) \in \mathcal{F}\right\}$ *is an ultrafilter on* $K$*. Show that since* $K$ *is compact Hausdorff, every ultrafilter on* $K$ *converges to a unique point in* $K$*. Set* $\widetilde{f}(\mathcal{F}) := \lim f^*(\mathcal{F})$*. For an open neighborhood* $U$ *of* $\lim f^*(\mathcal{F})$*, there is an open* $V$ *such that* $\lim f^*(\mathcal{F}) \in V \subset \overline{V} \subset U$*. Show that* $[f^{-1}(V)]$ *is an open neighborhood of* $\mathcal{F}$ *whose image under* $\widetilde{f}$ *lies in* $U$*.*

## 2. Measures

**Problem 17.** Let $X$ be a set. A *ring* $\mathcal{R} \subset P(X)$ is a collection of subsets of $X$ which is closed under unions and set differences. That is, $E, F \in \mathcal{R}$ implies $E \cup F \in \mathcal{R}$ and $E \setminus F \in \mathcal{R}$.

(1) Let $\mathcal{R} \subset P(X)$ be a ring.
   (a) Prove that $\emptyset \in \mathcal{R}$.
   (b) Show that $E, F \in \mathcal{R}$ implies the symmetric difference $E \triangle F \in \mathcal{R}$.
   (c) Show that $E, F \in \mathcal{R}$ implies $E \cap F \in \mathcal{R}$.

(2) Show that any ring $\mathcal{R} \subset P(X)$ is an algebraic ring where the addition is symmetric difference and multiplication is intersection.
   (a) What is $0_{\mathcal{R}}$?
   (b) Show that this algebraic ring has *characteristic* 2, i.e., $E + E = 0_{\mathcal{R}}$ for all $E \in \mathcal{R}$.
   (c) When is the algebraic ring $\mathcal{R}$ unital? In this case, what is $1_{\mathcal{R}}$?
   (d) Determine the relationship (if any) between an algebra of sets in the sense of measure theory and an algebra in the algebraic sense.
   (e) Sometimes an algebra in measure theory is called a *field*. Why?

**Problem 18.** Let $X$ be a set. A $\pi$-*system* on $X$ is a collection of subsets $\Pi \subset P(X)$ which is closed under finite intersections. A $\lambda$-*system* on $X$ is a collection of subsets $\Lambda \subset P(X)$ such that

- $X \in \Lambda$
- $\Lambda$ is closed under taking complements, and
- for every sequence of disjoint subsets $(E_i)$ in $\Lambda$, $\bigcup E_i \in \Lambda$.

(1) Show that $\mathcal{M}$ is a $\sigma$-algebra if and only if $\mathcal{M}$ is both a $\pi$-system and a $\lambda$-system.
(2) Suppose $\Lambda$ is a $\lambda$-system. Show that for every $E \in \Lambda$, the set

$$\Lambda(E) := \{F \subset X | F \cap E \in \Lambda\}$$

is also a $\Lambda$-system.

**Problem 19** ($\pi - \lambda$ Theorem)**.** Let $\Pi$ be a $\pi$-system, let $\Lambda$ be the smallest $\lambda$-system containing $\Pi$, and let $\mathcal{M}$ be the smallest $\sigma$-algebra containing $\Pi$.

(1) Show that $\Lambda \subseteq \mathcal{M}$.
(2) Show that for every $E \in \Pi$, $\Pi \subset \Lambda(E)$ where $\Lambda(E)$ was defined in Problem 18 above. Deduce that $\Lambda \subset \Lambda(E)$ for every $E \in \Pi$.
(3) Show that $\Pi \subset \Lambda(F)$ for every $F \in \Lambda$. Deduce that $\Lambda \subset \Lambda(F)$ for every $F \in \Lambda$.
(4) Deduce that $\Lambda$ is a $\sigma$-algebra, and thus $\mathcal{M} = \Lambda$.

**Problem 20.** Let $\Pi$ be a $\pi$-system, and let $\mathcal{M}$ be the smallest $\sigma$-algebra containing $\Pi$. Suppose $\mu, \nu$ are two measures on $\mathcal{M}$ whose restrictions to $\Pi$ agree.

(1) Suppose that $\mu, \nu$ are finite and $\mu(X) = \nu(X)$. Show $\mu = \nu$.
*Hint: Consider* $\Lambda := \{E \in \mathcal{M} | \nu(E) = \mu(E)\}$*.*

(2) Suppose that $X = \coprod_{j=1}^\infty X_j$ with $(X_j) \subset \Pi$ and $\mu(X_j) = \nu(X_j) < \infty$ for all $j \in \mathbb{N}$. (Observe that $\mu$ and $\nu$ are $\sigma$-finite.) Show $\mu = \nu$.

**Problem 21** (Folland §1.3, #14 and #15)**.** Given a measure $\mu$ on $(X, \mathcal{M})$, define $\nu$ on $\mathcal{M}$ by

$$\nu(E) := \sup \{\mu(F) | F \subset E \text{ and } \mu(F) < \infty\}.$$

(1) Show that $\nu$ is a semifinite measure. We call it the *semifinite part* of $\mu$.
(2) Suppose $E \in \mathcal{M}$ with $\nu(E) = \infty$. Show that for any $n > 0$, there is an $F \subset E$ such that $n < \nu(F) < \infty$.
*This is exactly Folland §1.3, #14 applied to $\nu$.*
(3) Show that if $\mu$ is semifinite, then $\mu = \nu$.
(4) Show there is a measure $\rho$ on $\mathcal{M}$ (which is generally not unique) which assumes only the values 0 and $\infty$ such that $\mu = \nu + \rho$.

**Problem 22.** Suppose $(\mu_i^*)_{i \in I}$ is a family of outer measures on $X$. Show that

$$\mu^*(E) := \sup_{i \in I} \mu_i^*(E)$$

is an outer measure on $X$.

**Problem 23.** Define the *h-intervals*

$$\mathcal{H} := \{\emptyset\} \cup \{(-a, b] | -\infty \leq a < b < \infty\} \cup \{(a, \infty) | a \in \mathbb{R}\}.$$

Let $\mathcal{A}$ be the collection of finite disjoint unions of elements of $\mathcal{H}$. Show *directly from the definitions* that $\mathcal{A}$ is an algebra. Deduce that the $\sigma$-algebra $\mathcal{M}(\mathcal{A})$ generated by $\mathcal{A}$ is equal to the Borel $\sigma$-algebra $\mathcal{B}_{\mathbb{R}}$.

**Problem 24.** Denote by $\overline{\mathbb{R}}$ the extended real numbers $[-\infty, \infty]$ with its usual topology. Prove the following assertions.

(1) The Borel $\sigma$-algebra on $\overline{\mathbb{R}}$ is generated by the open rays $(a, \infty]$ for $a \in \mathbb{R}$.
(2) If $\mathcal{E} \subset P(\mathbb{R})$ generates the Borel $\sigma$-algebra on $\mathbb{R}$, then $\mathcal{E} \cup \{\{\infty\}\}$ generates the Borel $\sigma$-algebra on $\overline{\mathbb{R}}$.

**Problem 25** (Adapted from Folland §1.4, #18 and #22)**.** Suppose $\mathcal{A}$ is an algebra on $X$, and let $\mathcal{M}$ be the $\sigma$-algebra generated by $\mathcal{A}$. Let $\mu_0$ be a $\sigma$-finite premeasure on $\mathcal{A}$, $\mu^*$ the induced outer measure, and $\mathcal{M}^*$ the $\sigma$-algebra of $\mu^*$-measurable sets. Show that the following are equivalent.

(1) $E \in \mathcal{M}^*$
(2) $E = F \setminus N$ where $F \in \mathcal{M}$ and $\mu^*(N) = 0$.
(3) $E = F \cup N$ where $F \in \mathcal{M}$ and $\mu^*(N) = 0$.

Deduce that if $\mu$ is a $\sigma$-finite measure on $\mathcal{M}$, then $\mu^*|_{\mathcal{M}^*}$ on $\mathcal{M}^*$ is the completion of $\mu$ on $\mathcal{M}$.

**Problem 26** (Folland §1.4, #20)**.** Let $\mu^*$ be an outer measure on $P(X)$, $\mathcal{M}^*$ the $\sigma$-algebra of $\mu^*$-measurable sets, and $\mu := \mu^*|_{\mathcal{M}^*}$. Let $\mu^+$ be the outer measure on $P(X)$ induced by the (pre)measure $\mu$ on the ($\sigma$-)algebra $\mathcal{M}^*$.

(1) Show that $\mu^*(E) \leq \mu^+(E)$ for all $E \subset X$ with equality if and only if there is an $F \in \mathcal{M}^*$ with $E \subset F$ and $\mu^*(E) = \mu^*(F)$.
(2) Show that if $\mu^*$ was induced from a premeasure $\mu_0$ on an algebra $\mathcal{A}$, then $\mu^* = \mu^+$.
(3) Construct an outer measure $\mu^*$ on the two point set $X = \{0, 1\}$ such that $\mu^* \neq \mu^+$.

**Problem 27** (Sarason)**.** Suppose $\mu_0$ is a finite premeasure on the algebra $\mathcal{A} \subset P(X)$, and let $\mu^* : P(X) \to [0, \infty]$ be the outer measure induced by $\mu_0$. Prove that the following are equivalent for $E \subset X$.

(1) $E \in \mathcal{M}^*$, the $\mu^*$-measurable sets.

(2) $\mu^*(E) + \mu^*(X \setminus E) = \mu(X)$.

*Hint: Use Problem 25.*

**Problem 28.** Assume the notation of Problem 23. Suppose $F : \mathbb{R} \to \mathbb{R}$ is non-decreasing and right continuous, and extend $F$ to a function $[-\infty, \infty] \to [-\infty, \infty]$ still denoted $F$ by

$$F(-\infty) := \lim_{a \to -\infty} F(a) \qquad \text{and} \qquad F(\infty) := \lim_{b \to \infty} F(b).$$

Define $\mu_0 : \mathcal{H} \to [0, \infty]$ by

- $\mu_0(\emptyset) := 0$,
- $\mu_0((a, b]) := F(b) - F(a)$ for all $-\infty \leq a < b < \infty$, and
- $\mu_0((a, \infty)) := F(\infty) - F(a)$ for all $a \in \mathbb{R}$.

Suppose $(a, \infty) = \coprod_{j=1}^{\infty} H_j$ where $(H_j) \subset \mathcal{H}$ is a sequence of disjoint h-intervals. Show that

$$\mu_0((a, \infty)) = \sum_{j=1}^{\infty} \mu_0(H_j).$$

**Problem 29** (Folland, §1.5, #28)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous, and let $\mu_F$ be the associated Lebesgue-Stieltjes Borel measure on $\mathcal{B}_{\mathbb{R}}$. For $a \in \mathbb{R}$, define

$$F(a-) := \lim_{r \nearrow a} F(r).$$

Prove that:

(1) $\mu_F(\{a\}) = F(a) - F(a-)$,
(2) $\mu_F([a, b)) = F(b-) - F(a-)$,
(3) $\mu_F([a, b]) = F(b) - F(a-)$, and
(4) $\mu_F((a, b)) = F(b-) - F(a)$.

**Problem 30.** Let $(X, \rho)$ be a metric (or simply a topological) space. A subset $S \subset X$ is called *nowhere dense* if $\overline{S}$ does not contain any open set in $X$. A subset $T \subset X$ is called *meager* if it is a countable union of nowhere dense sets.

Construct a meager subset of $\mathbb{R}$ whose complement is Lebesgue null.

**Problem 31** (Steinhaus Theorem, Folland §1.5, #30 and 31)**.** Suppose $E \in \mathcal{L}$ and $\lambda(E) > 0$.

(1) Show that for any $0 \leq \alpha < 1$, there is an open interval $I \subset \mathbb{R}$ such that $\lambda(E \cap I) > \alpha\lambda(I)$.
(2) Apply (1) with $\alpha = 3/4$ to show that the set

$$E - E = \{x - y | x, y \in E\}$$

contains the interval $(-\lambda(I)/2, \lambda(I)/2)$.

**Problem 32.** Let $\mathcal{B}_{\mathbb{R}}$ be the Borel $\sigma$-algebra of $\mathbb{R}$. Suppose $\mu$ is a translation invariant measure on $\mathcal{B}_{\mathbb{R}}$ such that $\mu((0, 1]) = 1$. Prove that $\mu = \lambda|_{\mathcal{B}_{\mathbb{R}}}$, the restriction of Lebesgue measure on $\mathcal{L}$ to $\mathcal{B}_{\mathbb{R}}$.

**Problem 33** (Sarason)**.** Suppose $E \in \mathcal{L}$ is Lebesgue null, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function (continuous with continuous derivative). Prove that $\varphi(E)$ is also Lebesgue null.

**Problem 34.** Find an uncountable subset of $\mathbb{R}$ with Hausdorff dimension zero.

**Problem 35.** Suppose $(X, \mathcal{M})$ is a measurable space and $(Y, \tau)$, $(Z, \theta)$ are topological spaces, $i : Y \to Z$ is a continuous injection which maps open sets to open sets, and $f : X \to Y$. Show that $f$ is $\mathcal{M} - \mathcal{B}_\tau$ measurable if and only if $i \circ f$ is $\mathcal{M} - \mathcal{B}_\theta$ measurable.

Deduce that if $f : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ only takes values in $\mathbb{R}$, then $f$ is $\mathcal{M} - \mathcal{B}_{\overline{\mathbb{R}}}$ measurable if and only if $f$ is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable.

**Problem 36.** Prove the following assertions.

(1) Suppose $f : X \to Y$ is a function. Define $\overleftarrow{f} : P(Y) \to P(X)$ by $\overleftarrow{f}(T) := \{x \in X | f(x) \in T\}$. Then $\overleftarrow{f}$ preserves unions, intersections, and complements.

(2) Suppose $f : X \to Y$ is a function. Define $\overrightarrow{f} : P(X) \to P(Y)$ by $\overrightarrow{f}(S) := \{f(s) | s \in S\}$. Then $\overrightarrow{f}$ preserves unions, but not intersections nor complements.

(3) Given $f : X \to Y$ and a topology $\theta$ on $Y$, $\overset{\rightleftarrows}{f}(\theta) = \left\{f^{-1}(U) \middle| U \in \theta\right\}$ is a topology on $X$. Moreover it is the weakest topology on $X$ such that $f$ is continuous.

(4) Given $f : X \to Y$ and a topology $\tau$ on $X$, $\overset{\leftleftarrows}{f}(\tau) = \left\{U \subset Y \middle| f^{-1}(U) \in \tau\right\}$ is a topology on $Y$. Moreover it is the strongest topology on $Y$ such that $f$ is continuous.

(5) Given $f : X \to Y$ and a $\sigma$-algebra $\mathcal{N}$ on $Y$, $\overset{\rightleftarrows}{f}(\mathcal{N}) = \left\{f^{-1}(F) \middle| F \in \mathcal{N}\right\}$ is a $\sigma$-algebra on $X$. Moreover it is the weakest $\sigma$-algebra on $X$ such that $f$ is measurable.

(6) Given $f : X \to Y$ and a $\sigma$-algebra $\mathcal{M}$ on $X$, $\overset{\leftleftarrows}{f}(\mathcal{M}) = \left\{F \subset Y \middle| f^{-1}(F) \in \mathcal{M}\right\}$ is a $\sigma$-algebra on $Y$. Moreover it is the strongest $\sigma$-algebra on $Y$ such that $f$ is measurable.

**Problem 37.** Let $(X, \mathcal{M})$ be a measurable space.

(1) Prove that the Borel $\sigma$-algebra $\mathcal{B}_{\mathbb{C}}$ on $\mathbb{C}$ is generated by the 'open rectangles'
$$\{z \in \mathbb{C} | a < \mathrm{Re}(z) < b \text{ and } c < \mathrm{Im}(z) < d\}.$$

(2) Prove directly from the definitions that $f : X \to \mathbb{C}$ is $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable.

(3) Prove that the $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable functions form a $\mathbb{C}$-vector space.

(4) Show that if $f : X \to \mathbb{C}$ is $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable, then $|f| : X \to [0, \infty)$ is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable.

(5) Show that if $(f_n)$ is a sequence of $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable functions $X \to \mathbb{C}$ and $f_n \to f$ pointwise, then $f$ is $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable.

**Problem 38.** Let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be the completion of the measure space $(X, \mathcal{M}, \mu)$.

(1) Show that if $f$ is $\overline{\mathcal{M}}$-measurable and $g = f$ a.e., then $g$ is $\overline{\mathcal{M}}$-measurable.
*Optional: Does this hold with $\overline{\mathcal{M}}$ replaced by $\mathcal{M}$?*

(2) Show that if $f$ is $\overline{\mathcal{M}}$-measurable, there exists an $\mathcal{M}$-measurable $g$ such that $f = g$ a.e.
*Hint: First do the case $f$ is $\mathbb{R}$-valued.*

(3) Show that if $(f_n)$ is a sequence of $\overline{\mathcal{M}}$-measurable functions and $f_n \to f$ a.e., then $f$ is $\overline{\mathcal{M}}$-measurable.
*Optional: Does this hold with $\overline{\mathcal{M}}$ replaced by $\mathcal{M}$?*

(4) Show that if $(f_n)$ is a sequence of $\mathcal{M}$-measurable functions and $f_n \to f$ a.e., then $f$ is $\overline{\mathcal{M}}$-measurable. Deduce that there is an $\mathcal{M}$-measurable function $g$ such that $f = g$ a.e., so $f_n \to g$ a.e.

For all parts, consider the cases of $\mathbb{R}$, $\overline{\mathbb{R}}$, and $\mathbb{C}$-valued functions.

**Problem 39.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Show that a simple function $\psi = \sum_{k=1}^{n} c_k \chi_{E_k}$ where $c_k > 0$ for all $k = 1, \ldots, n$ is integrable if and only if $\mu(E_k) < \infty$ for all $k = 1, \ldots, n$.
(2) Show that if a simple function $\psi = \sum_{k=1}^{n} c_k \chi_{E_k}$ is integrable with $\mu(E_k) < \infty$ for all $k = 1, \ldots, n$, then $\int \psi = \sum_{k=1}^{n} c_k \mu(E_k)$.

*In both parts of the question, we do not assume that $\psi$ is written in its standard form.*

**Problem 40.** Suppose $f : (X, \mathcal{M}, \mu) \to [0, \infty]$ is $\mathcal{M}$-measurable and $\{f > 0\}$ is $\sigma$-finite. Show that there exists a sequence of nonnegative simple functions $(\psi_n)$ such that

- $\psi_n \nearrow f$,
- $\psi_n$ is integrable for every $n \in \mathbb{N}$.

*Optional: In what sense can you say $\psi_n \nearrow f$ uniformly?*

**Problem 41.** Assume Fatou's Lemma and prove the Monotone Convergence Theorem from it.

**Problem 42.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Suppose $f \in L^+$ and $\int f < \infty$. Prove that $\{f = \infty\}$ is $\mu$-null and $\{f > 0\}$ is $\sigma$-finite.
(2) Suppose $f \in L^1(\mu, \mathbb{C})$. Prove that $\{f \neq 0\}$ is $\sigma$-finite.

**Problem 43.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f \in L^1(\mu, \mathbb{C})$. Prove that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $E \in \mathcal{M}$ with $\mu(E) < \delta$, $\int_E |f| < \varepsilon$.

**Problem 44.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Prove that $\|\cdot\|_1 : \mathcal{L}^1(\mu, \mathbb{C}) \to [0, \infty)$ given by $\|f\|_1 := \int |f|$ is a norm. That is, prove the following axioms hold:
    - (definite) $\|f\|_1 = 0$ if and only if $f = 0$.
    - (homogeneous) $\|\lambda \cdot f\|_1 = |\lambda| \cdot \|f\|_1$ for all $\lambda \in \mathbb{C}$.
    - (subadditive) $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.
(2) Suppose $(V, \|\cdot\|)$ is a $\mathbb{C}$-vector space with a norm (you may assume $V = \mathcal{L}^1(\mu, \mathbb{C})$ and $\|\cdot\| = \|\cdot\|_1$ if you wish). Prove that $\rho(x, y) := \|x - y\|$ defines a metric on $V$.
(3) Prove that the metric $\rho_1$ on $\mathcal{L}^1$ induced by $\|\cdot\|_1$ is complete. That is, prove every Cauchy sequence converges in $\mathcal{L}^1$.

**Problem 45.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space, and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. Find a canonical $\mathbb{C}$-vector space isomorphism $\mathcal{L}^1(\mu, \mathbb{C}) \cong \mathcal{L}^1(\overline{\mu}, \mathbb{C})$ which preserves $\|\cdot\|_1$.

**Problem 46.** Let $\mu$ be a Lebesgue-Stieltjes Borel measure on $\mathbb{R}$. Show that $C_c(\mathbb{R})$, the continuous functions of compact support ($\overline{\{f \neq 0\}}$ compact) is dense in $\mathcal{L}^1(\mu, \mathbb{R})$. Does the same hold for $\overline{\mathbb{R}}$ and $\mathbb{C}$-valued functions?
*Hint: You could proceed in this way:*

*(1) Reduce to the case $f \in L^1 \cap L^+$.*
*(2) Reduce to the case $f \in L^1 \cap SF^+$.*
*(3) Reduce to the case $f = \chi_E$ with $E \in \mathcal{B}_{\mathbb{R}}$ and $\mu(E) < \infty$.*
*(4) Reduce to the case $f = \chi_U$ with $U \subset \mathbb{R}$ open and $\mu(U) < \infty$.*
*(5) Reduce to the case $f = \chi_{(a,b)}$ with $a < b$ in $\mathbb{R}$.*

**Problem 47** (Lusin's Theorem)**.** Suppose $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\varepsilon > 0$. There is a compact set $E \subset [a, b]$ such that $\lambda(E^c) < \varepsilon$ and $f|_E$ is continuous.

**Problem 48.** Suppose $f \in \mathcal{L}^1([0, 1], \lambda)$ is an integrable non-negative function.

(1) Show that for every $n \in \mathbb{N}$, $\sqrt[n]{f} \in \mathcal{L}^1([0, 1], \lambda)$.
(2) Show that $(\sqrt[n]{f})$ converges in $\mathcal{L}^1$ and compute its limit.

*Hint for both parts: Consider $\{f \geq 1\}$ and $\{f < 1\}$ separately.*

**Problem 49.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f_n \to f$ in measure and $g_n \to g$ in measure (these functions are assumed to be measurable). Show that

(1) $|f_n| \to |f|$ in measure.
(2) $f_n + g_n \to f + g$ in measure.
(3) $f_n g_n \to fg$ if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.
*Hint: First show $f_n g \to fg$ in measure. To do so, one could follow the following steps.*
*(a) Show that for $g : X \to \mathbb{C}$ with $\mu(X) < \infty$, $\mu(\{|g| \geq n\}) \to 0$ as $n \to \infty$.*
*(b) Show that for any $\varepsilon > 0$, by step (a), $X = E \amalg E^c$ where $|g|_E| < M$ and $\mu(E^c) < \varepsilon/2$.*
*(c) For $\delta > 0$ and carefully chosen $M > 0$ and $E$,*

$$\begin{aligned}\{|f_n g - fg| > \delta\} &= (\{|f_n g - fg| > \delta\} \cap E) \amalg (\{|f_n g - fg| > \delta\} \cap E^c) \\ &\subseteq \left\{|f_n - f| > \frac{\delta}{M}\right\} \cup E^c.\end{aligned}$$

**Problem 50** (Folland §2.4, #33 and 34)**.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f_n \to f$ in measure (these functions are assumed to be measurable).

(1) Show that if $f_n \geq 0$ everywhere, then $\int f \leq \liminf \int f_n$.
(2) Suppose $|f_n| \leq g \in \mathcal{L}^1$. Prove that $\int f = \lim \int f_n$ and $f_n \to f$ in $\mathcal{L}^1$.

## 4. Product measures and differentiation

**Problem 51.** For the following statement, either provide a proof or a counterexample. Let $X, Y$ be topological spaces with Borel $\sigma$-algebras $\mathcal{B}_X, \mathcal{B}_Y$ respectively and regular Borel measures $\mu, \nu$. Then the product measure $\mu \times \nu$ is also regular.
*Optional: If you find a counterexample, can you find a weak modification under which it is true?*

**Problem 52.** Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is such that each $x$-section $f_x$ is Borel measurable and $f^y$ is continuous. Show $f$ is Borel measurable.
*Hint (Ratner): Let $(x_n)$ be a countable dense subset of $\mathbb{R}$. Prove that*

$$f^{-1}(-\infty, r] = \bigcap_m \bigcup_n \left\{(x,y) \middle| x \in B_{\frac{1}{m}}(x_n) \text{ and } f(x_n, y) < r + \frac{1}{m}\right\}.$$

**Problem 53.** Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces and $(E_n) \subset \mathcal{M} \times \mathcal{N}$. Prove the following assertions about $x$-sections.

(1) $(\bigcup E_n)_x = \bigcup (E_n)_x$.
(2) $(\bigcap E_n)_x = \bigcap (E_n)_x$.
(3) $(E_m \setminus E_n)_x = (E_m)_x \setminus (E_n)_x$.
(4) $\chi_{E_n}(x, y) = \chi_{(E_n)_x}(y)$ for all $x \in X$ and $y \in Y$.

**Problem 54** (Counterexamples: Folland §2.5, #46 and #48)**.**

(1) Let $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, $\mu = \lambda$ Lebesgue measure, and $\nu$ counting measure. Let $\Delta = \{(x, x) | x \in [0, 1]\}$ be the diagonal. Prove that $\int\int \chi_\Delta \, d\mu \, d\nu$, $\int\int \chi_\Delta \, d\nu \, d\mu$, and $\int \chi_\Delta \, d(\mu \times \nu)$ are all unequal.
(2) Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = P(\mathbb{N})$, and $\mu = \nu$ counting measure. Define

$$f(m, n) := \begin{cases} 1 & \text{if } m = n \\ -1 & \text{if } m = n + 1 \\ 0 & \text{else.} \end{cases}$$

Prove that $\int |f| d(\mu \times \nu) = \infty$, and $\int\int f \, d\mu \, d\nu$ and $\int\int f \, d\nu \, d\mu$ both exist and are unequal.

**Problem 55.** Show that the conclusions of the Fubini and Tonelli Theorems hold when $(X, \mathcal{M}, \mu)$ is an arbitrary measure space (not necessarily $\sigma$-finite) and $Y$ is a countable set, $\mathcal{N} = P(Y)$, and $\nu$ is counting measure.

**Problem 56.** Suppose $f, g \in \mathcal{L}^1(\mathbb{R}, \lambda)$.

(0) *(Not required for 2024) Show that the function $(x, y) \mapsto x - y$ is $\mathcal{L}^2 - \mathcal{L}$ measurable.*
Hint: *Recall that $\mathcal{L}$ is the completion of $\mathcal{B}_{\mathbb{R}}$. Show that the preimage of a $\lambda$-null set is $\lambda^2$-null by considering the preimage of a $G_\delta$ Borel set $N$ with measure zero, intersected with the unit square in $\mathbb{R}^2$. If $N = \bigcap U_n$ with $\lambda(U_n) \searrow 0$ and $V_n$ is the preimage of $U_n$ intersected with the unit square in $\mathbb{R}^2$, compute $\lim_n \int \chi_{V_n} d\lambda^2$.*

(1) Show that $y \mapsto f(x - y)g(y)$ is measurable for all $x \in \mathbb{R}$ and in $\mathcal{L}^1(\mathbb{R}, \lambda)$ for a.e. $x \in \mathbb{R}$.

(2) Define the *convolution* of $f$ and $g$ by

$$(f * g)(x) := \int_{\mathbb{R}} f(x - y)g(y)\, d\lambda.$$

Show that $f * g \in \mathcal{L}^1(\mathbb{R}, \lambda)$.

(3) Show that $\mathcal{L}^1(\mathbb{R}, \lambda)$ is a commutative $\mathbb{C}$-algebra under $\cdot, +, *$.

(4) Show that $\int_{\mathbb{R}} |f * g| \leq \int_{\mathbb{R}} |f| \int_{\mathbb{R}} |g|$, i.e., $\|\cdot\|_1$ is submultiplicative.

**Problem 57.** Suppose $f \in \mathcal{L}^1(\lambda^n)$. Prove that for all $T \in GL_n(\mathbb{R}) := \{T \in M_n(\mathbb{R}) | \det(T) \neq 0\}$, $f \circ T \in \mathcal{L}^1(\lambda^n)$ and

$$\int f(x)\, d\lambda^n(x) = |\det(T)| \cdot \int f(Tx)\, d\lambda^n(x).$$

Does this also hold when $\det(T) = 0$? Find a proof or counterexample.

**Problem 58** (Sarason)**.** For $f \in \mathcal{L}^1(\lambda^n)$, let $M$ be the Hardy-Littlewood maximal function

$$(Mf)(x) := \sup \left\{ \frac{1}{\lambda^n(Q)} \int_Q |f|\, d\lambda^n \middle| Q \in \mathcal{C}(x) \right\}$$

where $\mathcal{C}(x)$ is the set of all cubes of finite length which contain $x$. Define

$$f(x) := \begin{cases} \dfrac{1}{|x|(\ln |x|)^2} & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2} \end{cases}$$

Show that $f \in \mathcal{L}^1(\lambda)$, but $Mf \notin \mathcal{L}^1_{\text{loc}}$.

**Problem 59** (Sarason)**.** Suppose $E \subset \mathbb{R}^n$ (not assumed to be Borel measurable) and let $\mathcal{C}$ be a family of cubes covering $E$ such that

$$\sup \{\ell(Q) | Q \in \mathcal{C}\} < \infty.$$

Show there exists a sequence $(Q_k) \subset \mathcal{C}$ of disjoint cubes such that

$$\sum_{k=1}^{\infty} \lambda^n(Q_k) \geq 5^{-n} (\lambda_n)^*(E).$$

*Hint: Inductively choose $Q_k$ such that $2\ell(Q_k)$ is larger than the sup of the lengths of all cubes which do not intersect $Q_1, \ldots, Q_{k-1}$, with $Q_0 = \emptyset$ by convention.*

**Problem 60.** In this exercise, we will show that

$$M := M(X, \mathcal{M}, \mathbb{R}) := \{\text{finite signed measures on } (X, \mathcal{M})\}$$

is a Banach space with $\|\nu\| := |\nu|(X)$.

(1) Prove $\|\nu\| := |\nu|(X)$ is a norm on $M$.

(2) Show that $(\nu_n) \subset M$ Cauchy implies $(\nu_n(E)) \subset \mathbb{R}$ is uniformly Cauchy for all $E \in \mathcal{M}$. That is, show that for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and $E \in \mathcal{M}$, $|\nu_m(E) - \nu_n(E)| < \varepsilon$.
(3) Use part (2) to define a candidate limit signed measure $\mu$ on $\mathcal{M}$. Prove that $\nu$ is $\sigma$-additive. *Hint: first prove $\nu$ is finitely additive.*
(4) Prove that $\sum \nu(E_n)$ converges absolutely when $(E_n) \subset \mathcal{M}$ is disjoint, and thus $\nu$ is a finite signed measure.
(5) Show that $\nu_n \to \nu$ in $M$.

**Problem 61** (Folland §3.1, #3 and §3.2, #8)**.** Suppose $\mu$ is a positive measure on $(X, \mathcal{M})$ and $\nu$ is a signed measure on $(X, \mathcal{M})$.

(1) Prove that the following are equivalent.
 (a) $\nu \perp \mu$
 (b) $|\nu| \perp \mu$
 (c) $\nu_+ \perp \mu$ and $\nu_- \perp \mu$.
(2) Prove that the following are equivalent.
 (a) $\nu \ll \mu$
 (b) $|\nu| \ll \mu$
 (c) $\nu_+ \ll \mu$ and $\nu_- \ll \mu$.

**Problem 62** (Folland §3.1, #3)**.** Let $\nu$ be a signed measure on $(X, \mathcal{M})$. Prove the following assertions:

(a) $\mathcal{L}^1(\nu) = \mathcal{L}^1(|\nu|)$.
(b) If $f \in \mathcal{L}^1(\nu)$, $\left|\int f\,d\nu\right| \leq \int |f| d|\nu|$.
(c) If $E \in \mathcal{M}$, $|\nu|(E) = \sup\left\{\left|\int_E f\,d\nu\right| \middle| -1 \leq f \leq 1\right\}$.

**Problem 63** (Folland §3.1, #6)**.** Suppose

$$\nu(E) := \int_E f\,d\mu \qquad\qquad E \in \mathcal{M}$$

where $\mu$ is a positive measure on $(X, \mathcal{M})$ and and $f$ is an extended $\mu$-integrable function. Describe the Hahn decompositions of $\nu$ and the positive, negative, and total variations of $\nu$ in terms of $f$ and $\mu$.

**Problem 64** (Adapted from Folland §3.2, #9)**.** Suppose $\mu$ is a positive measure on $(X, \mathcal{M})$. Suppose $\{\nu_j\}$ is a sequence of positive measures on $(X, \mathcal{M})$ and $\mu$ is a positive measure on $(X, \mathcal{M})$. Prove the following assertions.

(a) If $\{\nu_j\}$ is a sequence of positive measures on $(X, \mathcal{M})$ with $\nu_j \perp \mu$ for all $j$, then $\sum_{j=1}^\infty \nu_j \perp \mu$.
(b) If $\nu_1, \nu_2$ are positive measures on $(X, \mathcal{M})$ with at least one of $\nu_1, \nu_2$ is finite and $\nu_j \perp \mu$ for $j = 1, 2$, then $(\nu_1 - \nu_2) \perp \mu$.
(c) If $\{\nu_j\}$ is a sequence of positive measures on $(X, \mathcal{M})$ with $\nu_j \ll \mu$ for all $j$, then $\sum_{j=1}^\infty \nu_j \ll \mu$.
(d) If $\nu_1, \nu_2$ are positive measures on $(X, \mathcal{M})$ with at least one of $\nu_1, \nu_2$ is finite and $\nu_j \ll \mu$ for $j = 1, 2$, then $(\nu_1 - \nu_2) \ll \mu$.

**Problem 65.** Suppose $F : [a, b] \to \mathbb{C}$.

(1) Show that if $F$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $F'$ is bounded, then $F \in \mathsf{BV}[a, b]$.
(2) Show that if $F$ is absolutely continuous, then $F \in \mathsf{BV}[a, b]$.

**Problem 66.** Suppose $F \in \mathsf{NBV}$, and let $\nu_F$ be the corresponding Lebesgue-Stieltjes complex Borel measure.

(1) Prove that $\nu_F$ is regular.

(2) Prove that $|\nu_F| = \nu_{T_F}$.
*One could proceed as follows.*
(a) *Define* $G(x) := |\nu_F|((-\infty, x])$*. Show that* $|\nu_F| = \nu_{T_F}$ *if and only if* $G = T_F$*.*
(b) *Show* $T_F \leq G$*.*
(c) *Show that* $|\nu_F(E)| \leq \nu_{T_F}(E)$ *whenever* $E$ *is an interval.*
(d) *Show that* $|\nu_F| \leq \nu_{T_F}$*.*

**Problem 67** (cf. Folland Thm. 3.22)**.** Denote by $\lambda^n$ Lebesgue measure on $\mathbb{R}^n$. Suppose $\nu$ is a regular signed or complex Borel measure on $\mathbb{R}^n$ which is finite on compact sets (and thus Radon and $\sigma$-finite). Let $d\nu = d\rho + f d\lambda^n$ be its Lebesgue-Radon-Nikodym representation. Then for $\lambda^n$-a.e. $x \in \mathbb{R}^n$,

$$\lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{\nu(Q)}{\lambda^n(Q)} = f(x).$$

*Hint: One could proceed as follows.*
(1) *Show that* $d|\nu| = d|\rho| + |f|d\lambda^n$*. Deduce that* $\rho$ *and* $f d\lambda^n$ *are regular, and* $f \in L^1_{\text{loc}}$*.*
(2) *Use the Lebesgue Differentiation Theorem to reduce the problem to showing*

$$\lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{|\rho|(Q)}{\lambda^n(Q)} = 0 \qquad \lambda^n\text{-a.e. } x \in \mathbb{R}^n.$$

*Thus we may assume* $\rho$ *is positive.*
(3) *Since* $\rho \perp \lambda^n$*, pick* $P \subset \mathbb{R}^n$ *Borel measurable such that* $\rho(P) = \lambda^n(P^c) = 0$*. For* $a > 0$*, define*

$$E_a := \left\{ x \in P \,\middle|\, \lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{|\rho|(Q)}{\lambda^n(Q)} > a \right\}.$$

*Let* $\varepsilon > 0$*. Since* $\rho$ *is regular, there is an open* $U_\varepsilon \supset P$ *such that* $\rho(U_\varepsilon) < \varepsilon$*. Adapt the proof of the HLMT to show there is a constant* $c > 0$*, depending only on* $n$*, such that for all* $a > 0$*,*

$$\lambda^n(E_a) \leq c \cdot \frac{\rho(U_\varepsilon)}{a} = c \cdot \frac{\varepsilon}{a}$$

*(Choose your family of cubes to be contained in* $U_\varepsilon$*.) Deduce that* $\lambda^n(E_a) = 0$*.*

**Problem 68.** Let $F : \mathbb{R} \to \mathbb{R}$ be a bounded, non-decreasing continuously differentiable function, and let $\mu_F$ be the corresponding Lebesgue-Stieltjes measure on $\mathbb{R}$.
(1) Denoting Lebesgue measure by $\lambda$, prove that

$$\mu_F(E) = \int_E F' \, d\lambda \qquad \forall E \in \mathcal{B}_{\mathbb{R}}.$$

*Hint: First prove the above equality for intervals. Then use Problem 20.*
(2) Deduce that $\mu_F \ll \lambda$ and $\frac{d\mu_F}{d\lambda} = F'$ a.e.

## 5. Functional analysis

**Problem 69.** Suppose $X$ is a normed space and $Y \subset X$ is a subspace. Define $Q : X \to X/Y$ by $Qx = x + Y$. Define

$$\|Qx\|_{X/Y} = \inf \{ \|x - y\|_X | y \in Y \}.$$

(1) Prove that $\|\cdot\|_{X/Y}$ is a well-defined seminorm.
(2) Show that if $Y$ is closed, then $\|\cdot\|_{X/Y}$ is a norm.

(3) Show that in the case of (2) above, $Q : X \to X/Y$ is continuous and open.
*Optional: is $Q$ continuous or open only in the case of (1)?*
(4) Show that if $X$ is Banach, so is $X/Y$.

**Problem 70.** Suppose $F$ is a finite dimensional vector space.

(1) Show that for any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $F$, there is a $c > 0$ such that $\|f\|_1 \leq c\|f\|_2$ for all $f \in F$. Deduce that all norms on $F$ induce the same vector space topology on $F$.
*Note: You need only prove the result for one of $\mathbb{R}$ or $\mathbb{C}$. You may use that the unit sphere in $\mathbb{K}^n$ is compact with respect to the usual Euclidean topology.*
(2) Show that for any two finite dimensional normed spaces $F_1$ and $F_2$, all linear maps $T : F_1 \to F_2$ are continuous.
*Optional: Show that for any two finite dimensional vector spaces $F_1$ and $F_2$ endowed with their vector space topologies from part (1), all linear maps $T : F_1 \to F_2$ are continuous.*
(3) Let $X, F$ be normed spaces with $F$ finite dimensional, and let $T : X \to F$ be a linear map. Prove that the following are equivalent:
(a) $T$ is bounded (there is an $R > 0$ such that $T(B_1(0_X)) \subset B_R(0_F)$), and
(b) $\ker(T)$ is closed.
*Hint: One way to do (b) implies (a) uses Problem 69 part (3) and part (2) of this problem.*

**Problem 71** (Folland §5.1, #7)**.** Suppose $X$ is a Banach space and $T \in \mathcal{L}(X) = \mathcal{L}(X, X)$. Let $I \in \mathcal{L}(X)$ be the identity map.

(1) Show that if $\|I - T\| < 1$, then $T$ is invertible.
*Hint: Show that $\sum_{n\geq 0}(I - T)^n$ converges in $\mathcal{L}(X)$ to $T^{-1}$.*
(2) Show that if $T \in \mathcal{L}(X)$ is invertible and $\|S - T\| < \|T^{-1}\|^{-1}$, then $S$ is invertible.
(3) Deduce that the set of invertible operators $GL(X) \subset \mathcal{L}(X)$ is open.

**Problem 72** (Folland §5.2, #19)**.** Let $X$ be an infinite dimensional normed space.

(1) Construct a sequence $(x_n)$ such that $\|x_n\| = 1$ for all $n$ and $\|x_m - x_n\| \geq 1/2$ for all $m \neq n$.
(2) Deduce $X$ is not locally compact.

**Problem 73.** Suppose $\varphi, \varphi_1, \ldots, \varphi_n$ are linear functionals on a vector space $X$. Prove that the following are equivalent.

(1) $\varphi \in \sum_{k=1}^n \alpha_k \varphi_k$ where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$.
(2) There is an $\alpha > 0$ such that for all $x \in X$, $|\varphi(x)| \leq \alpha \max_{k=1,\ldots,n} |\varphi_k(x)|$.
(3) $\bigcap_{k=1}^n \ker(\varphi_k) \subset \ker(\varphi)$.

**Problem 74.** Consider the following sequence spaces.

$$\ell^1 := \left\{ (x_n) \subset \mathbb{C}^\infty \middle| \sum |x_n| < \infty \right\} \qquad \|x\|_1 := \sum |x_n|$$
$$c_0 := \{(x_n) \subset \mathbb{C}^\infty | x_n \to 0 \text{ as } n \to \infty\} \qquad \|x\|_\infty := \sup |x_n|$$
$$c := \left\{ (x_n) \subset \mathbb{C}^\infty \middle| \lim_{n\to\infty} x_n \text{ exists} \right\} \qquad \|x\|_\infty := \sup |x_n|$$
$$\ell^\infty := \{(x_n) \subset \mathbb{C}^\infty | \sup |x_n| < \infty\} \qquad \|x\|_\infty := \sup |x_n|$$

(1) Show that every space above is a Banach space.
*Hint: First show $\ell^1$ and $\ell^\infty$ are Banach. Then show $c_0, c$ are closed in $\ell^\infty$.*
(2) Construct isometric isomorphisms $c_0^* \cong \ell^1 \cong c^*$ and $(\ell^1)^* \cong \ell^\infty$.
(3) Which of the above spaces are separable?
(4) (Folland §5.2, #25) Prove that if $X$ is a Banach space such that $X^*$ is separable, then $X$ is separable.
(5) Find a separable Banach space $X$ such that $X^*$ is not separable.

**Problem 75** (Folland §5.3, #42). Let $E_n \subset C([0,1])$ be the space of all functions $f$ such that there is an $x_0 \in [0,1]$ such that $|f(x) - f(x_0)| < n|x - x_0|$ for all $x \in [0,1]$.

(1) Prove that $E_n$ is nowhere dense in $C([0,1])$.
(2) Show that the subset of nowhere differentiable functions is residual in $C([0,1])$.

**Problem 76.** Provide examples of the following:

(1) Normed spaces $X, Y$ and a discontinuous linear map $T : X \to Y$ with closed graph.
(2) Normed spaces $X, Y$ and a family of linear operators $\{T_\lambda\}_{\lambda\in\Lambda}$ such that $(T_\lambda x)_{\lambda\in\Lambda}$ is bounded for every $x \in X$, but $(\|T_\lambda\|)_{\lambda\in\Lambda}$ is not bounded.

**Problem 77.** Suppose $X$ and $Y$ are Banach spaces and $T : X \to Y$ is a continuous linear map. Show that the following are equivalent.

(a) There exists a constant $c > 0$ such that $\|Tx\|_Y \geq c\|x\|_X$ for all $x \in X$.
(b) $T$ is injective and has closed range.

**Problem 78.** Let $X$ be a normed space.

(1) Show that every weakly convergent sequence in $X$ is norm bounded.
(2) Suppose in addition that $X$ is Banach. Show that every weak* convergent sequence in $X^*$ is norm bounded.
(3) Give a counterexample to (2) when $X$ is not Banach.
*Hint: Under $\|\cdot\|_\infty$, $c_c^* \cong \ell^1$, where $c_c$ is the space of sequences which are eventually zero.*

**Problem 79** (Goldstine's Theorem). Let $X$ be a normed vector space with closed unit ball $B$. Let $B^{**}$ be the unit ball in $X^{**}$, and let $i : X \to X^{**}$ be the canonical inclusion. Show that $i(B)$ is weak* dense in $B^{**}$.
*Note: recall that the weak* topology on $X^{**}$ is the weak topology induced by $X^*$.*
*Hint: You could use a Hahn-Banach separation theorem that we did not discuss in class. Or you could proceed as follows.*

*(1) Show that for every $z \in B^{**}$, $\varphi_1, \ldots, \varphi_n \in X^*$, and $\delta > 0$, there is an $x \in (1+\delta)B$ such that $\varphi_i(x) = z(\varphi_i)$ for all $1 \leq i \leq n$.*
*(2) Suppose $U$ is a basic open neighborhood of $z \in B^{**}$. Deduce that for every $\delta > 0$, $(1+\delta)i(B) \cap U \neq \emptyset$. That is, there is an $x_\delta \in (1+\delta)B$ such that $i(x_\delta) \in U$.*
*(3) By part (2), $(1+\delta)^{-1}x_\delta \in B$. Show that for $\delta$ sufficiently small (which can be expressed in terms of the basic open neighborhood $U$), $(1+\delta)^{-1}i(x_\delta) \in i(B) \cap U$.*

**Problem 80** (Banach Limits). Let $\ell^\infty(\mathbb{N}, \mathbb{R})$ denote the Banach space of bounded functions $\mathbb{N} \to \mathbb{R}$. Show that there is a $\varphi \in \ell^\infty(\mathbb{N}, \mathbb{R})^*$ satisfying the following two conditions:

(1) Letting $S : \ell^\infty(\mathbb{N}, \mathbb{R}) \to \ell^\infty(\mathbb{N}, \mathbb{R})$ be the shift operator $(Sx)_n = x_{n+1}$ for $x = (x_n)_{n\in\mathbb{N}}$, $\varphi = \varphi \circ S$.
(2) For all $x \in \ell^\infty$, $\liminf x_n \leq \varphi(x) \leq \limsup x_n$.

*Hint: One could proceed as follows.*

*(1) Consider the subspace $Y = \operatorname{im}(S - I) = \{Sx - x | x \in \ell^\infty\}$. Prove that for all $y \in Y$ and $r \in \mathbb{R}$, $\|y + r\cdot\mathbf{1}\| \geq |r|$, where $\mathbf{1} = (1)_{n\in\mathbb{N}} \in \ell^\infty$.*
*(2) Show that the linear map $f : Y \oplus \mathbb{R}\mathbf{1} \to \mathbb{R}$ given by $f(y + r\cdot\mathbf{1}) := r$ is well-defined, and $|f(z)| \leq \|z\|$ for all $z \in Y \oplus \mathbb{R}\mathbf{1}$.*
*(3) Use the Real Hahn-Banach Theorem to extend $f$ to a $g \in \ell^\infty(\mathbb{N}, \mathbb{R})^*$ which satisfies the desired properties.*

**Problem 81.** Let $X$ be a compact Hausdorff topological space. For $x \in X$, define $\operatorname{ev}_x : C(X) \to \mathbb{F}$ by $\operatorname{ev}_x(f) = f(x)$.

(1) Prove that $\operatorname{ev}_x \in C(X)^*$, and find $\|\operatorname{ev}_x\|$.

(2) Show that the map $\text{ev} : X \to C(X)^*$ given by $x \mapsto \text{ev}_x$ is a homeomorphism onto its image, where the image has the relative weak* topology.

**Problem 82.** Suppose $X, Y$ are Banach spaces and $T : X \to Y$ is a linear transformation.

(1) Show that if $T \in \mathcal{L}(X,Y)$, then $T$ is weak-weak continuous. That is, if $x_\lambda \to x$ in the weak topology on $X$ induced by $X^*$, then $Tx_\lambda \to Tx$ in the weak topology on $Y$ induced by $Y^*$.
(2) Show that if $T$ is norm-weak continuous, then $T \in \mathcal{L}(X,Y)$.
(3) Show that if $T$ is weak-norm continuous, then $T$ has finite rank, i.e., $TX$ is finite dimensional.

*Hint: For part (3), one could proceed as follows.*

*(a) First, reduce to the case that $T$ is injective by replacing $X$ with $Z = X/\ker(T)$ and $T$ with $S : Z \to Y$ given by $x + \ker(T) \mapsto Tx$. (You must show $S$ is weak-norm continuous on $Z$.)*
*(b) Take a basic open set $\mathcal{U} = \{z \in Z \,|\, |\varphi_i(z)| < \varepsilon \text{ for all } i = 1, \ldots, n\} \subset S^{-1}B_1(0_Y)$. Use that $S$ is injective to prove that $\bigcap_{i=1}^n \ker(\varphi_i) = (0)$.*
*(c) Use Problem 73 to deduce that $Z^*$ is finite dimensional, and thus that $Z$ and $TX = SZ$ are finite dimensional.*

**Problem 83.** Suppose $X$ is a Banach space. Prove the following are equivalent:

(1) $X$ is separable.
(2) The relative weak* topology on the closed unit ball of $X^*$ is metrizable.

Deduce that the closed unit ball of $X^*$ is weak* sequentially compact.

**Problem 84.** Suppose $X$ is a Banach space. Prove the following are equivalent:

(1) $X^*$ is separable.
(2) The relative weak topology on the closed unit ball of $X$ is metrizable.

Prove that in this case, $X$ is also separable.

**Problem 85** (Eberlein-Smulian)**.** Suppose $X$ is a Banach space. Prove the following are equivalent:

(1) $X$ is reflexive.
(2) The closed unit ball of $X$ is weakly compact.
(3) The closed unit ball of $X$ is weakly sequentially compact.

*Optional: How do you reconcile Problems 83, 84, and 85? That is, how do you reconcile the fact that there exist separable Banach spaces which are not reflexive?*

**Problem 86.** Consider the space $L^2(\mathbb{T}) := L^2(\mathbb{R}/\mathbb{Z})$ of $\mathbb{Z}$-periodic functions $\mathbb{R} \to \mathbb{C}$ such that $\int_{[0,1]} |f|^2 < \infty$. Define

$$\langle f, g \rangle := \int_{[0,1]} f\overline{g}.$$

(1) Prove that $L^2(\mathbb{T})$ is a Hilbert space.
(2) Show that the subspace $C(\mathbb{T}) \subset L^2(\mathbb{T})$ of continuous $\mathbb{Z}$-periodic functions is dense.
(3) Prove that $\{e_n(x) := \exp(2\pi i n x) | n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$.
*Hint: Orthonormality is easy. Use (2) and the Stone-Weierstrass Theorem to show the linear span is dense.*
(4) Define $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ by $\mathcal{F}(f)_n := \langle f, e_n \rangle_{L^2(\mathbb{T})} = \int_0^1 f(x) \exp(-2\pi i n x)\, dx$. Show that if $f \in L^2(\mathbb{T})$ and $\mathcal{F}(f) \in \ell^1(\mathbb{Z})$, then $f \in C(\mathbb{T})$, i.e., $f$ is a.e. equal to a continuous function.

**Problem 87.** Suppose $H$ is a Hilbert space, $E \subset H$ is an orthonormal set, and $\{e_1, \ldots, e_n\} \subset E$. Prove the following assertions.

(1) If $x = \sum_{i=1}^n c_i e_i$, then $c_i = \langle x, e_i \rangle$.
(2) The set $E$ is linearly independent.

(3) For every $x \in H$, $\sum_{i=1}^n \langle x, e_i\rangle e_i$ is the unique element of $\operatorname{span}\{e_1, \ldots, e_n\}$ minimizing the distance to $x$.
(4) (Bessel's Inequality) For every $x \in H$, $\|x\|^2 \geq \sum_{i=1}^n |\langle x, e_i\rangle|^2$.
(5) If $H$ is separable, then $E$ is countable.
(6) The set $E$ can be extended to an orthonormal basis for $H$.
(7) If $E$ is an orthonormal basis, then the map $H \to \ell^2(E)$ given by $x \mapsto (\langle x, \cdot\rangle : E \to \mathbb{C})$ is a unitary isomorphism of Hilbert spaces.

## 6. Radon measures

**Problem 88.** Let $X$ be a locally compact Hausdorff space and suppose $\varphi : C_0(X) \to \mathbb{C}$ is a linear functional such that $\varphi(f) \geq 0$ whenever $f \geq 0$. Prove that $\varphi$ is bounded.
*Hint: Prove that $\{\varphi(f) | 0 \leq f \leq 1\}$ is bounded.*

**Problem 89.** Suppose $X$ is an LCH space, $K \subset X$ is compact, and $U_1, \ldots, U_n$ are open sets such that $K \subset \bigcup_{i=1}^n U_i$. Show there exist $g_1, \ldots, g_n \in C_c(X)$ such that $g_i \prec U_i$ for all $i$ and $\sum_{i=1}^n g_i = 1$ on $K$.

**Problem 90.** Suppose $X$ is an LCH space, $\mu$ is a $\sigma$-finite Radon measure on $X$, and $E$ is a Borel set. Prove that for every $\varepsilon > 0$, there is an open set $U$ and a closed set $F$ with $F \subset E \subset U$ such that $\mu(U \setminus F) < \varepsilon$.

**Problem 91.** Suppose $X$ is an LCH space and $\varphi \in C_0(X)^*$. Prove there are finite Radon measures $\mu_0, \mu_1, \mu_2, \mu_3$ on $X$ such that

$$\varphi(f) = \sum_{k=1}^{3} i^k \int f \, d\mu_k \qquad \forall f \in C_0(X).$$