5.1. **Normed spaces and linear maps.** For this section, X will denote a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. (We will assume $\mathbb{F} = \mathbb{C}$ unless stated otherwise.)

Definition 5.1.1. A *seminorm* on X is a function $\|\cdot\| : X \to [0, \infty)$ which is

- (homogeneous) $\|\lambda x\| = |\lambda| \cdot \|x\|$
- (subadditive) $\|x + y\| \leq \|x\| + \|y\|$

We call $\|\cdot\|$ a *norm* if in addition it is

- (definite) $\|x\| = 0$ implies $x = 0$.

Recall that given a norm $\|\cdot\|$ on a vector space X, $d(x, y) := \|x - y\|$ is a metric which induces the *norm topology* on X. Two norms $\|\cdot\|_1, \|\cdot\|_2$ are called *equivalent* if there is a $c > 0$ such that

$$c^{-1}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1 \qquad \forall x \in X.$$

Exercise 5.1.2. Show that all norms on $\mathbb{F}^n$ are equivalent. Deduce that a finite dimensional subspace of a normed space is closed.
Note: You may assume that the unit ball of $\mathbb{F}^n$ is compact in the Euclidean topology.

Exercise 5.1.3. Show that two norms $\|\cdot\|_1, \|\cdot\|_2$ on X are equivalent if and only if they induce the same topology.

Definition 5.1.4. A *Banach space* is a normed vector space which is complete in the induced metric topology.

Examples 5.1.5.

(1) If X is an LCH topological space, then $C_0(X)$ and $C_b(X)$ are Banach spaces.
(2) If $(X, \mathcal{M}, \mu)$ is a measure space, $\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a Banach space.
(3) $\ell^1 := \{(x_n) \subset \mathbb{F}^\infty | \sum |x_n| < \infty\}$

Definition 5.1.6. Suppose $(X, \|\cdot\|)$ is a normed space and $(x_n) \subset (X, \|\cdot\|)$ is a sequence. We say $\sum x_n$ *converges to* $x \in X$ if $\sum^N x_n \to x$ as $N \to \infty$. We say $\sum x_n$ *converges absolutely* if $\sum \|x_n\| < \infty$.

Proposition 5.1.7. *The following are equivalent for a normed space $(X, \|\cdot\|)$.*

(1) X is Banach, and
(2) Every absolutely convergent sequence converges.

Proof.
(1) ⇒ (2): Suppose X is Banach and $\sum \|x_n\| < \infty$. Let $\varepsilon > 0$, and pick $N > 0$ such that $\sum_{n>N} \|x_n\| < \varepsilon$. Then for all $m \geq n > N$,

$$\left\|\sum^m x_i - \sum^n x_i\right\| = \left\|\sum_{n+1}^m x_i\right\| \leq \sum_{n+1}^m \|x_i\| \leq \sum_{n>N} \|x_i\| < \varepsilon.$$

(2) ⇒ (1): Suppose (x_n) is Cauchy, and choose $n_1 < n_2 < \cdots$ such that $\|x_m - x_n\| < 2^{-k}$ whenever $m, n > n_k$. Define $y_0 := 0$ (think of this as x_{n_0} by convention), and inductively define $y_k := x_{n_k} - x_{n_{k-1}}$ for all $k \in \mathbb{N}$. Then

$$\sum \|y_k\| \leq \|x_{n_1}\| + \sum_{k\geq 1} 2^{-k} = \|x_{n_1}\| + 1 < \infty.$$

Hence $x := \lim x_{n_k} = \sum y_k$ exists in X. Since (x_n) is Cauchy, $x_n \to x$. □

Proposition 5.1.8. *Suppose X, Y are normed spaces and $T : X \to Y$ is linear. The following are equivalent:*

(1) T is uniformly continuous (with respect to the norm topologies),
(2) T is continuous,
(3) T is continuous at 0_X, and
(4) T is bounded, *i.e., there exists a $c > 0$ such that $\|Tx\| \le c\|x\|$ for all $x \in X$.*

Proof.
(1) ⇒ (2) ⇒ (3): Trivial.
(3) ⇒ (4): Suppose T is continuous at 0_X. Then there is a neighborhood U of 0_X such that $TU \subset \{y \in Y | \|y\| \le 1\}$. Since U is open, there is a $\delta > 0$ such that $\{x \in X | \|x\| \le \delta\} \subset U$. Thus $\|x\| \le \delta$ implies $\|Tx\| \le 1$. Then for all $x \neq 0$

$$\left\| \delta \cdot \frac{x}{\|x\|} \right\| \le \delta \qquad \Longrightarrow \qquad \left\| \delta \cdot \frac{Tx}{\|x\|} \right\| \le 1 \qquad \Longrightarrow \qquad \|Tx\| \le \delta^{-1}\|x\|.$$

(4) ⇒ (1): Let $\varepsilon > 0$. If $\|x_1 - x_2\| < c^{-1}\varepsilon$, then

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \le c\|x_1 - x_2\| < \varepsilon.$$ □

Exercise 5.1.9. Suppose X is a normed space and $Y \subset X$ is a subspace. Define $Q : X \to X/Y$ by $Qx = x + Y$. Define

$$\|Qx\|_{X/Y} = \inf\{\|x - y\|_X | y \in Y\}.$$

(1) Prove that $\|\cdot\|_{X/Y}$ is a well-defined seminorm.
(2) Show that if Y is closed, then $\|\cdot\|_{X/Y}$ is a norm.
(3) Show that in the case of (2) above, $Q : X \to X/Y$ is continuous and open.
Optional: is Q continuous or open only in the case of (1)?
(4) Show that if X is Banach, so is X/Y.

Exercise 5.1.10.

(1) Show that for any two finite dimensional normed spaces F_1 and F_2, all linear maps $T : F_1 \to F_2$ are continuous.
Optional: Show that for any two finite dimensional vector spaces F_1 and F_2 endowed with their vector space topologies from Exercise 5.1.2, all linear maps $T : F_1 \to F_2$ are continuous.
(2) Let X, F be normed spaces with F finite dimensional, and let $T : X \to F$ be a linear map. Prove that the following are equivalent:
 (a) T is bounded (there is an $c > 0$ such that $T(B_1(0_X)) \subseteq B_c(0_F)$), and
 (b) $\ker(T)$ is closed.
Hint: One way to do (b) implies (a) uses Exercise 5.1.9 part (3) and part (1) of this problem.

Definition 5.1.11. Suppose X, Y are normed spaces. Let

$$\mathcal{L}(X \to Y) := \{\text{bounded linear } T : X \to Y\}.$$

Define the *operator norm* on $\mathcal{L}(X \to Y)$ by

$$\begin{aligned}\|T\| &:= \sup\{\|Tx\| \,|\, \|x\| \leq 1\} \\ &= \sup\{\|Tx\| \,|\, \|x\| = 1\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} \middle| \|x\| \neq 0\right\} \\ &= \inf\{c > 0 \,|\, \|Tx\| \leq c\|x\| \text{ for all } x \in X\},\end{aligned}$$

Observe that if $S \in \mathcal{L}(Y \to Z)$ and $T \in \mathcal{L}(X \to Y)$, then $ST \in \mathcal{L}(X \to Z)$ and

$$\|STx\| \leq \|S\| \cdot \|Tx\| \leq \|S\| \cdot \|T\| \cdot \|x\| \qquad \forall x \in X.$$

So $\|ST\| \leq \|S\| \cdot \|T\|$.

Proposition 5.1.12. *If Y is Banach, then so is $\mathcal{L}(X \to Y)$.*

Proof. If (T_n) is Cauchy, then so is $(T_n x)$ for all $x \in X$. Set $Tx := \lim T_n x$ for $x \in X$. One verifies that T is linear, T is bounded, and $T_n \to T$. □

Corollary 5.1.13. *If X is complete, then $\mathcal{L}(X) := \mathcal{L}(X \to X)$ is a* Banach algebra *(an algebra with a complete submultiplicative norm).*

Exercise 5.1.14 (Folland §5.1, #7)**.** Suppose X is a Banach space and $T \in \mathcal{L}(X)$. Let $I \in \mathcal{L}(X)$ be the identity map.

(1) Show that if $\|I - T\| < 1$, then T is invertible.
Hint: Show that $\sum_{n\geq 0}(I - T)^n$ converges in $\mathcal{L}(X)$ to T^{-1}.
(2) Show that if $T \in \mathcal{L}(X)$ is invertible and $\|S - T\| < \|T^{-1}\|^{-1}$, then S is invertible.
(3) Deduce that the set of invertible operators $GL(X) \subset \mathcal{L}(X)$ is open.

Exercise 5.1.15. Consider the measure space $(M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}, \lambda^{n^2})$. Show that $GL_n(\mathbb{C})^c \subset M_n(\mathbb{C})$ is λ^{n^2}-null.

Exercise 5.1.16 (Folland §5.2, #19)**.** Let X be an infinite dimensional normed space.

(1) Construct a sequence (x_n) such that $\|x_n\| = 1$ for all n and $\|x_m - x_n\| \geq 1/2$ for all $m \neq n$.
(2) Deduce X is not locally compact.

5.2. **Dual spaces.**

Definition 5.2.1. Let X be a (normed) vector space. A linear map $X \to \mathbb{F}$ is called a (linear) functional. The *dual space* of X is $X^* := \operatorname{Hom}(X \to \mathbb{F})$. Here, Hom means:

- linear maps if X is a vector space, and
- bounded linear maps if X is a normed space.

Exercise 5.2.2. Suppose $\varphi, \varphi_1, \ldots, \varphi_n$ are linear functionals on a vector space X. Prove that the following are equivalent.

(1) $\varphi = \sum_{k=1}^n \alpha_k \varphi_k$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$.
(2) There is an $\alpha > 0$ such that for all $x \in X$, $|\varphi(x)| \leq \alpha \max_{k=1,\ldots,n} |\varphi_k(x)|$.
(3) $\bigcap_{k=1}^n \ker(\varphi_k) \subset \ker(\varphi)$.

Exercise 5.2.3. Let X be a locally compact Hausdorff space and suppose $\varphi : C_0(X) \to \mathbb{C}$ is a linear functional such that $\varphi(f) \geq 0$ whenever $f \geq 0$. Prove that φ is bounded.
Hint: Argue by contradiction that $\{\varphi(f) | 0 \leq f \leq 1\}$ is bounded using Proposition 5.1.7.

Proposition 5.2.4. *Suppose X is a complex vector space.*

(1) *If $\varphi : X \to \mathbb{C}$ is $\mathbb{C}$-linear, then $\operatorname{Re}(\varphi) : X \to \mathbb{R}$ is $\mathbb{R}$-linear, and for all $x \in X$,*
$$\varphi(x) = \operatorname{Re}(\varphi)(x) - i \operatorname{Re}(\varphi)(ix).$$
(2) *If $f : X \to \mathbb{R}$ is $\mathbb{R}$-linear, then*
$$\varphi(x) := f(x) - if(ix)$$
defines a $\mathbb{C}$-linear functional.

(3) *Suppose X is normed and $\varphi : X \to \mathbb{C}$ is $\mathbb{C}$-linear.*
- *In Case (1), $\|\varphi\| < \infty$ implies $\|\operatorname{Re}(\varphi)\| \leq \|\varphi\|$*
- *In Case (2), $\|\operatorname{Re}(\varphi)\| < \infty$ implies $\|\varphi\| \leq \|\operatorname{Re}(\varphi)\|$.*

Thus $\|\varphi\| = \|\operatorname{Re}(\varphi)\|$.

Proof.
(1) Just observe $\operatorname{Im}(\varphi(x)) = -\operatorname{Re}(i\varphi(x)) = -\operatorname{Re}(\varphi)(ix)$.
(2) It is clear that φ is $\mathbb{R}$-linear. We now check
$$\varphi(ix) = f(ix) - if(i^2x) = f(ix) - if(-x) = if(x) + f(ix) = i(f(x) - if(ix)) = i\varphi(x).$$
(3, Case 1) Since $|\operatorname{Re}(\varphi)(x)| \leq |\varphi(x)|$ for all $x \in X$, $\|\operatorname{Re}(\varphi)\| \leq \|\varphi\|$.
(3, Case 2) If $\varphi(x) \neq 0$, then
$$|\varphi(x)| = \overline{\operatorname{sgn}(\varphi(x))}\varphi(x) = \varphi(\overline{\operatorname{sgn}(\varphi(x))} \cdot x) = \operatorname{Re}(\varphi)(\overline{\operatorname{sgn}(\varphi(x))} \cdot x).$$
Hence $|\varphi(x)| \leq \|\operatorname{Re}(\varphi)\| \cdot \|x\|$, which implies $\|\varphi\| \leq \|\operatorname{Re}(\varphi)\|$. □

Exercise 5.2.5. Consider the following sequence spaces.
$$\ell^1 := \left\{(x_n) \subset \mathbb{C}^\infty \middle| \sum |x_n| < \infty\right\} \qquad \|x\|_1 := \sum |x_n|$$
$$c_0 := \{(x_n) \subset \mathbb{C}^\infty | x_n \to 0 \text{ as } n \to \infty\} \qquad \|x\|_\infty := \sup |x_n|$$
$$c := \left\{(x_n) \subset \mathbb{C}^\infty \middle| \lim_{n\to\infty} x_n \text{ exists}\right\} \qquad \|x\|_\infty := \sup |x_n|$$
$$\ell^\infty := \{(x_n) \subset \mathbb{C}^\infty | \sup |x_n| < \infty\} \qquad \|x\|_\infty := \sup |x_n|$$
(1) Show that every space above is a Banach space.
Hint: First show ℓ^1 and ℓ^∞ are Banach. Then show c_0, c are closed in ℓ^∞.
(2) Construct isometric isomorphisms $c_0^* \cong \ell^1 \cong c^*$ and $(\ell^1)^* \cong \ell^\infty$.
(3) Which of the above spaces are separable?

Warning 5.2.6. If X is a normed space, constructing a non-zero bounded linear functional takes a considerable amount of work. One cannot get by simply choosing a basis for X as an ordinary linear space and mapping the basis to arbitrarily chosen elements of $\mathbb{F}$.

Definition 5.2.7. Suppose X is an $\mathbb{R}$-vector space. A *sublinear (Minkowski) functional* on X is a function $p : X \to \mathbb{R}$ such that
- (positive homogeneous) for all $x \in X$ and $r \geq 0$, $p(rx) = rp(x)$, and
- (subadditive) for all $x, y \in X$, $p(x + y) \leq p(x) + p(y)$.

Theorem 5.2.8 (Real Hahn-Banach)**.** *Let X be an $\mathbb{R}$-vector space, $p : X \to \mathbb{R}$ a sublinear functional, $Y \subset X$ a subspace, and $f : Y \to \mathbb{R}$ a linear functional such that $f(y) \leq p(y)$ for all $y \in Y$. Then there is an $\mathbb{R}$-linear functional $g : X \to \mathbb{R}$ such that $g|_Y = f$ and $g(x) \leq p(x)$ for all $x \in X$.*

Proof.

Step 1: For all $x \in X \setminus Y$, there is a linear $g : Y \oplus \mathbb{R}x \to \mathbb{R}$ such that $g|_Y = f$ and $g(z) \leq p(z)$ on $Y \oplus \mathbb{R}x$.

Proof. Any extension g of f to $Y \oplus \mathbb{R}x$ is determined by $g(y + rx) = f(y) + r\alpha$ for all $r \in \mathbb{R}$, where $\alpha = g(x)$. We want to choose $\alpha \in \mathbb{R}$ such that

$$f(y) + r\alpha \leq p(y + rx) \qquad \forall\, y \in Y \text{ and } \forall\, r \in \mathbb{R}. \tag{5.2.9}$$

Since f is $\mathbb{R}$-linear and p is positive homogeneous, we need only consider the cases $r = \pm 1$. Restricting to these 2 cases, (5.2.9) becomes:

$$f(y) - p(y - x) \leq \alpha \leq p(z + x) - f(z) \qquad \forall\, y, z \in Y.$$

Now observe that

$$p(z+x)-f(z)-f(y)+p(y-x) = p(z+x)+p(y-x)-f(y+z) \geq p(y+z)-f(y+z) \geq 0.$$

Hence there exists an α which lies in the interval

$$[\sup\{f(y) - p(y - x) | y \in Y\}, \inf\{p(z + x) - f(z) | z \in Y\}].$$ □

Step 2: Observe that Step 1 applies to any extension g of f to $Y \subset Z \subset X$ such that $g|_Y = f$ and $g \leq p$ on Z. Thus any maximal extension g of f satisfying $g|_Y = f$ and $g \leq p$ on its domain must have domain X. Note that

$$\left\{ (Z, g) \middle| \begin{array}{l} Y \subseteq Z \subseteq X \text{ is a subspace and } g : Z \to \mathbb{R} \\ \text{such that } g|_Y = f \text{ and } g \leq p \text{ on } Z \end{array} \right\}$$

is partially ordered by $(Z_1, g_1) \leq (Z_2, g_2)$ if $Z_1 \subseteq Z_2$ and $g_2|_{Z_1} = g_1$. Since every ascending chain has an upper bound, there is a maximal extension by Zorn's Lemma. □

Remark 5.2.10. Suppose p is a seminorm on X and $f : X \to \mathbb{R}$ is $\mathbb{R}$-linear. Then $f \leq p$ if and only if $|f| \leq p$. Indeed,

$$|f(x)| = \pm f(x) = f(\pm x) \leq p(\pm x) = p(x).$$

Theorem 5.2.11 (Complex Hahn-Banach)**.** *Let X be an $\mathbb{C}$-vector space, $p : X \to [0, \infty)$ a seminorm, $Y \subset X$ a subspace, and $\varphi : Y \to \mathbb{R}$ a linear functional such that $|\varphi(y)| \leq p(y)$ for all $y \in Y$. Then there is a $\mathbb{C}$-linear functional $\psi : X \to \mathbb{C}$ such that $\psi|_Y = \varphi$ and $|\psi(x)| \leq p(x)$ for all $x \in X$.*

Proof. By the Real Hahn-Banach Theorem 5.2.8 applied to $\operatorname{Re}(\varphi)$ which is bounded above by p, there is an $\mathbb{R}$-linear extension $g : X \to \mathbb{R}$ such that $g|_Y = \operatorname{Re}(\varphi)$ and $|g| \leq p$. Define $\psi(x) := g(x) - ig(ix)$. By Proposition 5.2.4, $\psi|_Y = \varphi$. Finally, for all $x \in X$,

$$|\psi(x)| = \overline{\operatorname{sgn}\psi(x)} \cdot \psi(x) = \psi(\overline{\operatorname{sgn}\psi(x)} \cdot x) = g(\overline{\operatorname{sgn}\psi(x)} \cdot x) \leq p(\overline{\operatorname{sgn}\psi(x)} \cdot x) = p(x).$$ □

Facts 5.2.12. Here are some corollaries of the Hahn-Banach Theorems 5.2.8 and 5.2.11. Let X be an $\mathbb{F}$-linear normed space.

(HB1) If $x \neq 0$, there is a $\varphi \in X^*$ such that $\varphi(x) = \|x\|$ and $\|\varphi\| = 1$.

> *Proof.* Define $f : \mathbb{F}x \to \mathbb{F}$ by $f(\lambda x) := \lambda\|x\|$, and observe that $|f| \leq \|\cdot\|$. Now apply Hahn-Banach. □

(HB2) If $Y \subset X$ is closed and $x \notin Y$, there is a $\varphi \in X^*$ such that $\|\varphi\| = 1$ and

$$\varphi(x) = \|x + Y\|_{X/Y} := \inf_{y \in Y} \|x - y\|.$$

> *Proof.* Apply (HB1) to $x + Y \in X/Y$ to get $f \in (X/Y)^*$ such that $\|f\| = 1$ and
>
> $$f(x + Y) = \|x + Y\| = \inf_{y \in Y} \|x - y\|.$$
>
> By Exercise 5.1.9, the canonical quotient map $Q : X \to X/Y$ is continuous. Since
>
> $$\|x + Y\| = \inf_{y \in Y} \|x - y\| \leq \|x\| \qquad \forall\, x \in X,$$
>
> we have $\|Q\| \leq 1$. Thus $\varphi := f \circ Q$ works. □

(HB3) X^* separates points of X.

> *Proof.* If $x \neq y$, then by (HB1), there is a $\varphi \in X^*$ such that $\varphi(x - y) = \|x - y\| \neq 0$. □

(HB4) For $x \in X$, define $\operatorname{ev}_x : X^* \to \mathbb{F}$ by $\operatorname{ev}_x(\varphi) := \varphi(x)$. Then $\operatorname{ev} : X \to X^{**}$ is a linear isometry.

> *Proof.* It is easy to see that ev is linear. For all $\varphi \in X^*$,
>
> $$\|\operatorname{ev}_x(\varphi)\| = |\varphi(x)| \leq \|\varphi\| \cdot \|x\| \qquad \Longrightarrow \qquad \|\operatorname{ev}_x\| \leq \|x\|.$$
>
> Thus $\operatorname{ev}_x \in X^{**}$. If $x \neq 0$, by (HB1) there is a $\varphi \in X^*$ such that $\varphi(x) = \|x\|$ and $\|\varphi\| = 1$. Thus $\|\operatorname{ev}_x\| = \|x\|$. □

Exercise 5.2.13 (Banach Limits)**.** Let $\ell^\infty(\mathbb{N}, \mathbb{R})$ denote the Banach space of bounded functions $\mathbb{N} \to \mathbb{R}$. Show that there is a $\varphi \in \ell^\infty(\mathbb{N}, \mathbb{R})^*$ satisfying the following two conditions:

(1) Letting $S : \ell^\infty(\mathbb{N}, \mathbb{R}) \to \ell^\infty(\mathbb{N}, \mathbb{R})$ be the shift operator $(Sx)_n = x_{n+1}$ for $x = (x_n)_{n \in \mathbb{N}}$, $\varphi = \varphi \circ S$.

(2) For all $x \in \ell^\infty$, $\liminf x_n \leq \varphi(x) \leq \limsup x_n$.

Hint: One could proceed as follows.

(1) Consider the subspace $Y = \operatorname{im}(S - I) = \{Sx - x | x \in \ell^\infty\}$*. Prove that for all* $y \in Y$ *and* $r \in \mathbb{R}$*,* $\|y + r \cdot \mathbf{1}\| \geq |r|$*, where* $\mathbf{1} = (1)_{n \in \mathbb{N}} \in \ell^\infty$*.*

(2) Show that the linear map $f : Y \oplus \mathbb{R}\mathbf{1} \to \mathbb{R}$ *given by* $f(y + r \cdot \mathbf{1}) := r$ *is well-defined, and* $|f(z)| \leq \|z\|$ *for all* $z \in Y \oplus \mathbb{R}\mathbf{1}$*.*

(3) Use the Real Hahn-Banach Theorem 5.2.8 to extend f *to a* $g \in \ell^\infty(\mathbb{N}, \mathbb{R})^*$ *which satisfies (1) and (2).*

Definition 5.2.14. For a normed space X, its *completion* is $\overline{X} := \overline{\mathrm{ev}(X)} \subset X^{**}$, which is always Banach. Observe that if X is Banach, then $\mathrm{ev}(X) \subset X^{**}$ is closed. In this case, if $\mathrm{ev}(X) = X^{**}$, we call X *reflexive.*

Exercise 5.2.15. Show that X is reflexive if and only if X^* is reflexive.
Hint: Instead of the converse, try proving the inverse, i.e., if X is not reflexive, then X^* is not reflexive.

Exercise 5.2.16.

(1) (Folland §5.2, #25) Prove that if X is a Banach space such that X^* is separable, then X is separable.
(2) Find a separable Banach space X such that X^* is not separable.

5.3. **The Baire Category Theorem and its consequences.**

Theorem 5.3.1 (Baire Category)**.** *Suppose X is either:*

(1) a complete metric space, or
(2) an LCH space.

Suppose (U_n) is a sequence of open dense subsets of X. Then $\bigcap U_n$ is dense in X.

Proof. Let $V_0 \subset X$ be non-empty and open. We will inductively construct for $n \in \mathbb{N}$ a non-empty open set $V_n \subset \overline{V_n} \subset U_n \cap V_{n-1}$.
Case 1: Take V_n to be a ball of radius $< 1/n$.
Case 2: Take V_n such that $\overline{V_n}$ is compact, so $(\overline{V_n})$ are non-empty nested compact sets.

Claim. *$K := \bigcap V_n$ is not empty.*

Proof of Claim.
Case 1: Let x_n be the center of V_n for all n. Then (x_n) is Cauchy, so it converges. The limit lies in K by construction.
Case 2: Observe $(\overline{V_n})$ is a family of closed sets with the finite intersection property. Since $\overline{V_1}$ is compact, we have $K \neq \emptyset$. □

Now observe $\emptyset \neq K \subset (\bigcap U_n) \cap V_0$. Thus $\bigcap U_n$ is dense in X. □

Corollary 5.3.2. *If X is as in the Baire Category Theorem 5.3.1, then X is not* meager, *i.e., a countable union of nowhere dense sets.*

Proof. If (Y_n) is a sequence of nowhere dense sets, then $(U_n := \overline{Y_n}^c)$ is a sequence of open dense sets. Then

$$\bigcap U_n = \bigcap \overline{Y_n}^c = \left(\bigcup \overline{Y_n}\right)^c \subseteq \left(\bigcup Y_n\right)^c$$

is dense in X, so $\bigcup Y_n \neq X$. □

Lemma 5.3.3. *Suppose X, Y are Banach spaces and $T \in \mathcal{L}(X \to Y)$. Let $U \subset X$ be an open ball centered at 0_X and $V \subset Y$ be an open ball centered at 0_Y. If $V \subset \overline{TU}$, then $V \subset TU$.*

Proof. Let $y \in V$. Take $r \in (0,1)$ such that $y \in rV$. Let $\varepsilon \in (0,1)$ to be decided later. Observe that

$$y \in \overline{rV} \subset \overline{rTU} = \overline{TrU},$$

so there is an $x_0 \in rU$ such that
$$y - Tx_0 \in \varepsilon r V \subset \overline{\varepsilon r TU} = \overline{T(\varepsilon r U)}.$$
Then there is an $x_1 \in \varepsilon r U$ such that
$$y - Tx_0 - Tx_1 \in \varepsilon^2 r V \subset \overline{T(\varepsilon^2 r U)}.$$
Hence by induction, we can construct a sequence (x_n) such that
$$x_n \in \varepsilon^n r U \qquad \text{and} \qquad y - \sum_{j=0}^{n} Tx_j \in \varepsilon^{n+1} r V.$$
Observe that $\sum x_j$ converges as $\|x_j\| < \varepsilon^j r R$ (which is summable!), where $R := \text{radius}(U)$. Moreover,
$$T \sum x_j = \lim_{n\to\infty} T \sum^{n} x_j = \lim_{n\to\infty} \sum^{n} Tx_j = y.$$
Finally, we have
$$\left\| \sum x_j \right\| \le \sum \|x_j\| < \sum_{j=0}^{\infty} \varepsilon^j r R = \frac{rR}{1-\varepsilon},$$
so $\sum x_j \in \frac{r}{1-\varepsilon} U$. Thus if $\varepsilon < 1 - r$, then $\sum x_n \in U$, so $y \in TU$. □

Theorem 5.3.4 (Open Mapping)**.** *Suppose X, Y are Banach spaces and $T \in \mathcal{L}(X \to Y)$ is surjective. Then T is an open map.*

Proof. It suffices to prove T maps an open neighborhood of 0_X to an open neighborhood of 0_Y. Note $Y = \bigcup_n \overline{TB_n(0_X)}$. By the Baire Category Theorem 5.3.1, there is an $n \in \mathbb{N}$ such that $\overline{TB_n(0)}$ contains a non-empty open set, say $Tx_0 + V$ where $x_0 \in TB_n(0_X)$ and V is an open ball in Y with center 0_Y. Then $V \subset \overline{TB_n(0)} - Tx_0 \subset \overline{TB_{2n}(0_X)}$. By Lemma 5.3.3, $V \subset TB_{2n}(0_X)$. □

Facts 5.3.5. Here are some corollaries of the Open Mapping Theorem 5.3.4.

(OMT1) Suppose X, Y are Banach spaces and $T \in \mathcal{L}(X \to Y)$ is bijective. Then $T^{-1} \in \mathcal{L}(Y \to X)$, and we call T an *isomorphism*.

> *Proof.* When T is bijective, T^{-1} is continuous if and only if T is open. □

(OMT2) Suppose X is Banach under $\|\cdot\|_1$ and $\|\cdot\|_2$. If there is a $c \ge 0$ such that $\|x\|_1 \le c\|x\|_2$ for all $x \in X$, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

> *Proof.* Apply (OMT1) to the identity map $\text{id} : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$. □

Definition 5.3.6. Suppose X, Y are normed spaces and $T : X \to Y$ is linear. The *graph* of T is the subspace
$$\Gamma(T) := \{(x, y) | Tx = y\} \subset X \times Y.$$
Here, we endow $X \times Y$ with the norm
$$\|(x, y)\|_\infty := \max\{\|x\|_X, \|y\|_Y\}.$$
We say T is *closed* if $\Gamma(T) \subset X \times Y$ is a closed subspace.

Remark 5.3.7. If $T \in \mathcal{L}(X \to Y)$, then $\Gamma(T)$ is closed. Indeed, $(x_n, Tx_n) \to (x, y)$ if and only if $x_n \to x$ and $Tx_n \to y$. Since T is continuous, $Tx_n \to Tx$. Since Y is Hausdorff, $Tx = y$.

Theorem 5.3.8 (Closed Graph)**.** *Suppose X, Y are Banach. If $T : X \to Y$ is a closed linear map, then $T \in \mathcal{L}(X \to Y)$, i.e., T is bounded.*

Proof. Since X, Y are Banach, so is $X \times Y$. Consider the canonical projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, which are continuous. Since $\pi_X|_{\Gamma(T)} : \Gamma(T) \to X$ by $(x, Tx) \mapsto x$ is norm decreasing and bijective, $\pi_X|_{\Gamma(T)}^{-1}$ is bounded by (OMT1). Now observe

$$x \xmapsto{\pi_X|_{\Gamma(T)}^{-1}} (x, Tx) \xmapsto{\pi_Y|_{\Gamma(T)}} Tx \qquad\qquad \Longrightarrow \qquad\qquad T = \pi_Y|_{\Gamma(T)} \circ \pi_X|_{\Gamma(T)}^{-1}$$

which is bounded as the composite of two bounded linear maps. □

Exercise 5.3.9. Suppose X, Y are Banach spaces and $S : X \to Y$ and $T : Y^* \to X^*$ are linear maps such that

$$\varphi(Sx) = (T\varphi)(x) \qquad\qquad \forall\, x \in X, \quad \forall\, \varphi \in Y^*.$$

Prove that S, T are bounded.

Definition 5.3.10. A subset S of a topological space $(X, \mathcal{T})$ is called:

- *meager* if S is a countable union of nowhere dense sets, and
- *residual* if S^c is meager.

Exercise 5.3.11. Construct a (non-closed) infinite dimensional meager subspace of ℓ^∞.

Theorem 5.3.12 (Banach-Steinhaus/Uniform Boundedness Principle)**.** *Suppose X, Y are normed spaces and $\mathcal{S} \subset \mathcal{L}(X \to Y)$.*

(1) If $\sup_{T \in \mathcal{S}} \|Tx\| < \infty$ for all x in a non-meager subset of X, then $\sup_{T \in \mathcal{S}} \|T\| < \infty$.
(2) If X is Banach and $\sup_{T \in \mathcal{S}} \|Tx\| < \infty$ for all $x \in X$, then $\sup_{T \in \mathcal{S}} \|T\| < \infty$.

Proof.
(1) Define

$$\begin{aligned} E_n := \left\{ x \in X \middle| \sup_{T \in \mathcal{S}} \|Tx\| \le n \right\} &= \bigcap_{T \in \mathcal{S}} \{x \in X | \|Tx\| \le n\} \\ &= \bigcap_{T \in \mathcal{S}} (\underbrace{\|\cdot\| \circ T}_{\text{cts}})^{-1}([0, n]), \end{aligned} \tag{5.3.13}$$

which is closed in X. Since $\bigcup E_n$ is a non-meager subset of X, some E_n is non-meager. Thus there is an $x_0 \in X$, $r > 0$, and $n > 0$ such that $\overline{B_r(x_0)} \subset E_n$. Then $\overline{B_r(0)} \subset E_{2n}$:

$$\|Tx\| \le \|T(\underbrace{x - x_0}_{\in \overline{B_r(x_0)} \subset E_n})\| + \|Tx_0\| \le 2n \qquad\qquad \text{when } \|x\| \le r.$$

Thus for all $T \in \mathcal{S}$ and $\|x\| \le r$, we have $\|Tx\| \le 2n$. This implies

$$\sup_{T \in \mathcal{S}} \|T\| \le \frac{2n}{r}.$$

(2) Define E_n as in (5.3.13) above. Since $X = \bigcup E_n$ is Banach, the sets cannot all be meager by Corollary 5.3.2 to the Baire Category Theorem 5.3.1. The result now follows from (1). ☐

Exercise 5.3.14. Provide examples of the following:

(1) Normed spaces X, Y and a discontinuous linear map $T : X \to Y$ with closed graph.
(2) Normed spaces X, Y and a family of linear operators $\{T_\lambda\}_{\lambda\in\Lambda}$ such that $(T_\lambda x)_{\lambda\in\Lambda}$ is bounded for every $x \in X$, but $(\|T_\lambda\|)_{\lambda\in\Lambda}$ is not bounded.

Exercise 5.3.15. Suppose X and Y are Banach spaces and $T : X \to Y$ is a continuous linear map. Show that the following are equivalent.

(1) There exists a constant $c > 0$ such that $\|Tx\|_Y \geq c\|x\|_X$ for all $x \in X$.
(2) T is injective and has closed range.

Exercise 5.3.16 (Folland §5.3, #42)**.** Let $E_n \subset C([0, 1])$ be the space of all functions f such that there is an $x_0 \in [0, 1]$ such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0, 1]$.

(1) Prove that E_n is nowhere dense in $C([0, 1])$.
(2) Show that the subset of nowhere differentiable functions is residual in $C([0, 1])$.

Exercise 5.3.17. Suppose X, Y are Banach spaces and $(T_n) \subset \mathcal{L}(X \to Y)$ is a sequence of bounded linear maps such that $(T_n x)$ converges for all $x \in X$.

(1) Show that $Tx := \lim T_n x$ defines a bounded linear map.
(2) Does $T_n \to T$ in norm? Give a proof or a counterexample.
Hint: Think about shift operators on a sequence space.

5.4. **Topological vector spaces.**

Definition 5.4.1. An $\mathbb{F}$-vector space X equipped with a topology $\mathcal{T}$ is called a *topological vector space* if

$$+ : X \times X \longrightarrow X$$
$$\cdot : \mathbb{F} \times X \longrightarrow X$$

are continuous.

A subset $C \subseteq X$ is called *convex* if if

$$x, y \in C \qquad \Longrightarrow \qquad tx + (1 - t)y \in C \quad \forall\, t \in [0, 1].$$

A topological vector space is called *locally convex* if for all $x \in X$ and open neighborhoods $U \subset X$ of x, there is a convex open neighborhood V of x such that $V \subseteq U$.

Facts 5.4.2. Suppose $\mathcal{P}$ is a family of seminorms on the $\mathbb{F}$-vector space X. For $x \in X$, $p \in \mathcal{P}$, and $\varepsilon > 0$, define

$$U_{x,p,\varepsilon} := \{y \in X | p(x - y) < \varepsilon\}.$$

Let $\mathcal{T}$ be the topology generated by the sets $U_{x,p,\varepsilon}$, i.e., arbitrary unions of finite intersections of sets of this form.

(LCnvx1) Suppose $x_1, \ldots, x_n \in X$, $p_1, \ldots, p_n \in \mathcal{P}$, and $\varepsilon_1, \ldots, \varepsilon_n > 0$ and $x \in \bigcap_{i=1}^n U_{x_i,p_i,\varepsilon_i}$. Then there is a $\varepsilon > 0$ such that

$$\bigcap_{i=1}^n U_{x,p_i,\varepsilon} = \{y \in X | p_i(x - y) < \varepsilon \ \forall\, p_1, \ldots, p_n \in \mathcal{P}\} \subset \bigcap_{i=1}^n U_{x_i,p_i,\varepsilon_i}.$$

Hence sets of the form $\bigcap_{i=1}^n U_{x,p_i,\varepsilon} = \{y \in X | p_i(x-y) < \varepsilon \ \forall\, p_1, \ldots, p_n \in \mathcal{P}\}$ form a neighborhood base for $\mathcal{T}$ at x.

Proof. Define $\varepsilon := \min\{\varepsilon_i - p_i(x - x_i) | i = 1, \ldots, n\}$. Then for all $y \in \bigcap_{i=1}^n U_{x,p_i,\varepsilon}$ and $j = 1, \ldots, n$,

$$p_j(x_j - y) \leq p_j(x_j - x) + p_j(x - y) \leq (\varepsilon_j - \varepsilon) + \varepsilon = \varepsilon_j.$$

Thus $y \in \bigcap_{i=1}^n U_{x_i,p_i,\varepsilon_i}$, and thus $\bigcap_{i=1}^n U_{x,p_i,\varepsilon} \subseteq \bigcap_{i=1}^n U_{x_i,p_i,\varepsilon_i}$. □

(LCnvx2) If $(x_i) \subset X$ is a net, $x_i \to x$ if and only if $p(x - x_i) \to 0$ for all $p \in \mathcal{P}$.

Proof. By (LCnvx1) $x_i \to x$ if and only if (x_i) is eventually in $U_{x,p,\varepsilon}$ for all $\varepsilon > 0$ and $p \in \mathcal{P}$ if and only if $p(x - x_i) \to 0$ for all $p \in \mathcal{P}$. □

(LCnvx3) $\mathcal{T}$ is the weakest topology such that the $p \in \mathcal{P}$ are continuous.

Proof. Exercise. □

(LCnvx4) $(X, \mathcal{T})$ is a topological vector space.

Proof.
<u>+ cts:</u> Suppose $x_i \to x$ and $y_i \to y$. Then for all $p \in \mathcal{P}$,

$$p(x + y - (x_i + y_i)) \leq p(x - x_i) + p(y - y_i) \to 0.$$

<u>· cts:</u> Suppose $x_i \to x$ and $\alpha_i \to \alpha$. Then for all $p \in \mathcal{P}$,

$$\begin{aligned} p(\alpha_i x_i - \alpha x) &\leq p(\alpha_i x_i - \alpha x_i) + p(\alpha x_i - \alpha x) \\ &\leq \underbrace{|\alpha_i - \alpha|}_{\to 0} \cdot \underbrace{p(x_i)}_{\to p(x)} + |\alpha| \cdot \underbrace{p(x_i - x)}_{\to 0}. \end{aligned}$$

□

(LCnvx5) $(X, \mathcal{T})$ is locally convex.

Proof. Observe that each $U_{x,p,\varepsilon}$ is convex. Indeed, if $y, z \in U_{x,p,\varepsilon}$, then for all $t \in [0, 1]$,

$$\begin{aligned} p(x - (ty + (1-t)z)) &= p((tx + (1-t)x) - (ty + (1-t)z)) \\ &= p((t(x - y) + (1-t)(x - z)) \\ &\leq t \cdot p(x - y) + (1-t) \cdot p(x - z) \\ &< t\varepsilon + (1-t)\varepsilon \\ &= \varepsilon. \end{aligned}$$

The result now follows from (LCnvx1) as the intersection of convex sets is convex. □

(LCnvx6) $(X, \mathcal{T})$ is Hausdorff if and only if $\mathcal{P}$ separates points if and only if for all $x \in X \setminus \{0\}$, there is a $p \in \mathcal{P}$ such that $p(x) \neq 0$.

Proof. Exercise. $\square$

(LCnvx7) If $(X, \mathcal{T})$ is Hausdorff and $\mathcal{P}$ is countable, then there exists a metric $d : X \times X \to [0, \infty)$ which is *translation invariant* ($d(x+z, y+z) = d(x, y)$ for all $x, y, z \in X$) which induces the same topology as $\mathcal{P}$.

Proof. Let $\mathcal{P} = (p_n)$ be an enumeration and set

$$d(x, y) := \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}$$

We leave it to the reader to verify that d is a translation invariant metric which induces the topology $\mathcal{T}$. $\square$

(LCnvx8) If $(X, \mathcal{T})$ is locally convex Hausdorff TVS, then $\mathcal{T}$ is given by a separating family of seminorms.

Proof. Beyond the scope of this course; take Functional Analysis 7211. $\square$

Proposition 5.4.3. *Suppose $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ are seminormed locally convex topological vector spaces. The following are equivalent for a linear map $T : X \to Y$:*

(1) T is continuous.
(2) T is continuous at 0_X.
(3) For all $q \in \mathcal{Q}$, there are $p_1, \dots, p_n \in \mathcal{P}$ and $c > 0$ such that $q(Tx) \le c \sum_{j=1}^{n} p_j(x)$ for all $x \in X$.

Proof.
<u>(1) ⇒ (2):</u> Trivial.
<u>(2) ⇒ (3):</u> Suppose T is continuous at 0_X and $q \in \mathcal{Q}$. Then there are $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ such that for all $x \in V := \bigcap_{i=1}^{n} U_{0, p_i, \varepsilon}$, we have $q(Tx) < 1$. Fix $x \in X$. If $p_i(x) = 0$ for all $i = 1, \dots, n$, then $rx \in V$ for all $r > 0$, so

$$rq(Tx) = q(\underbrace{Trx}_{\in V}) < 1 \qquad \forall\, r > 0.$$

This implies $q(Tx) = 0 \le c \sum_{i=1}^{n} p_i(x)$ for all $c > 0$, so we may assume $p_1(x) > 0$. Then

$$y := \left(\frac{\varepsilon}{2 \sum_{i=1}^{n} p_i(x)} \right) \cdot x \in V$$

as $p_i(y) \le \varepsilon/2 < \varepsilon$ for all $i = 1, \dots, n$. Thus

$$q(Tx) = \left(\frac{2}{\varepsilon} \sum_{i=1}^{n} p_i(x) \right) q(Ty) < \frac{2}{\varepsilon} \sum_{i=1}^{n} p_i(x)$$

as desired.

(3) ⇒ (1): We must show if $x_i \to x$ in X, then $q(Tx_i - Tx) \to 0$ for all $q \in \mathcal{Q}$. Since $x_i \to x$, $p(x_i - x) \to 0$ for all $p \in \mathcal{P}$. Fix $q \in \mathcal{Q}$. By (3), there are $p_1, \dots, p_n \in \mathcal{P}$ and $c > 0$ such that

$$q(T(x_i - x)) \leq c \sum_{j=1}^{n} p_j(x_i - x) \longrightarrow 0 \qquad \forall\, x \in X. \qquad \square$$

Definition 5.4.4. Let X be a normed space. Recall that X^* separates points of X by the Hahn-Banach Theorem 5.2.8 or 5.2.11. Consider the family of seminorms

$$\mathcal{P} := \{\, x \mapsto |\varphi(x)| \,|\varphi \in X^*\}$$

on X, which separates points. Hence $\mathcal{P}$ induces a locally convex Hausdorff vector space topology on X in which $x_i \to x$ if and only if $\varphi(x_i) \to \varphi(x)$ for all $\varphi \in X^*$ by (LCnvx2). We call this topology the *weak topology* on X.

Proposition 5.4.5. *If $U \subset X$ is weakly open then U is $\|\cdot\|$-open.*

Proof. Observe that every basic open set $U_{x,\varphi,\varepsilon} = \{y \in X |\, |\varphi(x-y)| < \varepsilon\}$ is norm open in X. Indeed, $y \mapsto |\varphi(x-y)|$ is norm continuous as $\varphi \in X^*$ is norm continuous, the vector space operations are norm-continuous, and $|\cdot| : \mathbb{C} \to [0,\infty)$ is continuous. $\square$

Exercise 5.4.6. Let X be a normed space. Prove that the weak and norm topologies agree if and only if X is finite dimensional.

Proposition 5.4.7. *A linear functional $\varphi : X \to \mathbb{F}$ is weakly continuous (continuous with respect to the weak topology) if and only if $\varphi \in X^*$ (continuous with respect to the norm topology).*

Proof. Suppose $\varphi \in X^*$. Then $\varphi^{-1}(B_\varepsilon(0_\mathbb{C})) = \{x \in X |\, |\varphi(x)| < \varepsilon\} = U_{0,\varepsilon,\varepsilon}$ is weakly open. Hence φ is continuous at 0_X and thus weakly continuous by Proposition 5.4.3.

Now suppose $\varphi : X \to \mathbb{C}$ is weakly continuous. Then for all $U \subset \mathbb{C}$ open, $\varphi^{-1}(U)$ is weakly open and thus norm open by Proposition 5.4.5. Thus φ is $\|\cdot\|$-continuous and thus in X^*. $\square$

Definition 5.4.8. The weak* topology on X^* is the locally convex Hausdroff vector space topology induced by the separating family of seminorms

$$\mathcal{P} = \{\varphi \mapsto |\operatorname{ev}_x(\varphi)| = |\varphi(x)| \,| x \in X\}.$$

Observe that $\varphi_i \to \varphi$ if and only if $\varphi_i(x) \to \varphi(x)$ for all $x \in X$.

Theorem 5.4.9 (Banach-Alaoglu)**.** *The norm-closed unit ball B^* of X^* is weak*-compact.*

Proof.

> **Trick.** For $x \in X$, let $D_x = \{z \in \mathbb{C} |\, |z| \leq \|x\|\,\}$. By Tychonoff's Theorem, $D := \prod_{x \in X} D_x$ is compact Hausdorff. The elements $(d_x) \in D$ are precisely functions $f : X \to \mathbb{C}$ (not necessarily linear) such that $|f(x)| \leq \|x\|$ for all $x \in X$.

Observe $B^* \subset D$ is the subset of linear functions. The relative product topology on B^* is the relative weak* topology, as both are pointwise convergence. It remains to prove $B^* \subset D$ is closed. If $(\varphi_i) \subset B^*$ is a net with $\varphi_i \to \varphi \in D$, then

$$\varphi(\alpha x + y) = \lim \varphi_i(\alpha x + y) = \lim \alpha \varphi_i(x) + \varphi_i(y) = \alpha\varphi(x) + \varphi(y). \qquad \square$$

Exercise 5.4.10. Let X be a normed space.

(1) Show that every weakly convergent sequence in X is norm bounded.
(2) Suppose in addition that X is Banach. Show that every weak* convergent sequence in X^* is norm bounded.
(3) Give a counterexample to (2) when X is not Banach.
Hint: Under $\|\cdot\|_\infty$, $c_c^* \cong \ell^1$, where c_c is the space of sequences which are eventually zero.

Exercise 5.4.11 (Goldstine's Theorem)**.** Let X be a normed vector space with closed unit ball B. Let B^{**} be the unit ball in X^{**}, and let $i : X \to X^{**}$ be the canonical inclusion. Recall that the weak* topology on X^{**} is the weak topology induced by X^*. In this exercise, we will prove that $i(B)$ is weak* dense in B^{**}.
Note: You may use a Hahn-Banach separation theorem that we did not discuss in class to prove the result directly if you do not choose to proceed along the following steps.

(1) Show that for every $x^{**} \in B^{**}$, $\varphi_1, \dots, \varphi_n \in X^*$, and $\delta > 0$, there is an $x \in (1+\delta)B$ such that $\varphi_i(x) = x^{**}(\varphi_i)$ for all $1 \le i \le n$.
Hint: Here is a walkthrough for this first part. Fix $\varphi_1, \dots, \varphi_n \in X^*$.
  (a) Find $x \in X$ such that $\varphi_i(x) = x^{**}(\varphi_i)$ for all $1 \le i \le n$.
  (b) Set $Y := \bigcap \ker(\varphi_i)$ and let $\delta > 0$. Show by contradiction that $(x+Y)\cap(1+\delta)B \neq \emptyset$. (This part uses the Hahn-Banach Theorem.)
(2) Suppose U is a basic open neighborhood of $x^{**} \in B^{**}$. Deduce that for every $\delta > 0$, $(1+\delta)i(B) \cap U \neq \emptyset$. That is, there is an $x_\delta \in (1+\delta)B$ such that $i(x_\delta) \in U$.
(3) By part (2), $(1+\delta)^{-1}x_\delta \in B$. Show that for δ sufficiently small (which can be expressed in terms of the basic open neighborhood U), $(1+\delta)^{-1}i(x_\delta) \in i(B) \cap U$.

Exercise 5.4.12. Suppose X is a Banach space. Prove that X is reflexive if and only if the unit ball of X is weakly compact.
Hint: Use the Banach-Alaoglu Theorem 5.4.9 and Exercise 5.4.11.

Exercise 5.4.13. Suppose X, Y are Banach spaces and $T : X \to Y$ is a linear transformation.

(1) Show that if $T \in \mathcal{L}(X,Y)$, then T is weak-weak continuous. That is, if $x_\lambda \to x$ in the weak topology on X induced by X^*, then $Tx_\lambda \to Tx$ in the weak topology on Y induced by Y^*.
(2) Show that if T is norm-weak continuous, then $T \in \mathcal{L}(X,Y)$.
(3) Show that if T is weak-norm continuous, then T has finite rank, i.e., TX is finite dimensional.

Hint: For part (3), one could proceed as follows.

(1) First, reduce to the case that T is injective by replacing X with $Z = X/\ker(T)$ and T with $S : Z \to Y$ given by $x + \ker(T) \mapsto Tx$. (You must show S is weak-norm continuous on Z.)
(2) Take a basic open set $\mathcal{U} = \{z \in Z \mid |\varphi_i(z)| < \varepsilon$ for all $i = 1, \dots, n\} \subset S^{-1}B_1(0_Y)$. Use that S is injective to prove that $\bigcap_{i=1}^n \ker(\varphi_i) = (0)$.
(3) Use Exercise 5.2.2 to deduce that Z^* is finite dimensional, and thus that Z and $TX = SZ$ are finite dimensional.

Exercise 5.4.14. Suppose X is a Banach space. Prove the following are equivalent:

(1) X is separable.
(2) The relative weak* topology on the closed unit ball of X^* is metrizable.

Deduce that if X is separable, the closed unit ball of X^* is weak* sequentially compact. *Hint: For* (1) $\Rightarrow$ (2)*, you could adapt either the proof of (LCnvx7) or the trick in the proof of the Banach-Alaoglu Theorem 5.4.9 using a countable dense subset. For* (2) $\Rightarrow$ (1)*, there a countable neighborhood base* $(U_n) \subset B^*$ *at* 0_X *such that* $\bigcap U_n = \{0\}$*. For each* $n \in \mathbb{N}$*, there is a finite set* $D_n \subset X$ *and an* $\varepsilon_n > 0$ *such that*

$$U_n \supseteq \{\varphi \in X^* | \, |\varphi(x)| < \varepsilon_n \text{ for all } x \in D_n\}.$$

Setting $D = \bigcup D_n$*, show that* $\operatorname{span}(D)$ *is dense in* X*. Deduce that* X *is separable.*

Exercise 5.4.15. Suppose X is a Banach space. Prove the following are equivalent:

(1) X^* is separable.
(2) The relative weak topology on the closed unit ball of X is metrizable.

Exercise 5.4.16. How do you reconcile Exercises 5.4.12, 5.4.14, and 5.4.15? That is, how do you reconcile the fact that there exist separable Banach spaces which are not reflexive?

Exercise 5.4.17.

(1) Prove that the norm closed unit ball of ℓ^∞ is weak* sequentially compact.
(2) Prove that the norm closed unit ball of ℓ^∞ is not weakly sequentially compact.
Hint: One could proceed as follows.
   (a) Prove that the weak topology on* $\ell^\infty \cong (\ell^1)^*$ *is contained in the weak topology, i.e., if* $x_i \to x$ *weakly, then* $x_i \to x$ *weak*.*
   (b) Consider the sequence $(x_n) \subset c \subset \ell^\infty$ *given by*

$$(x_n)(m) = \begin{cases} 0 & \text{if } n < m \\ 1 & \text{if } n \geq m. \end{cases}$$

   Show that $x_n \to 0$ *weak* in* ℓ^∞*.*
   (c) Show that (x_n) *does not converge weakly in* ℓ^∞ *by extending* $\lim : c \to \mathbb{C}$ *to* ℓ^∞*.*
   (d) Deduce no subsequence of (x_n) *converges weakly in* ℓ^∞*.*

Remark 5.4.18. The Eberlein-Šmulian Theorem (which we will not prove here) states that if X is a Banach space and $S \subset X$, the following are equivalent.

(1) S is weakly *precompact*, i.e., the weak closure of S is weakly compact.
(2) Every sequence of S has a weakly convergent subsequence (whose weak limit need not be in S).
(3) Every sequence of S has a weak cluster point.

Exercise 5.4.19. Let X be a compact Hausdorff topological space. For $x \in X$, define $\operatorname{ev}_x : C(X) \to \mathbb{F}$ by $\operatorname{ev}_x(f) = f(x)$.

(1) Prove that $\operatorname{ev}_x \in C(X)^*$, and find $\|\operatorname{ev}_x\|$.
(2) Show that the map $\operatorname{ev} : X \to C(X)^*$ given by $x \mapsto \operatorname{ev}_x$ is a homeomorphism onto its image, where the image has the relative weak* topology.

5.5. **Hilbert spaces.**

Definition 5.5.1. A *sesquilinear form* on an $\mathbb{F}$-vector space H is a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{F}$ which is

- linear in the first variable: $\langle \alpha x + y, z\rangle = \alpha\langle x, z\rangle + \langle y, z\rangle$ for all $\alpha \in \mathbb{F}$ and $x, y, z \in H$, and
- conjugate linear in the second variable: $\langle x, \alpha y + z\rangle = \overline{\alpha}\langle x, y\rangle + \langle x, z\rangle$ for all $\alpha \in \mathbb{F}$ and $x, y, z \in H$.

We call $\langle \cdot, \cdot \rangle$:

- *self-adjoint* if $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in H$,
- *non-degenerate* if $\langle x, y\rangle = 0$ for all $y \in H$ implies $x = 0$
- *positive* if $\langle x, x\rangle \geq 0$ for all $x \in H$. A positive sesquilinear form is called *definite* if moreover $\langle x, x\rangle = 0$ implies $x = 0$.

A self-adjoint positive definite sesquilinear form is called an *inner product*.

Exercise 5.5.2. Suppose $\langle \cdot, \cdot \rangle$ is a self-adjoint sesquilinear form on the $\mathbb{R}$-vector space H. Show that:

- ($\mathbb{R}$-polarization) $4\langle x, y\rangle = \langle x + y, x + y\rangle - \langle x - y, x - y\rangle$ for all $x, y \in H$.

Now suppose $\langle \cdot, \cdot \rangle$ is a sesquilinear form on the $\mathbb{C}$-vector space H. Prove the following.

(1) ($\mathbb{C}$-polarization) $4\langle x, y\rangle = \sum_{k=0}^{3} i^k\langle x + i^k y, x + i^k y\rangle$ for all $x, y \in H$.
(2) $\langle \cdot, \cdot \rangle$ is self-adjoint if and only if $\langle x, x\rangle \in \mathbb{R}$ for all $x \in H$.
(3) Positive implies self-adjoint.

Definition 5.5.3. Suppose that $\langle \cdot, \cdot \rangle$ is positive and self-adjoint (so $(H, \langle \cdot, \cdot \rangle)$ is a *pre-Hilbert space*). Define

$$\|x\| := \langle x, x\rangle^{1/2}.$$

Observe that $\|\cdot\|$ is *homogeneous*: $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $\alpha \in \mathbb{F}$ and $x \in H$.

We say that x and y are *orthogonal*, denoted $x \perp y$, if $\langle x, y\rangle = 0$.

Facts 5.5.4. We have the following facts about pre-Hilbert spaces:

(H1) (Pythagorean Theorem) $x \perp y$ implies $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Proof. $\|x + y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2 = \|x\|^2 + \|y\|^2$. □

(H2) $x \perp y$ if and only if $\|x\|^2 \leq \|x + \alpha y\|^2$ for all $\alpha \in \mathbb{F}$.

Proof.
$\Rightarrow$: $\|x + \alpha y\|^2 \underset{(H1)}{=} \|x\|^2 + |\alpha|^2\|y\|^2 \geq \|x\|^2$ for all $\alpha \in \mathbb{F}$.

$\Leftarrow$: Suppose

$$\|x\|^2 + 2\,\mathrm{Re}(\alpha\langle x, y\rangle) + |\alpha|^2\|y\|^2 = \|x + \alpha y\|^2 \geq \|x\|^2 \qquad \forall\, \alpha \in \mathbb{F}.$$

Then for all $\alpha \in \mathbb{F}$,

$$0 \leq 2\,\mathrm{Re}(\alpha\langle x, y\rangle) + |\alpha|^2\|y\|^2.$$

Taking $\alpha \in \mathbb{F}$ sufficiently close to $0_{\mathbb{F}}$, the term $2\,\mathrm{Re}(\alpha\langle x, y\rangle)$ dominates, and this can only be non-negative for all $\alpha \in \mathbb{F}$ if $\langle x, y\rangle = 0$. □

(H3) The properties of being definite and non-degenerate are equivalent.

Proof.
$\Rightarrow$: Trivial; just take $y = x$ in the definition of non-degeneracy.
$\Leftarrow$: If $\|x\|^2 = 0$, then for all $\alpha \in \mathbb{F}$ and $y \in H$, $\|x\|^2 = 0 \leq \|x + \alpha y\|^2$ by positivity. Hence $x \perp y$ for all $y \in H$ by (H2). Thus $x = 0$ by non-degeneracy. □

(H4) (Cauchy-Schwarz Inequality) For all $x, y \in H$, $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.

Proof. For all $r \in \mathbb{R}$,

$$0 \leq \|x - ry\|^2 = \|x\|^2 - 2r \operatorname{Re}\langle x, y\rangle + r^2\|y\|^2,$$

which is a non-negative quadratic in r. Therefore its discriminant

$$4(\operatorname{Re}\langle x, y\rangle)^2 - 4 \cdot \|x\|^2 \cdot \|y\|^2 \leq 0,$$

which implies $|\operatorname{Re}\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.

Trick. $|\langle x, y\rangle| = \alpha\langle x, y\rangle$ for some $\alpha \in U(1) = \{z \in \mathbb{C} | \, |z| = 1\}$.

Then

$$|\langle x, y\rangle| = \alpha\langle x, y\rangle = \langle \alpha x, y\rangle \leq \|\alpha x\| \cdot \|y\| = \|x\| \cdot \|y\|.$$ □

(H5) (Cauchy-Schwarz Definiteness) If $\langle \cdot\,, \cdot\rangle$ is definite, then $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ implies $\{x, y\}$ is linearly dependent.

Proof. We may assume $y \neq 0$. Set

$$\alpha := \frac{|\langle x, y\rangle|}{\|y\|^2}\overline{\operatorname{sgn}(\langle x, y\rangle)}.$$

Then we calculate

$$\begin{aligned}
\|x - \alpha y\|^2 &= \|x\|^2 - 2\operatorname{Re}(\alpha\langle x, y\rangle) + |\alpha|^2 \cdot \|y\|^2 \\
&= \|x\|^2 - 2\frac{|\langle x, y\rangle|^2}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^4}\|y\|^2 \\
&= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} \\
&= \|x\|^2 - \frac{\|x\|^2 \cdot \|y\|^2}{\|y\|^2} \\
&= 0.
\end{aligned}$$

This implies $x = \alpha y$ by definiteness.
(The essential idea here was to minimize a quadratic in α.) □

(H6) $\|\cdot\| : H \to [0, \infty)$ is a seminorm. It is a norm exactly when $\langle \cdot\,, \cdot\rangle$ is definite, i.e., an inner product.

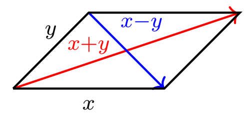
Proof. It remains to prove subadditivity of $\|\cdot\|$, which follows by the Cauchy-Schwarz Inequality (H4):

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \|x\|^2 + 2\operatorname{Re}\langle x,y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 && \text{(H4)} \\ &= (\|x\|+\|y\|)^2.\end{aligned}$$

Now take square roots. The final claim follows immediately. □

Proposition 5.5.5. *A norm $\|\cdot\|$ on a $\mathbb{C}$-vector space comes from an inner product if and only if it satisfies the parallelogram identity:*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$



Proof.
⇒: If $\|\cdot\|$ comes from an inner product, then add together

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\operatorname{Re}\langle x,y\rangle + \|y\|^2.$$

⇐: If the parallelogram identity holds, just define

$$\langle x,y\rangle := \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2$$

by polarization. One checks this works. □

Definition 5.5.6. A *Hilbert space* is an inner product space whose induced norm is complete, i.e., Banach.

Exercise 5.5.7. Verify the follows spaces are Hilbert spaces.

(1) $\ell^2 := \{(x_n) \in \mathbb{C}^\infty | \sum |x_n|^2 < \infty\}$ with $\langle x,y\rangle := \sum x_n \overline{y_n}$.
(2) Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Define

$$\mathcal{L}^2(X,\mu) := \frac{\left\{\text{measurable } f : X \to \mathbb{C} \middle| \int |f|^2\, d\mu < \infty\right\}}{\text{equality a.e.}}$$

with $\langle f, g\rangle := \int f\overline{g}\, d\mu$.

Exercise 5.5.8. Suppose H is a Hilbert space and $S, T : H \to H$ are linear operators such that for all $x, y \in H$, $\langle Sx, y\rangle = \langle x, Ty\rangle$. Prove that S and T are bounded.

From this point forward, H will denote a Hilbert space.

Theorem 5.5.9. *Suppose $C \subset H$ is a non-empty convex closed subset and $z \notin C$. There is a unique $x \in C$ such that*

$$\|x - z\| = \inf_{y \in C} \|y - z\|.$$

Proof. By translation, we may assume $z = 0 \notin C$. Suppose $(x_n) \subset C$ such that $\|x_n\| \to r := \inf_{y \in C} \|y\|$. Then by the parallelogram identity,

$$\left\|\frac{x_m - x_n}{2}\right\|^2 + \left\|\frac{x_m + x_n}{2}\right\|^2 = 2\left(\left\|\frac{x_m}{2}\right\|^2 + \left\|\frac{x_n}{2}\right\|^2\right)$$

Rearranging, we have

$$\|x_m - x_n\|^2 = 2\underbrace{\|x_m\|^2}_{\to r^2} + 2\underbrace{\|x_n\|^2}_{\to r^2} - 4\underbrace{\left\|\frac{x_m + x_n}{2}\right\|^2}_{\geq r^2}$$

where the last inequality follows since $(x_m + x_n)/2 \in C$ by convexity. This means that

$$\limsup_{m,n} \|x_m - x_n\|^2 \leq 2r^2 + 2r^2 - 4r^2 = 0,$$

and thus (x_n) is Cauchy. Since H is complete, there is an $x \in H$ such that $x_n \to x$, and $\|x\| = r$. Since C is closed, $x \in C$.

For uniqueness, observe that if $x' \in C$ satisfies $\|x'\| = r$, then $(x, x', x, x', \dots)$ is Cauchy by the above argument, and thus converges. We conclude that $x = x'$. ☐

Definition 5.5.10. For $S \subset H$, define the *orthogonal complement*

$$S^\perp := \{x \in H | \langle x, s\rangle = 0 \ , \forall\, s \in S\}.$$

Observe that $S^\perp$ is a closed subspace.

Facts 5.5.11. We have the following facts about orthogonal complements.

($\perp$1) If $S \subset T$, then $T^\perp \subset S^\perp$.

> *Proof.* Observe $x \in T^\perp$ if and only if $\langle x, t\rangle = 0$ for all $t \in T \supseteq S$. Hence $x \in S^\perp$. ☐

($\perp$2) $\overline{S} \subset S^{\perp\perp}$ and $S^\perp = S^{\perp\perp\perp}$.

> *Proof.* If $s \in S$, then $\langle s, x\rangle = \overline{\langle x, s\rangle} = 0$ for all $x \in S^\perp$. Thus $s \in S^{\perp\perp}$. Since $S^{\perp\perp}$ is closed, $\overline{S} \subset S^{\perp\perp}$.
> Now replacing S with $S^\perp$, we get $S^\perp \subset S^{\perp\perp\perp}$. But since $S \subseteq S^{\perp\perp}$, by ($\perp$1), we have $S^{\perp\perp\perp} \subseteq S^\perp$. ☐

($\perp$3) $S \cap S^\perp = \{0\}$.

> *Proof.* If $x \in S \cap S^\perp$, then $\langle x, x\rangle = 0$, so $x = 0$. ☐

($\perp$4) If $K \subset H$ is a subspace, then $H = \overline{K} \oplus K^\perp$.

Proof. By (⊥2) and (⊥3),

$$\{0\} \subseteq \overline{K} \cap K^{\perp} \subseteq K^{\perp\perp} \cap K^{\perp} = \{0\},$$

so equality holds everywhere.

Let $x \in H$. Since $\overline{K}$ is closed and convex, there is a unique $y \in \overline{K}$ minimizing the distance to x, i.e., $\|x - y\| \leq \inf_{k \in K} \|x - k\|$. We claim that $x - y \in K^{\perp}$, so that $x = y + (x - y)$, and $H = \overline{K} + K^{\perp}$. Indeed, for all $k \in K$ and $\alpha \in \mathbb{C}$,

$$\|x - y\|^2 \leq \|x - (y - \alpha k)\|^2 = \|(x - y) + \alpha k\|^2 \qquad \forall \alpha \in \mathbb{C}.$$

By (H2), we have $(x - y) \perp k$ for all $k \in K$, i.e., $x - y \in K^{\perp}$ as claimed. □

(⊥5) If $K \subset H$ is a subspace, then $\overline{K} = K^{\perp\perp}$.

Proof. Let $x \in K^{\perp\perp}$. By (⊥4), there are unique $y \in \overline{K}$ and $z \in K^{\perp}$ such that $x = y + z$. Then

$$0 = \langle x, z\rangle = \langle y + z, z\rangle = \underbrace{\langle y, z\rangle}_{=0 \text{ by } (\perp 2)} + \langle z, z\rangle.$$

Hence $z = 0$, and $x = y \in \overline{K}$. □

Notation 5.5.12 (Dirac bra-ket)**.** Let $(H, \langle \cdot , \cdot \rangle)$ be a Hilbert space, where $\langle \cdot , \cdot \rangle$ is linear on the left and conjugate linear on the right. Define $\langle \cdot | \cdot \rangle : H \times H \to \mathbb{F}$ by

$$\langle x|y\rangle := \langle y, x\rangle.$$

That is, $\langle \cdot | \cdot \rangle$ is the 'same' inner product, but linear on the right and conjugate linear on the left.

We may further denote a vector $x \in H$ by the *ket* $|x\rangle$. For $x \in H$, we denote the linear map $H \to \mathbb{F}$ by $y \mapsto \langle x|y\rangle$ by the *bra* $\langle x|$. Observe that the bra $\langle x|$ applied to the ket $|y\rangle$ gives the bracket $\langle x|y\rangle$.

Theorem 5.5.13 (Riesz Representation)**.** *Let H be a Hilbert space.*

(1) For all $y \in H$, $\langle y| \in H^*$ and $\|\langle y|\| = \|y\|$.
(2) For $\varphi \in H^*$, there is a unique $y \in H$ such that $\varphi = \langle y|$.
(3) The map $y \mapsto \langle y|$ is a conjugate-linear isometric isomorphism.

Proof.

(1) Clearly $\langle y|$ is linear. By Cauchy-Schwarz, $|\langle y|x\rangle| \leq \|x\| \cdot \|y\|$, so $\|\langle y|\| \leq \|y\|$. Taking $x = y$, we have $|\langle y|y\rangle| = \|y\|^2$, so $\|\langle y|\| = \|y\|$.

(2) If $\langle y| = \langle y'|$, then

$$\|y - y'\|^2 = \langle y - y'|y - y'\rangle = \langle y|y - y'\rangle - \langle y'|y - y'\rangle = 0,$$

and thus $y = y'$. Suppose now $\varphi \in H^*$. We may assume $\varphi \neq 0$. Then $\ker(\varphi) \subset H$ is a closed proper subspace. Pick $z \in \ker(\varphi)^{\perp}$ with $\varphi(z) = 1$. Now for all $x \in H$, $x - \varphi(x)z \in \ker(\varphi)$, so

$$\langle z|x\rangle = \langle z|x - \varphi(x)z + \varphi(x)z\rangle = \langle \underbrace{z}_{\in \ker(\varphi)^{\perp}} | \underbrace{x - \varphi(x)z}_{\in \ker(\varphi)}\rangle + \langle z|\varphi(x)z\rangle = \langle z|\varphi(x)z\rangle = \varphi(x)\|z\|^2.$$

We conclude that $\varphi = \left\langle \frac{z}{\|z\|^2} \right|$.

(3) $y \mapsto \langle y|$ is isometric by (1) and onto by (2). Conjugate linearity is straightforward. □

Exercise 5.5.14. Suppose H is a Hilbert space. Show that the dual space H^* with

$$\langle \langle x|, \langle y| \rangle_{H^*} := \langle y, x \rangle_H$$

is a Hilbert space whose induced norm is equal to the operator norm on H^*.

Definition 5.5.15. A subset $E \subset H$ is called *orthonormal* if $e, f \in E$ implies $\langle e, f \rangle = \delta_{e=f}$. Observe that $\|e - f\| = \sqrt{2}$ for all $e \neq f$ in E. Thus if H is separable, any orthonormal set is countable.

Exercise 5.5.16. Suppose H is a Hilbert space, $E \subset H$ is an orthonormal set, and $\{e_1, \ldots, e_n\} \subset E$. Prove the following assertions.

(1) If $x = \sum_{i=1}^n c_i e_i$, then $c_j = \langle x, e_j \rangle$ for all $j = 1, \ldots, n$.
(2) The set E is linearly independent.
(3) For every $x \in H$, $\sum_{i=1}^n \langle x, e_i \rangle e_i$ is the unique element of $\operatorname{span}\{e_1, \ldots, e_n\}$ minimizing the distance to x.
(4) (Bessel's Inequality) For every $x \in H$, $\|x\|^2 \geq \sum_{i=1}^n |\langle x, e_i \rangle|^2$.

Theorem 5.5.17. *For an orthonormal set $E \subset H$, the following are equivalent:*

(1) E is maximal,
(2) $\operatorname{span}(E)$*, the set of finite linear combinations of elements of E, is dense in H.*
(3) $\langle x, e \rangle = 0$ for all $e \in E$ implies $x = 0$.
(4) For all $x \in H$, $x = \sum_{e \in E} \langle x, e \rangle e$, where the sum on the right:
- *has at most countably many non-zero terms, and*
- *converges in the norm topology regardless of ordering.*

(5) For all $x \in H$, $\|x\|^2 = \sum_{e \in E} |\langle x, e \rangle|^2$.

If E satisfies the above properties, we call E an orthonormal basis *for H.*

Proof.

(1) ⇒ (2): If $\operatorname{span}(E)$ is not dense, there is an $e \in \operatorname{span}(E)^\perp$ with $\|e\| = 1$. Then $E \subsetneq E \cup \{e\}$, which is orthonormal.

(2) ⇒ (3): Suppose $\langle e, x \rangle = 0$ for all $e \in E$. Then $\langle x| = 0$ on $\operatorname{span}(E)$. Since $\operatorname{span}(E)$ is dense in H and $\langle x|$ is continuous, $\langle x| = 0$ on H, and thus $x = 0$ by the Riesz Representation Theorem 5.5.13.

(3) ⇒ (1): (3) is equivalent to $E^\perp = 0$. This means there is no strictly larger orthonormal set containing E.

(3) ⇒ (4): For all $e_1, \ldots, e_n \in E$, by Bessel's Inequality, $\|x\|^2 \geq \sum^n |\langle x, e_i \rangle|^2$. So for all countable subsets $F \subset E$, $\|x\|^2 \geq \sum_{f \in F} |\langle x, f \rangle|^2$. Hence $\{e \in E | \langle x, e \rangle \neq 0\}$ is countable. Let (e_i) be an enumeration of this set. Then

$$\left\| \sum_m^n \langle x, e_i \rangle e_i \right\|^2 = \sum_m^n |\langle x, e_i \rangle|^2 \xrightarrow{m,n \to \infty} 0.$$

So $\sum \langle x, e_i \rangle e_i$ converges as H is complete. Obsere that for all $e \in E$,

$$\left\langle x - \sum \langle x, e_i \rangle e_i, e \right\rangle = 0,$$

so $x = \sum \langle x, e_i \rangle e_i$ by (3).
(4) ⇒ (5): Let $x \in H$ and let $\{e_i\}$ be an enumeration of $\{e \in E | \langle x, e \rangle \neq 0\}$. Then

$$\|x\|^2 - \sum^n |\langle x, e_i \rangle|^2 = \left\| x - \sum^n \langle x, e_i \rangle e_i \right\|^2 \xrightarrow{n\to\infty} 0.$$

(Indeed, expand the term on the right into 4 terms to see you get the term on the left.)
(5) ⇒ (3): Immediate as $\|\cdot\|$ is definite. □

Exercise 5.5.18. Suppose H is a Hilbert space. Prove the following assertions.

(1) Every orthonormal set E can be extended to an orthonormal basis.
(2) H is separable if and only if it has a countable orthonormal basis.
(3) Two Hilbert spaces are isomorphic (there is an invertible $U \in \mathcal{L}(H \to K)$ such that $\langle Ux, Uy \rangle_K = \langle x, y \rangle$ for all $x, y \in H$) if and only if H and K have orthonormal bases which are the same size.
(4) If E is an orthonormal basis, the map $H \to \ell^2(E)$ given by $x \mapsto (\langle x, \cdot \rangle : E \to \mathbb{C})$ is a unitary isomorphism of Hilbert spaces. Here, $\ell^2(E)$ denotes square integrable functions $E \to \mathbb{C}$ with respect to counting measure.

Exercise 5.5.19. Consider the space $L^2(\mathbb{T}) := L^2(\mathbb{R}/\mathbb{Z})$ of $\mathbb{Z}$-periodic functions $\mathbb{R} \to \mathbb{C}$ such that $\int_{[0,1]} |f|^2 < \infty$. Define

$$\langle f, g \rangle := \int_{[0,1]} f\overline{g}.$$

(1) Prove that $L^2(\mathbb{T})$ is a Hilbert space.
(2) Show that the subspace $C(\mathbb{T}) \subset L^2(\mathbb{T})$ of continuous $\mathbb{Z}$-periodic functions is dense.
(3) Prove that $\{e_n(x) := \exp(2\pi i n x) | n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$.
Hint: Orthonormality is easy. Use (2) and the Stone-Weierstrass Theorem to show the linear span is dense.
(4) Define $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ by $\mathcal{F}(f)_n := \langle f, e_n \rangle_{L^2(\mathbb{T})} = \int_0^1 f(x) \exp(-2\pi i n x)\, dx$. Show that if $f \in L^2(\mathbb{T})$ and $\mathcal{F}(f) \in \ell^1(\mathbb{Z})$, then $f \in C(\mathbb{T})$, i.e., f is a.e. equal to a continuous function.

5.6. **The dual of $C_0(X)$.** Let X be an LCH space. In this section, we prove the Reisz Representation Theorem which characterizes the dual of $C_0(X)$ in terms of Radon measures on X.

Definition 5.6.1. A *Radon measure* on X is a Borel measure which is

- finite on compact subsets of X,
- outer regular on all Borel subsets of X, and
- inner regular on all open subsets of X.

Facts 5.6.2. Recall the following facts about Radon measures on an LCH space X.

(R1) If μ is a Radon measure on X and $E \subset X$ is σ-finite, then μ is σ-finite on E by Exercise 2.5.24(1). Hence every σ-finite Radon measure is regular.
(R2) If X is σ-compact, every Radon measure is σ-finite and thus regular.
(R3) Finite Radon measures on X are exactly finite regular Borel measures on X.

Exercise 5.6.3. Suppose X is LCH and μ is a Radon measure on X. Prove $C_c(X)$ is dense in $\mathcal{L}^1(\mu)$.

Notation 5.6.4. Recall that the *support* of $f : X \to \mathbb{C}$ is $\operatorname{supp}(f) := \{f \neq 0\}$. We say f has *compact support* if $\operatorname{supp}(f) := \overline{\{f \neq 0\}}$ is compact, and we denote the (possibly non-unital) algebra of all continuous functions of compact support by $C_c(X)$. For an open set $U \subset X$, we write $f \prec U$ to denote $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subset U$. Observe that if $f \prec U$, then $f \leq \chi_U$, but the converse need not be true.

Definition 5.6.5. A *Radon integral* on X is a *positive* linear functional $\varphi : C_c(X) \to \mathbb{C}$, i.e., $\varphi(f) \geq 0$ for all $f \in C_c(X)$ such that $f \geq 0$.

Lemma 5.6.6. *Radon integrals are bounded on compact subsets. That is, if $K \subset X$ is compact, there is a $c_K > 0$ such that for all $f \in C_c(X)$ with $\operatorname{supp}(f) \subset K$, $|\varphi(f)| \leq c_K \cdot \|f\|_\infty$.*

Proof. Let $K \subset X$ be compact. Choose $g \in C_c(X)$ such that $g = 1$ on K by the LCH Urysohn Lemma (Exercise 1.2.11(2)).

Step 1: If $f \in C_c(X, \mathbb{R})$ with $\operatorname{supp}(f) \subset K$, then $|f| \leq \|f\|_\infty \cdot g$ on X. So $\|f\|_\infty \cdot g - |f| \geq 0$, and $\|f\|_\infty \cdot g \pm f \geq 0$. Thus $\|f\|_\infty \cdot \varphi(g) \pm \varphi(f) \geq 0$. Hence

$$|\varphi(f)| \leq \varphi(g) \cdot \|f\|_\infty \qquad \forall\, f \in C_c(X, \mathbb{R}) \text{ with } \operatorname{supp}(f) \subset K.$$

Taking $c_K := \varphi(g)$ works for all $f \in C_c(X, \mathbb{R})$.

Step 2: Taking real and imaginary parts, we see $c_K := 2\varphi(g)$ works for all $f \in C_c(X)$. Indeed,

$$|\varphi(f)| \leq |\varphi(\operatorname{Re}(f))| + |\varphi(\operatorname{Im}(f))| \leq \varphi(g)\|\operatorname{Re}(f)\|_\infty + \varphi(g)\|\operatorname{Im}(f)\|_\infty \leq 2\varphi(g)\|f\|_\infty$$

for all $f \in C_c(X)$ with $\operatorname{supp}(f) \subset K$. □

Theorem 5.6.7 (Riesz Representation)**.** *If φ is a Radon integral on X, there is a unique Radon measure μ_φ on X such that*

$$\varphi(f) = \int f \, d\mu_\varphi \qquad \forall\, f \in C_c(X).$$

Moreover, μ_φ satisfies:

(μ_φ1) For all open $U \subset X$, $\mu_\varphi(U) = \sup\{\varphi(f) | f \in C_c(X) \text{ with } f \prec U\}$, and
(μ_φ2) For all compact $K \subset X$, $\mu_\varphi(K) = \inf\{\varphi(f) | f \in C_c(X) \text{ with } \chi_K \leq f\}$.

Proof.

Uniqueness: Suppose μ is a Radon measure such that $\varphi(f) = \int f \, d\mu$ for all $f \in C_c(X)$. If $U \subset X$ is open, then $\varphi(f) \leq \mu(U)$ for all $f \in C_c(X)$ with $f \prec U$. If $K \subset U$ is compact, then by the LCH Urysohn Lemma (Exercise 1.2.11(2)), there is an $f \in C_c(X)$ such that $f \prec U$ and $f|_K = 1$, and

$$\mu(K) \leq \int f \, d\mu = \varphi(f) \leq \mu(U).$$

But μ is inner regular on U as it is Radon, and thus

$$\mu(U) = \sup\{\mu(K) | U \supset K \text{ is compact}\} \leq \sup\{\varphi(f) | f \in C_c(X) \text{ with } f \prec U\} \leq \mu(U).$$

Hence μ satisfies (μ_φ1), so μ is determined on open sets. But since μ is outer regular, μ is determined on all Borel sets.

Existence: For $U \subset X$ open, define $\mu(U) := \sup\{\varphi(f) | f \in C_c(X) \text{ with } f \prec U\}$ and

$$\mu^*(E) := \inf\{\mu(U) | U \text{ is open and } E \subset U\} \qquad E \subset X.$$

Step 1: μ is monotone on inclusions of open sets, i.e., $U \subset V$ both open implies $\mu(U) \leq \mu(V)$. Hence $\mu^*(U) = \mu(U)$ for all open U.

Proof. Just observe that if $U \subseteq V$ are open, then $f \in C_c(X)$ with $f \prec U$ implies $f \prec V$. Hence $\mu(U) \leq \mu(V)$ are we are taking sup over a super set. □

Step 2: μ^* is an outer measure on X.

Proof. It suffices to prove that if (U_n) is a sequence of open sets, then $\mu\left(\bigcup U_n\right) \leq \sum \mu(U_n)$. This shows that

$$\mu^*(E) = \inf\left\{\sum \mu(U_n) \middle| \text{the } U_n \text{ are open and } E \subset \bigcup U_n\right\},$$

which we know is an outer measure by Proposition 2.3.3. Suppose $f \in C_c(X)$ with $f \prec \bigcup U_n$. Since $\operatorname{supp}(f)$ is compact, $\operatorname{supp}(f) \subset \bigcup_{n=1}^N U_n$ for some $N \in \mathbb{N}$.

Trick. By Exercise 1.2.17, there are $g_1, \ldots, g_N \in C_c(X)$ such that $g_n \prec U_n$ and $\sum_{n=1}^N g_n = 1$ on $\operatorname{supp}(f)$.

Then $f = f \sum_{n=1}^N g_n$ and $fg_n \prec U_n$ for each n, so

$$\varphi(f) = \sum_{n=1}^N \varphi(fg_n) \leq \sum_{n=1}^N \varphi(\chi_{U_n}) = \sum_{n=1}^N \mu(U_n) \leq \sum \mu(U_n).$$

Since $f \prec U$ was arbitrary,

$$\mu\left(\bigcup U_n\right) = \sup\left\{\varphi(f) \middle| f \in C_c(X) \text{ with } f \prec \bigcup U_n\right\} \leq \sum \mu(U_n).$$ □

Step 3: Every open set is μ^*-measurable, and thus $\mathcal{B}_X \subset \mathcal{M}^*$, the μ^*-measurable sets. Hence $\mu_\varphi := \mu^*|_{\mathcal{B}_X}$ is a Borel measure which is by definition outer regular and satisfies (μ_φ1).

Proof. Suppose $U \subset X$ is open. We must prove that for every $E \subset X$ such that $\mu^*(E) < \infty$, $\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$.

Case 1: If E is open, then $E \cap U$ is open. Given $\varepsilon > 0$, there is a $f \in C_c(X)$ with $f \prec E \cap U$ such that $\varphi(f) > \mu(E \cap U) - \varepsilon/2$. Since $E \setminus \operatorname{supp}(f)$ is open, there is a $g \prec E \setminus \operatorname{supp}(f)$ such that $\varphi(g) > \mu(E \setminus \operatorname{supp}(f)) - \varepsilon/2$. Then $f + g \prec E$, so

$$\begin{aligned}
\mu(E) &\geq \varphi(f) + \varphi(g) \\
&> \mu(E \cap U) + \mu(E \setminus \operatorname{supp}(f)) - \varepsilon \\
&\geq \mu^*(E \cap U) + \mu^*(E \setminus U) - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the result follows.

Case 2: For a general E, given $\varepsilon > 0$, there is an open $V \supseteq E$ such that $\mu(V) < \mu^*(E) + \varepsilon$. Thus

$$\begin{aligned}\mu^*(E) + \varepsilon &> \mu(V) \\ &\geq \mu^*(V \cap U) + \mu^*(V \setminus U) \\ &\geq \mu^*(E \cap U) + \mu^*(E \setminus U).\end{aligned}$$

Again, as $\varepsilon > 0$ was arbitrary, the result follows. □

Step 4: μ_φ satisfies ($\mu_\varphi 2$) and is thus finite on compact sets.

Proof. Suppose $K \subset X$ is compact and $f \in C_c(X)$ with $\chi_K \leq f$. Let $\varepsilon > 0$, and set $U_\varepsilon := \{1 - \varepsilon < f\}$, which is open. If $g \in C_c(X)$ with $g \prec U_\varepsilon$, then $(1-\varepsilon)^{-1} f - g \geq 0$, so $\varphi(g) \leq (1-\varepsilon)^{-1}\varphi(f)$. Hence

$$\mu_\varphi(K) \leq \mu_\varphi(U_\varepsilon) = \sup\{\varphi(g) | g \prec U_\varepsilon\} \leq (1-\varepsilon)^{-1}\varphi(f).$$

As $\varepsilon > 0$ was arbitrary, we conclude that $\mu_\varphi(K) \leq \varphi(f)$.
Now, for all open $U \supset K$, by the LCH Urysohn Lemma (Exercise 1.2.11(2)), there is an $f \prec U$ such that $\chi_K \leq f$ ($f|_K = 1$), and by definition, $\varphi(f) \leq \mu_\varphi(U)$. Since μ_φ is outer regular on K by definition,

$$\mu_\varphi(K) = \inf\{\mu_\varphi(U) | K \subset U \text{ open}\} = \inf\{\varphi(f) | f \geq \chi_K\}.$$ □

Step 5: μ_φ is inner regular on open sets and thus Radon.

Proof. If $U \subset X$ is open and $0 \leq \alpha < \mu(U)$, choose $f \in C_c(X)$ such that $f \prec U$ and $\varphi(f) > \alpha$. For all $g \in C_c(X)$ with $\chi_{\operatorname{supp}(f)} \leq g$, we have $g - f \geq 0$, so $\alpha < \varphi(f) \leq \varphi(g)$. Since ($\mu_\varphi 2$) holds, $\alpha < \mu(\operatorname{supp}(f)) \leq \mu(U)$. Hence μ is inner regular on U. □

Step 6: For all $f \in C_c(X)$, $\varphi(f) = \int f \, d\mu_\varphi$.

Proof. We may assume $f \in C_c(X, [0,1])$ as this set spans $C_c(X)$. Fix $N \in \mathbb{N}$, and set $K_j := \{f \geq j/N\}$ for $j = 1, \ldots, N+1$ and $K_0 := \operatorname{supp}(f)$ so that

$$\emptyset = K_{N+1} \subset K_N \subset \cdots \subset K_1 \subset K_0 = \operatorname{supp}(f).$$

for $j = 1, \ldots, N$, define

$$f_j := \left(\left(f - \frac{j-1}{N}\right) \vee 0\right) \wedge \frac{1}{N}$$

which is equivalent to

$$f_j(x) = \begin{cases} 0 & \text{if } x \notin K_{j-1} \\ f(x) - \frac{j-1}{N} & \text{if } x \in K_{j-1} \setminus K_j \\ N^{-1} & \text{if } x \in K_j. \end{cases}$$

Observe that this implies:

- $\frac{\chi_{K_j}}{N} \leq f_j \leq \frac{\chi_{K_{j-1}}}{N}$ for all $j = 1, \ldots, N$, and
- $\sum_{j=1}^N f_j = f$,

which gives us the inequalities

$$\frac{1}{N}\mu_\varphi(K_j) \leq \int f_j \, d\mu_\varphi \leq \frac{1}{N}\mu_\varphi(K_{j-1}). \tag{5.6.8}$$

Now for all open $U \supset K_{j-1}$, $Nf_j \prec U$, so $N\varphi(f_j) \leq \mu_\varphi(U)$. By ($\mu_\varphi 2$) and outer regularity of μ_φ, we have the inequalities

$$\frac{1}{N}\mu_\varphi(K_j) \leq \varphi(f_j) \leq \frac{1}{N}\mu_\varphi(K_{j-1}). \tag{5.6.9}$$

Now summing over $j = 1, \ldots N$ for both (5.6.8) and (5.6.9), we have the inequalities

$$\frac{1}{N}\sum_{j=1}^{N}\mu_\varphi(K_j) \leq \int f \, d\mu_\varphi \leq \frac{1}{N}\sum_{j=0}^{N-1}\mu_\varphi(K_j)$$
$$\frac{1}{N}\sum_{j=1}^{N}\mu_\varphi(K_j) \leq \varphi(f) \leq \frac{1}{N}\sum_{j=0}^{N-1}\mu_\varphi(K_j).$$

This implies that

$$\left|\varphi(f) - \int f \, d\mu_\varphi\right| \leq \frac{\mu_\varphi(K_0) - \mu_\varphi(K_N)}{N} \leq \frac{\mu_\varphi(\operatorname{supp}(f))}{N} \xrightarrow{N\to\infty} 0$$

as $\mu_\varphi(\operatorname{supp}(f)) < \infty$ and $N \in \mathbb{N}$ was arbitrary. □

This completes the proof. □

The following corollary is the upgrade of Proposition 2.5.22 promised in Remark 2.5.26.

Corollary 5.6.10. *Suppose X is LCH and every open subset of X is σ-compact (e.g., if X is second countable). Then every Borel measure on X which is finite on compact sets is Radon.*

Proof. Suppose μ is such a Borel measure. Since $C_c(X) \subset L^1(\mu)$, $\varphi(f) := \int f \, d\mu$ is a positive linear functional on $C_c(X)$. By the Riesz Representation Theorem 5.6.7, there is a unique Radon measure ν on C such that $\varphi(f) = \int f \, d\nu$ for all $C_c(X)$. It remains to prove $\mu = \nu$.

For an open $U \subset X$, write $U = \bigcup K_j$ with K_j compact for all j. We may inductively find $f_n \in C_c(X)$ such that $f_n \prec U$ and $f_n = 1$ on the compact set $\bigcup^n K_j \cup \bigcup^{n-1} \operatorname{supp}(f_j)$. Then $f_n \nearrow \chi_U$ pointwise, so by the MCT 3.3.9,

$$\mu(U) = \lim \int f_n \, d\mu = \lim \varphi(f_n) = \lim \int f_n \, d\nu = \nu(U).$$

Now suppose $E \in \mathcal{B}_X$ is arbitrary. By (R2), ν is a regular Borel measure, so by Exercise 2.5.23, given $\varepsilon > 0$, there are $F \subset E \subset U$ with F closed, U open, and $\nu(U \setminus F) < \varepsilon$. But since $U \setminus F$ is open,

$$\mu(U \setminus F) = \nu(U \setminus F) < \varepsilon,$$

and thus $\mu(U) - \varepsilon \leq \mu(E) \leq \mu(U)$. Hence μ is outer regular, and thus $\mu = \nu$. □

Lemma 5.6.11. *Suppose X is LCH and μ is a Radon measure on X. Define $\varphi(f) := \int f\, d\mu$ on $C_c(X)$. The following are equivalent:*

(1) φ extends continuously to $C_0(X)$.
(2) φ is bounded with respect to $\|\cdot\|_\infty$.
(3) $\mu(X)$ is finite.

Proof.
(1) $\Leftrightarrow$ (2): This follows as $C_c(X) \subset C_0(X)$ is dense with respect to $\|\cdot\|_\infty$ by the LCH Urysohn Lemma (Exercise 1.2.11(2)).
(2) $\Leftrightarrow$ (3): This follows as $\mu(X) = \sup\left\{\varphi(f) = \int f\, d\mu \middle| f \in C_c(X) \text{ with } 0 \leq f \leq 1\right\}$. □

Corollary 5.6.12. *A positive linear functional in $C_0(X)^*$ is of the form $\int \cdot\, d\mu$ for some finite Radon measure μ.*

Proposition 5.6.13. *If $\varphi \in C_0(X, \mathbb{R})^*$, there are positive $\varphi_\pm \in C_0(X, \mathbb{R})^*$ such that $\varphi = \varphi_+ - \varphi_-$. Hence there are finite Radon measures μ_1, μ_2 on X such that*

$$\varphi(f) = \int f\, d\mu_1 - \int f\, d\mu_2 = \int f\, d(\mu_1 - \mu_2) \qquad \forall f \in C_0(X, \mathbb{R}).$$

Proof. For $f \in C_0(X, [0\infty))$, define $\varphi_+(f) := \sup\{\varphi(g) | 0 \leq g \leq f\}$. For $f \in C_0(X, \mathbb{R})$, define $\varphi_+(f) := \varphi_+(f_+) - \varphi_+(f_-)$ as $f_\pm \in C_0(X, [0, \infty))$.
Step 1: For all $f_1, f_2 \in C_0(X, [0, \infty))$ and $c \geq 0$, $\varphi_+(cf_1 + f_2) = c\varphi_+(f_1) + \varphi_+(f_2)$.

Proof. It suffices to show additivity. Whenever $0 \leq g_1 \leq f_1$ and $0 \leq g_2 \leq f_2$, $0 \leq g_1 + g_2 \leq f_1 + f_2$. This implies $\varphi_+(f_1 + f_2) \geq \varphi_+(f_1) + \varphi_+(f_2)$.
Now if $0 \leq g \leq f_1 + f_2$, set $g_1 := g \wedge f_1$ and $g_2 := g - g_1$. Then $0 \leq g_1 \leq f_1$ and $0 \leq g_2 \leq f_2$, so

$$\varphi(g) = \varphi(g_1) + \varphi(g_2) \leq \varphi_+(f_1) + \varphi_+(f_2).$$

Taking sup over such g gives $\varphi_+(f_1 + f_2) \leq \varphi_+(f_1) + \varphi_+(f_2)$. □

Step 2: If $f \in C_0(X, \mathbb{R})$ with $f = g - h$ where $g, h \geq 0$, then $\varphi_+(f) = \varphi_+(g) - \varphi_+(h)$.

Proof. Observe that $g + f_- = h + f_+ \geq 0$, so $\varphi_+(g) + \varphi(f_-) = \varphi_+(h) + \varphi_+(f_+)$ by Step 1. Rearranging gives the result. □

Step 3: φ_+ is linear on $C_0(X, \mathbb{R})$.

Proof. Suppose $c \in \mathbb{R}$ and $f, g \in C_0(X, \mathbb{R})$. If $c \geq 0$, then $cf + g = cf_+ + g_+ - (cf_- + g_-)$ where $cf_\pm + g_\pm \geq 0$. Then

$$\begin{aligned}
\varphi_+(cf + g) &= \varphi_+(cf_+ + g_+) - \varphi_+(cf_- + g_-) && \text{(Step 2)}\\
&= c\varphi_+(f_+) + \varphi_+(g_+) - c\varphi_+(f_-) - \varphi_-(g_-) && \text{(Step 1)}\\
&= c(\varphi_+(f_+) - \varphi_+(f_-)) + (\varphi_+(g_+) - \varphi_+(g_-)) &&\\
&= c\varphi_+(f) + \varphi_+(g) && \text{(Step 2)}.
\end{aligned}$$

□

Step 4: $\varphi_+ \in C_0(X,\mathbb{R})^*$ is positive with $\|\varphi_+\| \leq \|\varphi\|$.

Proof. First suppose $f \in C_0(X,[0,\infty))$. Since

$$|\varphi(g)| \leq \|\varphi\| \cdot \|g\|_\infty \leq \|\varphi\| \cdot \|f\|_\infty \qquad \forall\, 0 \leq g \leq f,$$

we have that

$$0 = \varphi(0) \leq \varphi_+(f) \leq \|\varphi\| \cdot \|f\|_\infty \qquad \forall\, f \in C_0(X,[0,\infty)).$$

Now if $f \in C_0(X,\mathbb{R})$ is arbitrary,

$$|\varphi_+(f)| \leq \max\{\varphi_+(f_+), \varphi_+(f_-)\} \leq \|\varphi\| \cdot \max\{\|f_+\|_\infty, \|f_-\|_\infty\} \leq \|\varphi\| \cdot \|f\|_\infty.$$

Hence $\|\varphi_+\| \leq \|\varphi\|$. □

Step 5: Finally, the linear functional $\varphi_- := \varphi_+ - \varphi \in C_0(X,\mathbb{R})^*$ is also positive as $\varphi_+(f) \geq \varphi(f)$ for all $f \in C_0(X,[0,\infty))$ by definition of φ_+. □

Exercise 5.6.14. For $\varphi \in C_0(X)^*$, there are finite Radon measures $\mu_0, \mu_1, \mu_2, \mu_3$ on X such that

$$\varphi(f) = \sum_{k=0}^{3} i^k \int f\, d\mu_k = \int f\, d\left(\sum_{k=0}^{3} i^k \mu_k\right) \qquad \forall\, f \in C_0(X).$$

Definition 5.6.15. Let X be an LCH space.

- A signed Borel measure ν on X is called a *signed Radon measure* if $\nu_\pm$ are Radon, where $\nu = \nu_+ - \nu_-$ is the Jordan decomposition of ν. We denote by $\mathsf{RM}(X,\mathbb{R}) \subset M(X,\mathbb{R})$ the subset of finite signed Radon measures.
- A complex Borel measure ν on X is called a *complex Radon measure* if $\operatorname{Re}(\nu), \operatorname{Im}(\nu)$ are Radon. We denote by $\mathsf{RM}(X,\mathbb{C}) \subset M(X,\mathbb{C})$ the subset of complex Radon measures.

Exercise 5.6.16 (Lusin's Theorem)**.** Suppose X is LCH and μ is a Radon measure on X. If $f : X \to \mathbb{C}$ is measurable and vanishes outside a set of finite measure, then for all $\varepsilon > 0$, there is an $E \in \mathcal{B}_X$ with $\mu(E^c) < \varepsilon$ and a $g \in C_c(X)$ such that $g = f$ on E. Moreover:

- If $\|f\|_\infty < \infty$, we can arrange that $\|g\|_\infty \leq \|f\|_\infty$.
- If $\operatorname{im}(f) \subset \mathbb{R}$, we can arrange that $\operatorname{im}(g) \subset \mathbb{R}$.

Theorem 5.6.17 (Real Riesz Representation)**.** *Suppose X is LCH. Define $\Phi : \mathsf{RM}(X,\mathbb{R}) \to C_0(X,\mathbb{R})^*$ by $\nu \mapsto \varphi_\nu$ where $\varphi_\nu(f) := \int f\, d\nu$. Then Φ is an isometric linear isomorphism.*

Proof. Clearly Φ is linear. By Proposition 5.6.13, Φ is surjective. It remains to prove Φ is isometric, which also implies injectivity. Fix $\nu \in \mathsf{RM}$.

$\|\varphi_\nu\| \leq \|\nu\|$: For all $f \in C_0(X,\mathbb{R})$,

$$\begin{aligned} |\varphi_\nu(f)| = \left|\int f\, d\nu\right| &= \left|\int f\, d\nu_+ - \int f\, d\nu_-\right| \leq \left|\int f\, d\nu_+\right| + \left|\int f\, d\nu_-\right| \\ &\leq \int |f|\, d\nu_+ + \int |f|\, d\nu_- = \int |f|\, d|\nu| \leq \|f\|_\infty \cdot \|\nu\|_{\mathsf{RM}}. \end{aligned}$$

Hence $\|\varphi_\nu\| \leq \|\nu\|$.

$\underline{\|\varphi_\nu\| \geq \|\nu\|}$: Since ν is finite, by Exercise 4.2.11, $\left|\frac{d\nu}{d|\nu|}\right| = 1$ on X $|\nu|$-a.e. Let $\varepsilon > 0$. Since $|\nu|$ is finite, by Lusin's Theorem (Exercise 5.6.16), there is an $f \in C_c(X, \mathbb{R})$ such that $\|f\|_\infty = 1$ and $f = \overline{\frac{d\nu}{d|\nu|}}$ on $E \in \mathcal{B}_X$ where $|\nu|(E^c) < \varepsilon/2$. Then

$$\begin{aligned}\|\nu\| &= \int d|\nu| = \int \left|\frac{d\nu}{d|\nu|}\right|^2 d|\nu| = \int \overline{\frac{d\nu}{d|\nu|}} \cdot \frac{d\nu}{d|\nu|}\, d|\nu| \underset{\text{(Ex. 4.2.11)}}{=} \int \overline{\frac{d\nu}{d|\nu|}}\, d\nu \\ &\leq \left|\int f\, d\nu\right| + \left|\int f - \overline{\frac{d\nu}{d|\nu|}}\, d\nu\right| \leq \|\varphi_\nu\| \cdot \underbrace{\|f\|_\infty}_{=1} + \int \left|f - \overline{\frac{d\nu}{d|\nu|}}\right| d|\nu| \\ &\leq \|\varphi_\nu\| + 2|\nu|(E^c) \leq \|\varphi_\nu\| + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\|\nu\| \leq \|\varphi_\nu\|$. □

Exercise 5.6.18 (Complex Riesz Representation)**.** Suppose X is LCH. Define $\Phi : \mathsf{RM}(X, \mathbb{C}) \to C_0(X, \mathbb{C})^*$ by $\nu \mapsto \varphi_\nu$ where $\varphi_\nu(f) := \int f\, d\nu$. Show that Φ is an isometric linear isomorphism.