Basics of Functional Analysis

$\mathfrak{X}$ will denote a v. space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ ← Assume $\mathbb{C}$ unless stated otherwise.

Def: A seminorm on $\mathfrak{X}$ is a fct $\|\cdot\| : \mathfrak{X} \longrightarrow [0,\infty)$ s.t.

- (homogeneous) $\|\lambda x\| = |\lambda| \cdot \|x\|$ $\forall \lambda \in \mathbb{K}$ and $x \in \mathfrak{X}$
- (subadditive) $\|x+y\| \le \|x\| + \|y\|$ $\forall x,y \in \mathfrak{X}$. $\Longrightarrow \|0\| = 0$.

Call $\|\cdot\|$ a norm if in addition it is

- (definite) $\|x\| = 0 \Longrightarrow x = 0$.

Recall: $\rho(x,y) := \|x-y\|$ is a metric on a normed v. space which induces the norm topology on $\mathfrak{X}$. Two norms are called equivalent if $\exists C > 0$ s.t.

$$C^{-1}\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \qquad \forall x \in \mathfrak{X}$$

Exercise:

① Show all norms on $\mathbb{R}^n$ are equivalent.

② Call two norms topologically equivalent if they induce the same topology. Show two norms on a $\mathbb{K}$ v space which are topologically equivalent are equivalent norms

Def: A Banach space is a normed v. space which is complete in the induced topology.

Examples:

① $C_0(\mathfrak{X})$ or $C_b(\mathfrak{X})$ for $\mathfrak{X}$ LCH.

② $L^1(\mathfrak{X}, \mathcal{M}, \mu)$

Def: Suppose $(x_n) \subset X$ is a seq. Say $\sum x_n$ <u>converges to x</u> if $\sum^N x_n \to x$ as $N \to \infty$. Say $\sum x_n$ <u>converges absolutely</u> if $\sum \|x_n\| < \infty$.

Prop: For a normed space X, TFAE:

① X is Banach

② Every absolutely convergent series converges.

Pf: ①⇒②: Suppose X is Banach and $\sum \|x_n\| < \infty$. Let $\varepsilon > 0$, and pick $N > 0$ s.t. $\sum_{n > N} \|x_n\| < \varepsilon$. Then $\forall m \geq n > N$,

$$\left\| \sum_{i=1}^m x_i - \sum_{i=1}^n x_i \right\| = \left\| \sum_{n+1}^m x_n \right\| \leq \sum_{n+1}^m \|x_n\| \leq \sum_{n > N} \|x_n\| < \varepsilon.$$

②⇒①: Suppose (x_n) is Cauchy. Choose $n_1 < n_2 < \cdots$ s.t. $\|x_m - x_n\| < 2^{-k}$ $\forall m, n > n_k$. Define $y_0 := 0$ $(= x_{n_0})$ convention and $\forall k \in \mathbb{N}$, $y_k := x_{n_k} - y_{k-1} = x_{n_k} - x_{n_{k-1}}$. Then $\sum^k y_j = x_{n_k}$ and $\sum \|y_k\| \leq \|x_{n_1}\| + \sum 2^{-k} = \|x_{n_1}\| + 1 < \infty$. Hence $x := \lim_{k \to \infty} x_{n_k} = \sum y_k$ exists in X. Since (x_n) Cauchy, $x_n \to x$.

<u>Product + Quotient spaces</u>: HW!

Prop: Suppose X, Y are normed spaces and $T: X \to Y$ linear. TFAE:

① T is cts

② T is cts at 0_X

③ T is <u>bdd</u>: $\exists c > 0$ s.t. $\|Tx\| \leq c\|x\|$ $\forall x \in X$.

Pf: ① ⇒ ②: trivial.

② ⇒ ③: Suppose T cts at $0_{\mathfrak{X}}$. $\exists$ nbhd $\mathcal{U}$ of $0_{\mathfrak{X}}$ s.t. $T\mathcal{U} \subset \{y \in Y \mid \|y\| \leq 1\}$. Since $\mathcal{U}$ open, $\exists\, \delta > 0$ s.t. $\{x \mid \|x\| \leq \delta\} \subset \mathcal{U}$. Thus $\|x\| \leq \delta \Rightarrow \|Tx\| \leq 1$, and so $\forall a > 0$,

$$\|x\| \leq a \Rightarrow \frac{\delta}{a}\|x\| \leq \delta \Rightarrow \frac{\delta}{a}\|Tx\| \leq 1 \Rightarrow \|Tx\| \leq \frac{a}{\delta}.$$

Taking inf over $\|x\| \leq a$ gives $\|Tx\| \leq \delta^{-1}\|x\|$.

③ ⇒ ①: Let $\varepsilon > 0$. If $\|x_1 - x_2\| < C^{-1}\varepsilon$, then

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq C\|x_1 - x_2\| < \varepsilon.$$

Def: Let $\mathcal{L}(\mathfrak{X}, Y) := \{$ bdd linear maps $\mathfrak{X} \to Y\}$. Define

$$\begin{aligned}\|T\| &:= \sup\{\|Tx\| \mid \|x\| \leq 1\} \\ &= \sup\{\|Tx\| \mid \|x\| = 1\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} \;\middle|\; x \neq 0\right\} \\ &= \inf\{c \geq 0 \mid \|Tx\| \leq c\|x\| \ \forall x \in \mathfrak{X}\}.\end{aligned}$$

called the operator norm. For HW, you'll show it's a norm.

Observe if $S \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}(\mathfrak{X}, Y)$, $ST \in \mathcal{L}(\mathfrak{X}, Z)$ w/

$$\|STx\| \leq \|S\| \cdot \|Tx\| \leq \|S\| \cdot \|T\| \cdot \|x\| \quad \forall x \in \mathfrak{X}$$

So $\|ST\| \leq \|S\| \cdot \|T\|$.

Prop: If Y complete, so is $\mathcal{L}(\mathfrak{X}, Y)$.

Pf: If (T_n) Cauchy, so is $(T_n x)$ $\forall x \in \mathfrak{X}$. Set $Tx := \lim T_n x$. One verifies that (1) T is linear, (2) T bdd, and (3) $T_n \to T$.

Cor: $\mathfrak{X}$ complete $\Rightarrow \mathcal{L}(\mathfrak{X})$ a Banach algebra.

Dual Spaces:

A linear map $\mathfrak{X} \to \mathbb{K}$ is called a (linear) <u>functional</u>.

Def: The <u>dual space</u> is $\mathfrak{X}^* := \mathcal{L}(\mathfrak{X}, \mathbb{K})$ ← <u>Banach!</u>

Prop: Suppose $\mathfrak{X}$ is a $\mathbb{C}$-v.sp and $\varphi: \mathfrak{X} \to \mathbb{C}$ linear.

① $\mathrm{Re}(\varphi): \mathfrak{X} \to \mathbb{R}$ is $\mathbb{R}$-linear, and $\forall x \in \mathfrak{X}$,

$$\varphi(x) = \mathrm{Re}(\varphi)(x) - i\,\mathrm{Re}(\varphi)(ix).$$

② If $f: \mathfrak{X} \to \mathbb{R}$ is $\mathbb{R}$-linear, then setting

$$\varphi(x) := f(x) - i f(ix)$$

defines a $\mathbb{C}$-linear ftl.

③ Suppose $\mathfrak{X}$ is normed.

- In case ①, $\|\varphi\| < \infty \implies \|\mathrm{Re}(\varphi)\| \le \|\varphi\|$
- In case ②, $\|\mathrm{Re}(\varphi)\| < \infty \implies \|\varphi\| \le \|\mathrm{Re}(\varphi)\|$

<u>$\|\varphi\| = \|\mathrm{Re}(\varphi)\|$</u>

Pf: ① Just observe $\mathrm{Im}(\varphi(x)) = -\mathrm{Re}(i\varphi(x)) = -\mathrm{Re}(\varphi)(ix)$.

② It's clear φ is $\mathbb{R}$-linear. Check:

$$\varphi(ix) = f(ix) - i f(i(ix)) = f(ix) - i f(-x) = i f(x) + f(ix)$$
$$= i\big(f(x) - i f(ix)\big) = i\varphi(x).$$

③ <u>Case 1:</u> $|\mathrm{Re}(\varphi)(x)| \le |\varphi(x)| \implies \|\mathrm{Re}(\varphi)\| \le \|\varphi\|$.

<u>Case 2:</u> If $\varphi(x) \neq 0$,

$$|\varphi(x)| = \overline{\mathrm{sign}\,\varphi(x)}\,\varphi(x) = \varphi(\overline{\mathrm{sign}\,\varphi(x)}\,x) = \mathrm{Re}(\varphi)[\overline{\mathrm{sign}\,\varphi(x)}\,x].$$

Hence $|\varphi(x)| \le \|\mathrm{Re}(\varphi)\| \cdot \|x\| \implies \|\varphi\| \le \|\mathrm{Re}(\varphi)\|$.

<u>Q:</u> Do cts linear ftl's exist?

Def: Suppose $\mathfrak{X}$ is an $\mathbb{R}$-v. space. A sublinear (Minkowski) fctl on $\mathfrak{X}$ is a fct $p: \mathfrak{X} \to \mathbb{R}$ s.t.

- (positive homogeneous) $\forall x \in \mathfrak{X}$ and $\lambda \geq 0$, $p(\lambda x) = \lambda p(x)$.
- (subadditive) $\forall x, y \in \mathfrak{X}$, $p(x+y) \leq p(x) + p(y)$.

$\mathbb{R}$ Hahn-Banach Thm: Let $\mathfrak{X}$ be an $\mathbb{R}$-vector space, $p: \mathfrak{X} \to \mathbb{R}$ a sublinear fctl, $\mathcal{M} \subseteq \mathfrak{X}$ a subspace, and $f: \mathcal{M} \to \mathbb{R}$ a linear fctl s.t. $f(x) \leq p(x)$ $\forall x \in \mathcal{M}$. Then $\exists$ linear fctl $\varphi: \mathfrak{X} \to \mathbb{R}$ s.t. $\varphi|_{\mathcal{M}} = f$ and $\varphi(x) \leq p(x)$ $\forall x \in \mathfrak{X}$.

Pf: Step 1: $\forall x \in \mathfrak{X} \setminus \mathcal{M}$, $\exists$ linear $g: \mathcal{M} \oplus \mathbb{R}x \to \mathbb{R}$ s.t. $g|_{\mathcal{M}} = f$ and $g(y) \leq p(y)$ on $\mathcal{M} \oplus \mathbb{R}x$.

Pf: Any extension g of f to $\mathcal{M} \oplus \mathbb{R}x$ is determined by $g(m + \lambda x) = f(m) + \lambda\alpha$ where $\alpha = g(x)$ $\forall \lambda \in \mathbb{R}$. We want to choose $\alpha \in \mathbb{R}$ s.t. $f(m) + \lambda\alpha \leq p(m + \lambda x)$ $\forall m \in \mathcal{M}, \lambda \in \mathbb{R}$. Since f is linear and p is positive homog., we need only consider $\lambda = \pm 1$ [exercise]. These 2 conditions are:

$$f(m) - p(m - x) \underset{\substack{\uparrow \\ \lambda=-1}}{\leq} \alpha \underset{\substack{\uparrow \\ \lambda=+1}}{\leq} p(n + x) - f(n) \qquad \forall m, n \in \mathcal{M}.$$

Now

$$\begin{aligned} p(n+x) - f(n) - f(m) + p(m-x) &= p(n+x) + p(m-x) - f(m+n) \\ &\geq p(m+n) - f(m+n) \geq 0. \end{aligned}$$

$\Rightarrow \exists\, \alpha \in [\sup\{f(m) - p(m-x) \mid m \in \mathcal{M}\}, \inf\{p(n+x) - f(n) \mid n \in \mathcal{M}\}]$.

Step 2: Observe Step 1 applies to any extension g of f to $\mathcal{M} \subseteq \mathcal{N}$ s.t. $g|_{\mathcal{M}} = f$ and $g \leq p$ on $\mathcal{N}$. Thus any maximal extension φ of f satisfying $\varphi|_{\mathcal{M}} = f$ and $\varphi \leq p$ on its domain must have domain $\mathfrak{X}$. Now

$$\left\{ (\mathcal{N}, g) \;\middle|\; \begin{array}{l} \mathcal{M} \subseteq \mathcal{N} \subseteq \mathfrak{X} \text{ and } g : \mathcal{N} \to \mathbb{R} \text{ satisfying} \\ g|_{\mathcal{M}} = f \text{ and } g \leq p \text{ on } \mathcal{N} \end{array} \right\}$$

is partially ordered by $(\mathcal{N}_1, g_1) \leq (\mathcal{N}_2, g_2)$ if $\mathcal{N}_1 \subseteq \mathcal{N}_2$ and $g_2|_{\mathcal{N}_1} = g_1$. Since every ascending chain has an upper bound [exercise], $\exists$ a maximal elt by Zorn.

Remark: Suppose p is a seminorm on $\mathfrak{X}$ and $f : \mathfrak{X} \to \mathbb{R}$ is $\mathbb{R}$-linear. Then $f \leq p \iff |f| \leq p$. Indeed,

$$|f(x)| = \pm f(x) = f(\pm x) \leq p(\pm x) = p(x).$$

$\mathbb{C}$ Hahn-Banach Thm: Suppose $\mathfrak{X}$ is a $\mathbb{C}$-vector space, $p : \mathfrak{X} \to [0,\infty)$ is a seminorm, $\mathcal{M} \subseteq \mathfrak{X}$ a subspace, and $f : \mathcal{M} \to \mathbb{C}$ a $\mathbb{C}$-linear ftl s.t. $|f| \leq p$ on $\mathcal{M}$. Then $\exists$ a $\mathbb{C}$-linear ftl $\varphi : \mathfrak{X} \to \mathbb{C}$ s.t. $\varphi|_{\mathcal{M}} = f$ and $|\varphi| \leq p$.

Pf: By $\mathbb{R}$-HB applied to $\mathrm{Re}(f)$, $\exists$ $\mathbb{R}$-linear extension $g : \mathfrak{X} \to \mathbb{R}$ s.t. $|g| \leq p$. Define $\varphi(x) := g(x) - i g(ix)$. Then $\varphi|_{\mathcal{M}} = f$. Then

$$|\varphi(x)| = \overline{\operatorname{sign} \varphi(x)}\, \varphi(x) = \varphi(\overline{\operatorname{sign} \varphi(x)}\, x) = g(\overline{\operatorname{sign} \varphi(x)}\, x)$$
$$\leq p(\overline{\operatorname{sign} \varphi(x)}\, x) = p(x) \qquad \forall x \in \mathfrak{X}.$$

Corollaries of Hahn-Banach: Let $\mathfrak{X}$ be a normed space.

① If $x \neq 0$, $\exists \varphi \in \mathfrak{X}^*$ s.t. $\varphi(x) = \|x\|$ and $\|\varphi\| = 1$.

Pf: Define $f : \mathbb{K}x \to \mathbb{K}$ by $f(\lambda x) = \lambda \cdot \|x\|$, and note $|f| \leq \|\cdot\|$. Now apply HB.

② If $\mathcal{M} \subset \mathfrak{X}$ closed and $x \in \mathcal{M}^c$, $\exists \varphi \in \mathfrak{X}^*$ s.t. $\varphi(x) = \inf_{m \in \mathcal{M}} \|x - m\|$ and $\|\varphi\| = 1$.

Pf: Apply ① to $x + \mathcal{M} \in \mathfrak{X}/\mathcal{M}$ to get $\tilde{\varphi} \in [\mathfrak{X}/\mathcal{M}]^*$ s.t. $\|\tilde{\varphi}(x + \mathcal{M})\| = \|x + \mathcal{M}\| = \inf_{m \in \mathcal{M}} \|x - m\|$ and $\|\tilde{\varphi}\| = 1$. By HW, the canonical quotient map $Q : \mathfrak{X} \to \mathfrak{X}/\mathcal{M}$ is cts, and clearly $\|x + \mathcal{M}\| \leq \|x\|$ $\forall x \Rightarrow \|Q\| \leq 1$. Then $\varphi := \tilde{\varphi} \circ Q$ works.

③ $\mathfrak{X}^*$ seperates pts of $\mathfrak{X}$.

Pf: If $x \neq y$, $\exists \varphi \in \mathfrak{X}^*$ s.t. $\varphi(x - y) = \|x - y\| \neq 0$ by ①.

④ For $x \in \mathfrak{X}$, define $ev_x : \mathfrak{X}^* \to \mathbb{K}$ by $\varphi \mapsto \varphi(x)$. Then $ev : \mathfrak{X} \to \mathfrak{X}^{**}$ is a linear isometry.

Pf: $\forall \varphi \in \mathfrak{X}^*$, $\|ev_x \varphi\| = |\varphi(x)| \leq \|\varphi\| \cdot \|x\|$, so $\|ev_x\| \leq \|x\|$, and $ev_x \in \mathfrak{X}^{**}$. If $x \neq 0$, $\exists \varphi$ s.t. $\varphi(x) = \|x\|$ and $\|\varphi\| = 1$ by ①, so $\|ev_x\| = \|x\|$.

Def: $\overline{\mathfrak{X}} := \overline{ev(\mathfrak{X})} \subset \mathfrak{X}^{**}$ is the <u>completion</u> of $\mathfrak{X}$, which is always Banach. If $\mathfrak{X}$ is Banach, $ev(\mathfrak{X}) \subset \mathfrak{X}^{**}$ is closed. In this case, if $ev(\mathfrak{X}) = \mathfrak{X}^{**}$, call $\mathfrak{X}$ <u>reflexive</u>.

Baire Category Thm + Consequences:

Thm (Baire Category): Suppose X is either:

① a LCH space or

② a complete metric space.

Let (U_n) be a sequence of open dense subsets of X. Then $\bigcap U_n$ is dense in X.

Pf: Let $V_0 \subset X$ be nonempty + open. We'll inductively construct $\forall n \in \mathbb{N}$ a non-empty open set $V_n \subset \overline{V_n} \subset U_n \cap V_{n-1}$.

For ①: Take V_n s.t. $\overline{V_n}$ cpt so $(\overline{V_n})$ are non-empty nested cpt sets.

For ②: Take V_n to be a ball of radius $\frac{1}{n}$.

Claim: $K := \bigcap V_n$, we have $K \neq \emptyset$.

Pf: **For ①:** $(\overline{V_n})$ has FIP $\Rightarrow \bigcap \overline{V_n} \neq \emptyset$.

For ②: Let x_n be the center of V_n. Then (x_n) Cauchy $\Rightarrow$ converges.

Now observe $\emptyset \neq K \subset [\bigcap U_n] \cap V_0$, so $\bigcap U_n$ dense in X.

Cor: If X is as above, X is not the countable union of nowhere dense sets. [X is not meager.]

Pf: If (S_n) is a sequence of nowhere dense sets, $(U_n := \overline{S_n}^c)$ is a sequence of open dense sets. Then

$$\bigcap U_n = \bigcap \overline{S_n}^c = [\bigcup \overline{S_n}]^c \subset [\bigcup S_n]^c$$

is dense in X, so $\bigcup S_n \neq X$.

Lemma: Suppose $\mathbb{X}, Y$ are Banach spaces and $T \in \mathcal{L}(\mathbb{X}, Y)$. Let $\mathcal{U} \subset \mathbb{X}$ be an open ball centred at $0_{\mathbb{X}}$ and $V \subset Y$ be an open ball centred at 0_Y. If $V \subset \overline{T\mathcal{U}}$, then $V \subset T\mathcal{U}$.

Pf: Let $y \in V$. Take $r \in (0,1)$ s.t. $y \in rV$. Let $\varepsilon \in (0,1)$ to be decided later. Observe:

$$y \in rV \subset r\overline{T\mathcal{U}} = \overline{T(r\mathcal{U})}$$

$$\Rightarrow \exists x_0 \in r\mathcal{U} \text{ s.t. } y - Tx_0 \in \varepsilon r V \subset \varepsilon_r \overline{T\mathcal{U}} = \overline{T(\varepsilon r \mathcal{U})}$$

$$\Rightarrow \exists x_1 \in \varepsilon r \mathcal{U} \text{ s.t. } y - Tx_0 - Tx_1 \in \varepsilon^2 r V \subset \overline{T(\varepsilon^2 r \mathcal{U})}$$

$$\Rightarrow \dots$$

By induction, get a seq. (x_n) s.t. $x_n \in \varepsilon^n r\mathcal{U}$ and $y - \sum_{j=0}^{n} Tx_j \in \varepsilon^{n+1} rV$ $\forall n \in \mathbb{N}$. Then $\sum x_n$ converges as $\|x_n\| < \varepsilon^n r R$ [summable!] for $R := \text{radius}(\mathcal{U})$, and $T\sum x_n = \lim_N T\sum^N x_n = \lim_N \sum^N Tx_n = y$. Moreover, we have $\|\sum x_n\| \leq \sum \|x_n\| < \sum_{n=0}^{\infty} \varepsilon^n r R = \frac{rR}{1-\varepsilon}$, so $\sum x_n \in \frac{r}{1-\varepsilon}\mathcal{U}$.

Thus if $\varepsilon < 1-r$, $\frac{r}{1-\varepsilon} < 1 \Rightarrow \sum x_n \in \mathcal{U} \Rightarrow y \in T\mathcal{U}$.

Thm (Open Mapping): Suppose $\mathbb{X}, Y$ Banach and $T \in \mathcal{L}(\mathbb{X}, Y)$ is surjective. Then T is an open map.

Pf: It suffices to prove T maps a nbhd of $0_{\mathbb{X}}$ to a nbhd of 0_Y. Note $Y = \bigcup_n \overline{TB_n(0)}$. By Baire Category Thm, $\exists n \in \mathbb{N}$ s.t. $\overline{TB_n(0)}$ contains a nonempty open set, say $Tx_0 + V$ where $x_0 \in TB_n(0)$ and V is an open ball in Y w/ center 0_Y. Then $V \subset \overline{TB_n(0)} - Tx_0 \subset \overline{TB_{2n}(0)}$. By the lemma, $V \subset TB_{2n}(0)$.

Cor: Suppose $\mathfrak{X}, \mathcal{Y}$ Banach and $T \in \mathcal{L}(\mathfrak{X}, \mathcal{Y})$ is bijective. Then $T^{-1} \in \mathcal{L}(\mathfrak{X}, \mathcal{Y})$. [T is an isomorphism.]

Pf: If T is bijective, T^{-1} is cts $\iff$ T is open.

Cor: Suppose $\mathfrak{X}$ is Banach under $\|\cdot\|_1$ and $\|\cdot\|_2$. If $\exists$ $C > 0$ s.t. $\|x\|_1 \leq C \|x\|_2$ $\forall x \in \mathfrak{X}$, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equiv.

Pf: Apply the corollary to $id : (\mathfrak{X}, \|\cdot\|_2) \longrightarrow (\mathfrak{X}, \|\cdot\|_1)$.

Def: Suppose $\mathfrak{X}, \mathcal{Y}$ are normed spaces and $T : \mathfrak{X} \to \mathcal{Y}$ is linear. The graph of T is

subspace!

$$\Gamma(T) := \{(x,y) \mid Tx = y\} \subset \mathfrak{X} \times \mathcal{Y}.$$

Here, we endow $\mathfrak{X} \times \mathcal{Y}$ w/ the norm

$$\|(x,y)\|_\infty := \max\{\|x\|_{\mathfrak{X}}, \|y\|_{\mathcal{Y}}\}.$$

We say T is closed if $\Gamma(T) \subset \mathfrak{X} \times \mathcal{Y}$ is closed.

Fact: If $T \in \mathcal{L}(\mathfrak{X}, \mathcal{Y})$, $\Gamma(T)$ is closed.

Pf: $(x_n, Tx_n) \to (x,y) \iff x_n \to x$ and $Tx_n \to y$.

Since T is cts, $Tx_n \to Tx$. Since $\mathcal{Y}$ Hausdorff, $Tx = y$.

Thm (Closed Graph): Suppose $\mathfrak{X}, \mathcal{Y}$ Banach. If $T : \mathfrak{X} \to \mathcal{Y}$ is a closed linear map, then $T \in \mathcal{L}(\mathfrak{X}, \mathcal{Y})$.

Pf: Consider $\pi_{\mathfrak{X}}, \pi_{\mathcal{Y}}$ the canonical projection maps, cts. Since $\pi_{\mathfrak{X}} : \Gamma(T) \longrightarrow \mathfrak{X}$ by $(x, Tx) \longmapsto x$ is norm decreasing + bijective, $\pi_{\mathfrak{X}}^{-1}$ bdd by cor of OMT. Now observe $T = \pi_{\mathcal{Y}} \circ \pi_{\mathfrak{X}}^{-1}$ is bdd. $[x \xmapsto{\pi_{\mathfrak{X}}^{-1}} (x, Tx) \xmapsto{\pi_{\mathcal{Y}}} Tx.]$

Thm (Banach–Steinhaus / Uniform Boundedness Principle):

Suppose X, Y are normed spaces and $\mathcal{S} \subset \mathcal{L}(X, Y)$.

① If $\sup_{T \in \mathcal{S}} \|Tx\| < \infty$ $\forall x$ in a non-meager subset of X,

then $\sup_{T \in \mathcal{S}} \|T\| < \infty$.

② If X Banach and $\sup_{T \in \mathcal{S}} \|Tx\| < \infty$ $\forall x \in X$, then

$\sup_{T \in \mathcal{S}} \|T\| < \infty$.

Pf: ① Define $E_n := \{x \in X \mid \sup_{T \in \mathcal{S}} \|Tx\| \le n\} = \bigcap_{T \in \mathcal{S}} \underbrace{\{x \in X \mid \|Tx\| \le n\}}_{\text{closed}}$,

which is closed in X. By assumption, $\exists x_0 \in X$, $r > 0$ and $n > 0$ s.t. $\overline{B_r(x_0)} \subset E_n$. Then as in the Lemma for OMT, $\overline{B_r(0)} \subset E_{2n}$:

$$\|Tx\| \le \|T(x - x_0)\| + \|Tx_0\| \le 2n \quad \text{when } \|x\| \le r.$$

Thus $\forall T \in \mathcal{S}$, $\forall \|x\| \le r$, $\|Tx\| \le 2n \Rightarrow \sup_{T \in \mathcal{S}} \|T\| \le \frac{2n}{r}$.

② If X is Banach, the sets E_n in part ① cannot all be meager by Baire Category as $X = \bigcup E_n$.

Topological vector spaces

A vector space X over $\mathbb{K}$ equipped w/ a topology is called a <u>topological vector space</u> (TVS) if $+ : X \times X \to X$ and $\cdot : \mathbb{K} \times X \to X$ are cts. A subset $C \subseteq X$ is called <u>convex</u> if $x, y \in C \Rightarrow tx + (1-t)y \in C$ $\forall t \in [0,1]$. We'll focus on <u>locally convex</u> TVS's, i.e., X s.t. $\forall x \in X$ and open nbhd $\mathcal{U}$ of X, $\exists$ cnvx open nbhd V of x s.t. $V \subseteq \mathcal{U}$.

Let $\{p_i\}_{i\in I}$ be a family of seminorms on $\mathfrak{X}$. For $x\in\mathfrak{X}$, $i\in I$ and $\varepsilon>0$, define

$$\mathcal{U}_{x,i,\varepsilon} := \{y\in\mathfrak{X} \mid p_i(x-y)<\varepsilon\}.$$

Let τ be the topology generated by these sets.

Facts:

⓪ τ is the weakest topology s.t. $p_i:\mathfrak{X}\to[0,\infty)$ is cts $\forall i\in I$.

Pf: Exercise!

① $\forall x\in\mathfrak{X}$, the finite intersections $\{y\in\mathfrak{X} \mid p_i(x-y)<\varepsilon \;\; \forall i\in\{i_1,\dots,i_n\}\}$ form a nbhd base at x

Pf: τ consists of arbitrary unions of finite intersections of the $\mathcal{U}_{x,i,\varepsilon}$. Suppose $x\in\mathfrak{X}$. Then $\exists x_1,\dots,x_n\in\mathfrak{X}$, $i_1,\dots,i_n\in I$, and $\varepsilon_1,\dots,\varepsilon_n>0$ s.t. $x\in\bigcap_{j=1}^n \mathcal{U}_{x_j,i_j,\varepsilon_j}$. Define $\varepsilon := \min_j[\varepsilon_j - p_{i_j}(x-x_j)]$. Then $\forall y\in\bigcap_{j=1}^n \mathcal{U}_{x,i_j,\varepsilon}$, $p(x-y)\le p(x-x_j)+p(x_j-y)\le(\varepsilon_j-\varepsilon)+\varepsilon=\varepsilon_j$, so $y\in\bigcap_{j=1}^n \mathcal{U}_{x_j,i_j,\varepsilon_j}$.

② If $(x_\lambda)\subset\mathfrak{X}$ is a net, $x_\lambda\to x \iff p_i(x-x_\lambda)\to0 \;\; \forall i\in I$.

Pf: Observe $x_\lambda\to x \iff (x_\lambda)$ is eventually in $\mathcal{U}_{x,i,\varepsilon}$ $\forall\varepsilon>0$, $\forall i\in I$.
$\iff p_i(x-x_\lambda)\to0 \;\; \forall i\in I$.

③ $(\mathfrak{X},\tau)$ is a TVS.

Pf: + cts: Suppose $x_\lambda\to x$ and $y_\lambda\to y$. Then $\forall i\in I$,

$$p_i(x+y-x_\lambda+y_\lambda)\le p_i(x-x_\lambda)+p_i(y-y_\lambda)\longrightarrow0.$$

• cts: If $\alpha_\lambda\to\alpha$ and $x_\lambda\to x$,

$$p(\alpha_\lambda x_\lambda-\alpha x)\le p(\alpha_\lambda x_\lambda-\alpha x_\lambda)+p(\alpha x_\lambda-\alpha x)$$

$$\le \underbrace{|\alpha_\lambda-\alpha|}_{\to0}\,\underbrace{p(x_\lambda)}_{\to p(x)}+|\alpha|\underbrace{p(x_\lambda-x)}_{\to0}.$$

Hence $(\mathfrak{X},\tau)$ is a TVS.

④ $(\mathfrak{X}, \tau)$ is locally cnvx.

Pf: Observe the $\mathcal{U}_{x,i,\varepsilon}$ are cnvx: if $y, z \in \mathcal{U}_{x,i,\varepsilon}$, then $\forall t \in [0,1]$,

$$\begin{aligned} p(x - [ty + (1-t)z]) &= p(tx + (1-t)x - ty - (1-t)z)) \\ &\le t p(x-y) + (1-t) p(x-z) \\ &< t\varepsilon + (1-t)\varepsilon \\ &= \varepsilon. \end{aligned}$$

Hence $ty + (1-t)z \in \mathcal{U}_{x,i,\varepsilon}$. The result follows from ①.

Exercises:

⑤ $\mathfrak{X}$ is Hausdorff $\iff \{p_i\}_{i \in I}$ separates pts

$\iff \forall x \in \mathfrak{X}$ with $x \neq 0$, $\exists i \in I$ s.t. $p_i(x) \neq 0$.

⑥ If $\mathfrak{X}$ Hausdorff and I countable, then $\exists$ a translation invariant metric $\rho : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ s.t. $\rho(x+z, y+z) = \rho(x,y)$ $\forall z \in \mathfrak{X}$ which induces the same topology as $\{p_i\}_{i \in I}$.

Prop: Suppose $(\mathfrak{X}, \{p_i\}_{i \in I}, \tau)$ and $(Y, \{q_j\}_{j \in J}, \theta)$ are LCnvx TVS's and $T : \mathfrak{X} \to Y$ is linear. TFAE:

① T is cts.

② T is cts at 0.

③ $\forall j \in J$, $\exists i_1, \dots, i_n \in I$ and $C > 0$ s.t. $q_j(Tx) \le C \sum_{k=1}^{n} p_{i_k}(x)$ $\forall x \in \mathfrak{X}$.

Pf: ③ ⇒ ①: Suppose (3) for a fixed $j \in J$. If $x_\lambda \to x$, $p_{i_k}(x - x_\lambda) \to 0$ $\forall k = 1, \dots, n$, so $q_j(T(x - x_\lambda)) \le C \sum_{k=1}^{n} p_{i_k}(x - x_\lambda) \to 0$. Hence $Tx_\lambda \to Tx$.

① ⇒ ②: Trivial.

② ⇒ ③: Suppose T is cts at 0 and $j \in J$. $\exists i_1, \dots, i_n \in I$ and $\varepsilon > 0$ s.t. $\forall x$ in $V := \bigcap_{k=1}^{n} \mathcal{U}_{0, i_k, \varepsilon}$, $q_j(Tx) < 1$. Fix $x \in \mathfrak{X}$. If $p_{i_k}(x) = 0$ $\forall k = 1, \dots, n$, then $rx \in V$ $\forall r > 0$, so $q_j(Trx) = r q_j(Tx) < 1$ $\forall r > 0$, and $q_j(Tx) = 0$.

Assume $p_{i_1}(x) > 0$. Then $y = \varepsilon x / 2 \sum_{k=1}^{n} p_{i_k}(x) \in V$ as $p_{i_k}(y) \le \frac{\varepsilon}{2} < \varepsilon$ $\forall k$.

Thus $q_j(Tx) = \left[2\varepsilon^{-1} \sum_{k=1}^{n} p_{i_k}(x) \right] \underbrace{q_j(Ty)}_{<1} < 2\varepsilon^{-1} \sum_{k=1}^{n} p_{i_k}(x)$ as desired.

Example: Let $\mathfrak{X}$ be a normed space. Recall that $\mathfrak{X}^*$ separates pts of $\mathfrak{X}$ by HB. Consider the family of seminorms $\mathcal{S} := \{P_\varphi(x) := |\varphi(x)|\}_{\varphi \in \mathfrak{X}^*}$ on $\mathfrak{X}$. Then $\mathcal{S}$ separates pts, so the locally cnvx t.v.sp. top τ on $\mathfrak{X}$ gen by $\mathcal{S}$ is Hausdorff. Call this the weak topology on $\mathfrak{X}$. Observe a net (x_λ) converges to $x \in \mathfrak{X} \iff x_\lambda \to x \ \forall x \in \mathfrak{X}$.

Prop: If $\mathcal{U} \subseteq \mathfrak{X}$ is weakly open, $\mathcal{U}$ is $\|\cdot\|$-open.

Pf: Every basic open set $\mathcal{U}_{x,\varphi,\varepsilon} = \{y \in \mathfrak{X} \mid |\varphi(x-y)| < \varepsilon\}$ is norm open as $\varphi \in \mathfrak{X}^*$ is norm cts and $|\cdot| : \mathbb{C} \to [0,\infty)$ is cts.

Exercise: The weak top is the norm top $\iff \mathfrak{X}$ finite dim'l.

Prop: A linear ft $\varphi : \mathfrak{X} \to \mathbb{C}$ is cts wrt the weak top $\iff \varphi \in \mathfrak{X}^*$.

Pf: Suppose $\varphi \in \mathfrak{X}^*$. Then $\varphi^{-1}[B_\varepsilon^{\mathbb{C}}(0)] = \{x \mid |\varphi(x)| < \varepsilon\} = \mathcal{U}_{0,\varphi,\varepsilon}$ is weakly open. Hence φ is cts at 0, and thus φ is weakly cts.
Now suppose φ is weakly cts. Then $\forall \mathcal{U} \subset \mathbb{C}$ open, $\varphi^{-1}(\mathcal{U})$ is wkly open $\Rightarrow \|\cdot\|$ open. So φ is $\|\cdot\|$-cts and thus $\varphi \in \mathfrak{X}^*$.

Def: The weak* top on $\mathfrak{X}^*$ is induced by $\mathfrak{X} \subseteq \mathfrak{X}^{**}$, i.e., $\varphi_\lambda \to \varphi$ weak* $\iff \varphi_\lambda(x) \to \varphi(x) \ \forall x \in \mathfrak{X}$.

Thm (Banach-Alaoglu): The norm-closed unit ball $B^* \subset \mathfrak{X}^*$ is weak*-cpt.

Pf: For $x \in \mathfrak{X}$, let $D_x = \{z \in \mathbb{C} \mid |z| \le \|x\|\}$. By Tychonoff's thm, $D = \prod_{x \in \mathfrak{X}} D_x$ is cpt. The elts $(d_x) \in D$ are precisely fcts $f : \mathfrak{X} \to \mathbb{C}$ s.t. $|f(x)| \le \|x\| \ \forall x \in \mathfrak{X}$, and $B^* \subset D$ are the linear fcts. The relative top on B^* is the relative wk* top, as both are ptwise convergence. It remains to prove B^* is closed. If $(\varphi_\lambda) \subset B^*$ is a net w/ $\varphi_\lambda \to \varphi \in D$, then

$$\varphi(\alpha x+y) = \lim \varphi_\lambda(\alpha x+y) = \lim \alpha\varphi_\lambda(x) + \varphi_\lambda(y) = \alpha \lim \varphi_\lambda(x) + \lim \varphi_\lambda y = \alpha\varphi(x) + \varphi(y).$$

Hilbert spaces: A sesquilinear form on a $\mathbb{K}$-v.sp. H is a fct $\langle\cdot,\cdot\rangle : H\times H \to \mathbb{K}$ which is:

- linear in the first variable $\langle \alpha x+y, z\rangle = \alpha\langle x,z\rangle + \langle y,z\rangle$
- conjugate linear in the second variable $\langle x, \alpha y+z\rangle = \bar{\alpha}\langle x,y\rangle + \langle x,z\rangle$.

we call $\langle\cdot,\cdot\rangle$:

- self-adjoint if $\langle x,y\rangle = \overline{\langle y,x\rangle}$ $\forall x,y\in H$
- non-degenerate if $\langle x,y\rangle = 0$ $\forall y\in H \Rightarrow x=0$.
- positive if $\langle x,x\rangle \geq 0$ $\forall x\in H$.

  ↳ a positive sesq. form is called definite if $\langle x,x\rangle = 0 \Rightarrow x=0$.

Exercise: Suppose $\langle\cdot,\cdot\rangle$ is a sesq. form on H and $\mathbb{K}=\mathbb{C}$.

a) (polarization) $4\langle x,y\rangle = \sum_{k=0}^{3} i^k \langle x+i^k y, x+i^k y\rangle$

b) $\langle\cdot,\cdot\rangle$ self-adjoint $\Leftrightarrow \langle x,x\rangle\in\mathbb{R}$ $\forall x\in H$.

c) positive $\Rightarrow$ self-adjoint by polarization.

Suppose now $\langle\cdot,\cdot\rangle$ is positive and self-adjoint [pre-Hilbert space]

Define $\|x\| := \langle x,x\rangle^{1/2}$. Say $x\perp y$ if $\langle x,y\rangle = 0$.

⓪ ($\mathbb{R}$-polarization) if $\mathbb{K}=\mathbb{R}$, $4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2$.

① (Pythagorean) $x\perp y \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Pf: $\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x,y\rangle + \|y\|^2 = \|x\|^2 + \|y\|^2$.

② $x\perp y \Leftrightarrow \|x\|^2 \leq \|x+\alpha y\|^2$ $\forall \alpha\in\mathbb{C}$.

Pf: $\Rightarrow$: $\|x+\alpha y\|^2 = \|x\|^2 + |\alpha|^2\|y\|^2 \geq \|x\|^2$ $\forall\alpha\in\mathbb{C}$.

$\Leftarrow$: $\|x+\alpha y\|^2 = \|x\|^2 + 2\operatorname{Re}(\alpha\langle x,y\rangle) + |\alpha|^2\|y\|^2 \geq \|x\|^2$ $\forall\alpha\in\mathbb{C}$

$\Rightarrow 0 \leq 2\operatorname{Re}(\alpha\langle x,y\rangle) + |\alpha|^2\|y\|^2$ $\forall\alpha\in\mathbb{C}$ $\Rightarrow 0\leq 2\operatorname{Re}(\alpha\langle x,y\rangle)$ $\forall\alpha\in\mathbb{C}$

take α sufficiently small!

$\Rightarrow \langle x,y\rangle = 0$.

③ Definite $\Longleftrightarrow$ non-degenerate.

Pf: $\Leftarrow$: Obvious.

$\Rightarrow$: If $\|x\|^2 = 0$, $\forall \alpha \in \mathbb{C}$, $\forall y \in H$, $\|x\|^2 = 0 \le \|x + \alpha y\|^2 \Rightarrow x \perp y$ by ②

④ (Cauchy-Schwarz) $\forall x, y \in H$, $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$.

Pf: $\forall t \in \mathbb{R}$, $0 \le \|x - ty\|^2 = \|x\|^2 - 2t \operatorname{Re}\langle x, y\rangle + t^2 \|y\|^2$. $(*)$

$\Rightarrow$ discriminant $4[\operatorname{Re}\langle x, y\rangle]^2 - 4\|x\|^2 \cdot \|y\|^2 \le 0$

$\Rightarrow$ $|\operatorname{Re}\langle x, y\rangle| \le \|x\| \cdot \|y\|$.

Now $|\langle x, y\rangle| = \alpha \langle x, y\rangle$ for some $\alpha \in U(1)$

$\Rightarrow$ $|\langle x, y\rangle| = \alpha \langle x, y\rangle = \langle \alpha x, y\rangle \le \|\alpha x\| \cdot \|y\| = \|x\| \cdot \|y\|$. $(**)$

⑤ If $\langle \cdot, \cdot \rangle$ definite, $|\langle x, y\rangle| = \|x\| \cdot \|y\| \Rightarrow \{x, y\}$ lin. dependent.

Pf: We may assume $y \neq 0$. Set $\alpha := \dfrac{|\langle x, y\rangle|}{\|y\|^2} \operatorname{sign}(\langle x, y\rangle)$. Then

$$\|x - \alpha y\|^2 = \|x\|^2 - 2\operatorname{Re} \alpha \langle x, y\rangle + \|y\|^2 = \|x\|^2 - 2\frac{|\langle x, y\rangle|^2}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^4}\|y\|^2$$

$$= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} = \|x\|^2 - \frac{\|x\|^2 \cdot \|y\|^2}{\|y\|^2} = 0 \Rightarrow \underbrace{x = \alpha y}_{\text{by definiteness}}$$

⑤ $\|\cdot\| : H \longrightarrow [0, \infty)$ is a seminorm. It is a norm exactly when $\langle \cdot, \cdot \rangle$ is an inner product.

Pf: Homogeneity: $\|\lambda x\|^2 = \langle \lambda x, \lambda x\rangle = |\lambda|^2 \langle x, x\rangle = |\lambda|^2 \|x\|^2$.

Sub-additivity: $\|x + y\|^2 = \langle x + y, x + y\rangle = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$

$\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$

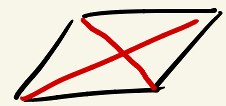
(C-S) $\le \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2$

$= (\|x\| + \|y\|)^2$.

The final claim follows immediately.

Prop: A norm $\|\cdot\|$ on a $\mathbb{C}$ v.sp. comes from an inner product
$\iff$ $\|\cdot\|$ satisfies the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right]$$



Pf: $\Rightarrow$: Add $\|x\pm y\|^2 = \|x\|^2 \pm 2\operatorname{Re}\langle x,y\rangle + \|y\|^2$.

$\Leftarrow$: Define $\langle x,y\rangle := \frac{1}{4}\sum_{k=0}^{3} i^k \|x+i^k y\|^2$ using polarization id.

Def: A Hilbert space is an inner product space which is complete wrt $\|\cdot\|$. [Hilbert $\Rightarrow$ Banach]

Examples:

① $\ell^2 := \{(x_n) \mid \sum |x_n|^2 < \infty\}$ with $\langle x,y\rangle := \sum x_n \bar{y}_n$.

② $L^2(X,\mu) := \{\text{measurable } f: X \to \mathbb{C} \mid \int |f|^2 \, d\mu < \infty\}$ with

$$\langle f,g\rangle := \int f\bar{g}\, d\mu.$$

From here on, H will denote a Hilbert space.

Thm: Suppose $C \subset H$ is a convex, closed subset and $z \notin C$.
$\exists!\, x \in C$ s.t. $\|x-z\| = \inf_{y\in C} \|y-z\|$.

Pf: By translation, we may assume $z = 0 \notin C$. Suppose $(x_n) \subset C$ s.t. $\|x_n\| \to r := \inf_{y\in C} \|y\|$. Then by parallelogram id,

$$(**)\quad \begin{aligned}
&\left\|\frac{x_m - x_n}{2}\right\|^2 + \Big\|\underbrace{\frac{x_m+x_n}{2}}_{\in C!}\Big\|^2 = 2\left[\left\|\frac{x_m}{2}\right\|^2 + \left\|\frac{x_n}{2}\right\|^2\right] \\
\Rightarrow\ &\|x_m - x_n\|^2 = 2\underbrace{\|x_m\|^2}_{\to r^2} + 2\underbrace{\|x_n\|^2}_{\to r^2} - 4\underbrace{\left\|\frac{x_m+x_n}{2}\right\|^2}_{\geq r^2}. \\
\Rightarrow\ &\limsup_{m,n} \|x_m - x_n\|^2 \leq 2r^2 + 2r^2 - 4r^2 = 0 \\
\Rightarrow\ &(x_n) \text{ Cauchy.}
\end{aligned}$$

Since H is complete $\exists\, x$ s.t. $x_n \to x$. Since C closed, $x \in C$, and $\|x\| = r$.
If $x' \in C$ satisfies $\|x'\| = r$, $(x, x', x, x', \dots)$ is Cauchy by $(**)$, so $x = x'$.

For $S \subseteq H$, let $S^\perp := \{x \in H \mid \langle x, s\rangle = 0 \ \forall s \in S\}$.

Observe $S^\perp \subseteq H$ is a closed subspace.

① $S \subseteq T \Rightarrow T^\perp \subseteq S^\perp$

Pf: $x \in T^\perp \iff \langle x, t\rangle = 0 \ \forall t \in T \supseteq S \Rightarrow x \in S^\perp$.

② $\overline{S} \subseteq S^{\perp\perp}$ and $S^\perp = S^{\perp\perp\perp}$

Pf: $s \in S \Rightarrow \langle s, x\rangle = \overline{\langle x, s\rangle} = 0 \ \forall x \in S^\perp \Rightarrow s \in S^{\perp\perp}$. Now since $S^{\perp\perp}$ is closed, $\overline{S} \subset S^{\perp\perp}$. Replacing S w/ $S^\perp$, get $S^\perp \subseteq S^{\perp\perp\perp}$. But since $S \subseteq S^{\perp\perp}$, by ①, $S^{\perp\perp\perp} \subseteq S^\perp$.

Let $M \subseteq H$ be a subspace.

③ $M \cap M^\perp = \{0\}$

Pf: If $x \in M \cap M^\perp$, $\langle x, x\rangle = 0 \Rightarrow x = 0$.

④ $H = \overline{M} \oplus M^\perp$

Pf: Let $x \in H$. Since $\overline{M}$ is closed + cnvx, $\exists!\, m \in \overline{M}$ minimizing distance to x, i.e., $\|x - m\| \leq \inf_{n \in M} \|x - n\|$. We claim that $m \perp (x-m)$, so $x = \underbrace{m}_{\in \overline{M}} + \underbrace{(x-m)}_{\in M^\perp}$, and $H = \overline{M} + M^\perp$. This w/ ③ completes the pf. Indeed, $\forall n \in M$ and $\alpha \in \mathbb{C}$

$$\|x - m\|^2 \leq \|x - (m - \alpha n)\|^2 = \|(x-m) + \alpha n\|^2 \Rightarrow x - m \perp n.$$

Hence $x - m \in M^\perp$, and we are finished.

⑤ $\overline{M} = M^{\perp\perp}$

Pf: Let $x \in M^{\perp\perp}$. Then $\exists!\, m \in \overline{M}$ and $y \in M^\perp$ s.t. $x = m + y$. Then

$$0 = \langle x, y\rangle = \langle m + y, y\rangle = \underbrace{\langle m, y\rangle}_{=0} + \langle y, y\rangle = \langle y, y\rangle \Rightarrow y = 0.$$

Thm (Riesz-Representation): Let H be a Hilbert space. For $y \in H$, define $\varphi_y : H \longrightarrow \mathbb{K}$ by $\varphi_y(x) := \langle x, y\rangle$.

① $\varphi_y \in H^*$ and $\|\varphi_y\| = \|y\|$.

② $\forall \varphi \in H^*$, $\exists!\, y \in H$ s.t. $\varphi = \varphi_y$.

③ $y \longmapsto \varphi_y$ is a conjugate linear isometric isomorphism $H \to H^*$.

Pf: ① Clearly φ_y is linear. By C-S: $|\varphi_y(x)| = |\langle x,y\rangle| \le \|x\|\cdot\|y\|$, so $\|\varphi_y\| \le \|y\|$. Taking $x=y$, $|\varphi_y(y)| = |\langle y,y\rangle| = \|y\|^2$, so $\|\varphi_y\| = \|y\|$.

② If $\varphi_y = \varphi_{y'}$, then $\|y-y'\|^2 = \langle y-y', y-y'\rangle = \varphi_y(y-y') - \varphi_{y'}(y-y') = 0$, so $y=y'$. Suppose $\varphi \in H^*$. If $\varphi = 0$, $y=0$ works. Assume $\varphi \ne 0$. Then $M := \{x\in H \mid \varphi(x) = 0\} \subseteq H$ is a closed proper subspace. Pick $z \in M^\perp$ w/ $\varphi(z) = 1$. $\forall x \in H$, $x - \varphi(x)z \in M$, so $\langle x, z\rangle = \langle x - \varphi(x)z + \varphi(x)z, z\rangle = \langle \varphi(x)z, z\rangle = \varphi(x)\|z\|^2$.
Hence $\varphi = \varphi_{\frac{z}{\|z\|^2}}$.

③ $y \mapsto \varphi_y$ is isometric by ① and onto by ②. Finally, observe
$$\varphi_{\alpha x+y}(z) = \langle z, \alpha x + y\rangle = \bar\alpha\langle z, x\rangle + \langle z, y\rangle = [\bar\alpha\varphi_x + \varphi_y](z).$$
Hence the map is a conjugate-linear isometric iso.

Exercise: H^* w/ $\langle \varphi_x, \varphi_y\rangle := \langle y, x\rangle$ is a Hilbert space.

Def: A subset $E \subset H$ is orthonormal if $e,f \in E \Rightarrow \langle e,f\rangle = \delta_{e=f}$.

Observe $\|e-f\| = \sqrt{2}$ $\forall e\ne f$ in E. Thus H separable $\Rightarrow$ any o.n. set is countable.

Exercise: Suppose $E \subset H$ is o.n. and $\{e_1,\dots,e_n\} \subset E$.

① If $x = \sum_{i=1}^n c_i e_i$, $c_j = \langle x, e_j\rangle$ $\forall j$ and $\|x\|^2 = \sum_{i=1}^n |c_i|^2$.

② $\{e_1,\dots,e_n\}$ is linearly independent.

③ $\forall x \in H$, $\sum_{i=1}^n \langle x, e_i\rangle e_i$ is the !elt of $M := \operatorname{span}\{e_1,\dots,e_n\}$ s.t. $\|x - \sum_{i=1}^n \langle x, e_i\rangle e_i\| = \inf_{m\in M}\|x-m\|$.

④ (Bessel's Inequality) $\forall x \in H$, $\|x\|^2 \ge \sum_{i=1}^n |\langle x, e_i\rangle|^2$.

Thm: For an O.N. set $E \subseteq H$, TFAE:

① E is maximal [E an **ONB**: orthonormal basis]

② $M := \{$finite linear comb. of elts of $E\}$ is dense in H

③ $\langle x, e\rangle = 0 \ \forall e \in H \Rightarrow x = 0$.

④ $\forall x \in H$, $x = \sum_{e} \langle x, e\rangle e$, where the sum on the right

- has at most countably many non-zero terms, and
- converges in the norm topology regardless of ordering.

⑤ $\forall x \in H$, $\|x\|^2 = \sum_{e} |\langle x, e\rangle|^2$

Pf: ① ⇒ ②: If $M \subseteq H$ not dense, $\exists e \in M^{\perp}$ w/ $\|e\| = 1$. Then $E \subsetneq E \cup \{e\}$ is O.N.

② ⇒ ③: Observe $\langle e, x\rangle = 0 \ \forall e \in E$, so $\varphi_x|_M = 0$. Since M is dense, $\varphi_x = 0$. Then $x = 0$.

③ ⇒ ①: Let $x \in E^{\perp}$. Then $\langle x, e\rangle = 0 \ \forall e \in E$, so $x = 0$. So E maximal.

③ ⇒ ④: $\forall e_1, \dots, e_n \in E$, $\|x\|^2 \geq \sum^n |\langle x, e_i\rangle|^2$, so $\forall$ countable subset $F \subset E$, $\|x\|^2 \geq \sum_f |\langle x, f\rangle|^2$. Hence $\{e \in E \mid \langle x, e\rangle \neq 0\}$ is countable. Let (e_i) be an enumeration. Then

$$\left\| \sum_m^n \langle x, e_i\rangle e_i \right\|^2 = \sum_m^n |\langle x, e_i\rangle|^2 \longrightarrow 0 \quad \text{as } m, n \to \infty.$$

So $\sum \langle x, e_i\rangle e_i$ converges as H is complete. Setting

$y := x - \sum \langle x, e_i\rangle e_i$, $\langle y, e\rangle = 0 \ \forall e \in E \Rightarrow y = 0$.

④ ⇒ ⑤: $\|x\|^2 - \sum^n |\langle x, e_i\rangle|^2 = \left\| x - \sum^n \langle x, e_i\rangle e_i \right\|^2 \longrightarrow 0$ as $n \to \infty$.

⑤ ⇒ ③: Immediate.

Facts: ① Every O.N. set can be extended to an ONB.

② H separable $\Leftrightarrow$ $\exists$ countable ONB.

③ $H \cong K$ $\Leftrightarrow$ they have ONB's of same size.

④ If $E \subset H$ an ONB, $H \cong \ell^2(E, \text{counting measure})$.