3. Integration

3.1. Measurable functions.

Definition 3.1.1. If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, we say $f : X \to Y$ is $(\mathcal{M} - \mathcal{N})$ *measurable* if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

Exercise 3.1.2. Prove the following assertions.

(1) Given $f : X \to Y$ and a σ-algebra $\mathcal{N}$ on Y, $\{f^{-1}(E) | E \in \mathcal{N}\}$ is a σ-algebra on X. Moreover it is the smallest σ-algebra on X such that f is measurable.

(2) Given $f : X \to Y$ and a σ-algebra $\mathcal{M}$ on X, $\{E \subset Y | f^{-1}(E) \in \mathcal{M}\}$ is a σ-algebra on Y. Moreover it is the largest σ-algebra on Y such that f is measurable.

Exercise 3.1.3. Prove that the composite of two measurable functions is measurable. More precisely, if $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is $\mathcal{M} - \mathcal{N}$ measurable and $g : (Y, \mathcal{N}) \to (Z, \mathcal{P})$ is $\mathcal{N} - \mathcal{P}$ measurable, then $g \circ f$ is $\mathcal{M} - \mathcal{P}$ measurable. Deduce that measurable spaces and measurable functions form a category.

Proposition 3.1.4. *Suppose* $(X, \mathcal{M})$ *and* $(Y, \mathcal{N})$ *are measurable spaces,* $f : X \to Y$*, and* $\mathcal{N} = \langle \mathcal{E} \rangle$ *for some* $\mathcal{E} \subset P(Y)$*. Then* f *is measurable if and only if* $f^{-1}(E) \in \mathcal{M}$ *for all* $E \in \mathcal{E}$*.*

Proof. The forward direction is trivial. Suppose $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$. Then $\mathcal{E}$ is contained in the σ-algebra $\mathcal{N}_f$ on Y co-induced by $\mathcal{M}, f$, i.e., the largest σ-algebra such that f is measurable. Since $\mathcal{N}_f$ is a σ-algebra containing $\mathcal{E}$, we see that $\mathcal{N}_f$ contains $\mathcal{N}$. Since f is $\mathcal{M} - \mathcal{N}_f$ measurable, f is $\mathcal{M} - \mathcal{N}$ measurable. □

Exercise 3.1.5. Show that every monotone increasing function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

Definition 3.1.6. Suppose X, Y are topological spaces. We call $f : X \to Y$ *Borel measurable* if it is $\mathcal{B}_X - \mathcal{B}_Y$ measurable.

Corollary 3.1.7. *Continuous functions are Borel measurable.*

Proof. Observe $f : X \to Y$ is continuous if and only if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X \subset \mathcal{B}_X$. This implies f is Borel measurable by Proposition 3.1.4. □

Corollary 3.1.8. *Suppose* $(X, \mathcal{M})$ *is a measurable space and* $f : X \to \mathbb{R}$ *(where* $\mathbb{R}$ *is equipped with the Borel* σ*-algebra). The following are equivalent:*

(1) f *is* $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ *measurable.*
(2) $f^{-1}(a, \infty) \in \mathcal{M}$ *for all* $a \in \mathbb{R}$*.*
(3) $f^{-1}[a, \infty) \in \mathcal{M}$ *for all* $a \in \mathbb{R}$*.*
(4) $f^{-1}(-\infty, a) \in \mathcal{M}$ *for all* $a \in \mathbb{R}$*.*
(5) $f^{-1}(-\infty, a] \in \mathcal{M}$ *for all* $a \in \mathbb{R}$*.*

Observe that we can also use collections of intervals $(a, b), [a, b), (a, b], [a, b]$ *for all* $a, b \in \mathbb{R}$*.*

Corollary 3.1.9. *If* $(X, \mathcal{M})$ *is a measurable space and* $f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$*, then Corollary 3.1.8 holds replacing* $\mathbb{R}$ *with* $\overline{\mathbb{R}}$ *and intervals excluding* $\pm\infty$ *with intervals including* $\pm\infty$ *respectively.*

Proof. Use Exercise 2.1.12. □

Definition 3.1.10. Suppose $(X, \mathcal{M})$ is a measurable space. We say a function $f : X \to \mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}$ is $\mathcal{M}$-*measurable* if f is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$, $\mathcal{M} - \mathcal{B}_{\overline{\mathbb{R}}}$, $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable respectively.

Warning 3.1.11. If $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable (i.e., $\mathcal{L} - \mathcal{B}_{\mathbb{R}}$ measurable), then $f \circ g$ need not be Lebesgue measurable!

Exercise 3.1.12. Find examples of $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable with $f \circ g$ not Lebesgue measurable.
Note: First find an $E \in \mathcal{L} \setminus \mathcal{B}_{\mathbb{R}}$ and an $\mathcal{L}$-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(E) \notin \mathcal{L}$. Then set $g := \chi_E$.

Exercise 3.1.13. Suppose $(X, \mathcal{M})$ is a measurable space and X, Y are topological spaces, $i : Y \to Z$ is a continuous injection which maps open sets to open sets, and $f : X \to Y$. (For example, $Y = \mathbb{R}$ and $Z = \overline{\mathbb{R}}$.)

$$\begin{array}{ccc} & & Z \\ & \overset{i\circ f}{\nearrow} & \uparrow i \\ X & \xrightarrow{f} & Y \end{array}$$

Show that f is $\mathcal{M} - \mathcal{B}_Y$ measurable if and only if $i \circ f$ is $\mathcal{M} - \mathcal{B}_Z$ measurable. Deduce that if $f : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ only takes values in $\mathbb{R}$, then f is $\mathcal{M} - \mathcal{B}_{\overline{\mathbb{R}}}$ measurable if and only if f is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable. Hence we can say f is $\mathcal{M}$-measurable without any confusion.

Exercise 3.1.14. Let $(X, \mathcal{M})$ be a measurable space.

(1) Prove that the Borel σ-algebra $\mathcal{B}_{\mathbb{C}}$ on $\mathbb{C}$ is generated by the 'open rectangles'
$$\{z \in \mathbb{C} | a < \operatorname{Re}(z) < b \text{ and } c < \operatorname{Im}(z) < d\}.$$
(2) Prove directly from the definitions that $f : X \to \mathbb{C}$ is $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable if and only if $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable.

Definition 3.1.15. Suppose $(X, \mathcal{M}, \mu)$ is a measure space. We say that a property P of a measurable function f from X into $\mathbb{R}$, $\overline{\mathbb{R}}$, or $\mathbb{C}$ holds *almost everywhere (a.e.)* if there is a μ-null set $E \in \mathcal{M}$ such that P holds on E^c. For example, $f \geq 0$ a.e. if there is a μ-null set $E \in \mathcal{M}$ such that $f|_{E^c} \geq 0$.

Exercise 3.1.16. Define a relation on the set of $\mathcal{M}$-measurable functions (into $\mathbb{R}$, $\overline{\mathbb{R}}$, or $\mathbb{C}$) by $f \sim g$ if and only if $f = g$ a.e. Prove $\sim$ is an equivalence relation.

Notation 3.1.17. Given $f : X \to \overline{\mathbb{R}}$, we write $\{a < f\} := f^{-1}(a, \infty]$. We define $\{a \leq f\}, \{f < b\}, \{f \leq b\}, \{a < f < b\}$, etc. similarly.

Facts 3.1.18. Suppose $(X, \mathcal{M})$ is a measurable space and $f, g : X \to \overline{\mathbb{R}}$ are $\mathcal{M}$-measurable. The following functions are all $\mathcal{M}$-measurable:

($\mathcal{M}$-meas1) $(f \vee g)(x) := \max\{f(x), g(x)\}$ and $(f \wedge g)(x) := \min\{f(x), g(x)\}$

Proof. If $a \in \mathbb{R}$, then
$$\{a < f \vee g\} = \{a < f\} \cup \{a < g\} \in \mathcal{M}$$
$$\{a < f \wedge g\} = \{a < f\} \cap \{a < g\} \in \mathcal{M}. \qquad \square$$

($\mathcal{M}$-meas2) any well-defined linear combination of f, g, where by convention, $0 \cdot \pm\infty = 0$ and $\pm\infty \pm \infty = \pm\infty$, but $\pm\infty \mp \infty$ is not defined.

Proof.
Step 1: For $a, c \in \mathbb{R}$,

$$\{cf > a\} = \begin{cases} \emptyset & \text{if } c = 0 \leq a \\ X & \text{if } c = 0 > a \\ \left\{\frac{a}{c} < f\right\} & \text{if } c > 0 \\ \left\{\frac{a}{c} > f\right\} & \text{if } c < 0 \end{cases} \qquad \text{which are all in } \mathcal{M}.$$

Step 2: If $f + g$ is well-defined, then for $a \in \mathbb{R}$,

$$\{a < f + g\} = \bigcup_{\substack{r,s\in\mathbb{Q} \\ a<r+s}} \{r < f\} \cap \{s < g\} \in \mathcal{M}.$$ □

($\mathcal{M}$-meas3) fg

Proof.
Step 1: Suppose f, g are non-negative. Then for all $a \geq 0$,

$$\{a < fg\} = \bigcup_{\substack{r,s\in\mathbb{Q}_{>0} \\ a<rs}} \{r < f\} \cap \{s < g\} \in \mathcal{M}.$$

Also, for all $a < 0$, $\{a < fg\} = X \in \mathcal{M}$.
Step 2: For f, g arbitrary, we use the following trick:

Trick. $f = f_+ - f_-$ where $f_+ := f \vee 0$ and $f_- := -(f \wedge 0)$. Observe that $f_\pm \cdot f_\mp = 0$.

Similarly, we can write $g = g_+ - g_-$. Then

$$fg = f_+g_+ - f_-g_+ - f_+g_- + f_-g_-,$$

all of which have disjoint support. Hence each of the summands of fg is measurable by Step 1, and the linear combination is measurable by (3) as it is well-defined. □

Exercise 3.1.19. Suppose $f : X \to \overline{\mathbb{R}}$. Show that $f = f_+ - f_-$ is the unique decomposition of f as $g - h$ such that $g, h \geq 0$ and $gh = 0$.

Exercise 3.1.20. Let $(X, \mathcal{M})$ be a measurable space.

(1) Prove that the $\mathcal{M} - \mathcal{B}_\mathbb{C}$ measurable functions form a $\mathbb{C}$-vector space.
(2) Show that if $f : X \to \mathbb{C}$ is $\mathcal{M} - \mathcal{B}_\mathbb{C}$ measurable, then $|f| : X \to [0, \infty)$ is $\mathcal{M} - \mathcal{B}_\mathbb{R}$ measurable.
(3) Show that if (f_n) is a sequence of $\mathcal{M} - \mathcal{B}_\mathbb{C}$ measurable functions $X \to \mathbb{C}$ and $f_n \to f$ pointwise, then f is $\mathcal{M} - \mathcal{B}_\mathbb{C}$ measurable.

Facts 3.1.21. Suppose (f_n) is a sequence of $\mathcal{M}$-measurable functions $X \to \overline{\mathbb{R}}$. The following functions are $\mathcal{M}$-measurable.

($\mathcal{M}$-meas4) $\sup f_n$ and $\inf f_n$

Proof. For all $a \in \mathbb{R}$,
$$\{a < \sup f_n\} = \bigcup_n \{a < f_n\} \in \mathcal{M}$$
$$\{a < \inf f_n\} = \bigcap_n \{a < f_n\} \in \mathcal{M}. \qquad \square$$

($\mathcal{M}$-meas5) $\limsup f_n$ and $\liminf f_n$

Proof. Observe that
$$\limsup f_n = \lim_{n\to\infty} \sup_{k>n} f_k = \inf_n \underbrace{\sup_{k>n} f_k}_{\text{measurable by } (\mathcal{M}\text{-meas4})}$$
$$\liminf f_n = \lim_{n\to\infty} \inf_{k>n} f_k = \sup_n \underbrace{\inf_{k>n} f_k}_{\text{measurable by } (\mathcal{M}\text{-meas4})}$$
Applying ($\mathcal{M}$-meas4) again, we see that $\limsup f_n$ and $\liminf f_n$ are $\mathcal{M}$-measurable. $\square$

3.2. **Measurable simple functions.** For this section, fix a measurable space $(X, \mathcal{M})$.

Definition 3.2.1. An $\mathcal{M}$-measurable function $\psi : X \to \mathbb{R}$ is *simple* if it takes finitely many values. Observe that if ψ is simple, we can write
$$\psi = \sum_{k=1}^{n} c_k \chi_{E_k} \qquad c_1, \ldots, c_n \in \mathbb{R} \qquad E_1, \ldots, E_n \in \mathcal{M}.$$
Here, we write χ_E for the *characteristic function* of E:
$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in E^c. \end{cases}$$
Observe that there is exactly one such expression of a simple function, called its *standard form*, such that

- $c_1, \ldots, c_n$ are distinct, and
- $E_1, \ldots, E_n$ are disjoint and non-empty such that $X = \coprod_{k=1}^n E_k$.

Denote by SF the collection of simple ($\mathcal{M}$-measurable) functions. Define $\mathsf{SF}^+ := \{\psi \in \mathsf{SF} | \psi \geq 0\}$.

Exercise 3.2.2. Verify the uniqueness of standard form of an simple function.

Exercise 3.2.3.

(1) Prove that SF is an $\mathbb{R}$-algebra and SF^+ is closed under addition, multiplication, and non-negative scalar multiplication.
(2) Prove SF is a lattice (closed under max and min) and $\mathsf{SF}^+ \subset \mathsf{SF}$ is a sublattice.

Proposition 3.2.4. *Suppose $f : X \to [0,\infty]$ is $\mathcal{M}$-measurable. There is a sequence $(\psi_n) \subset \mathsf{SF}^+$ such that*
- *$\psi_n(x) \nearrow f(x)$ for all $x \in X$, and*
- *for all $N \in \mathbb{N}$, $\psi_n \to f$ uniformly on $\{f \leq N\}$.*

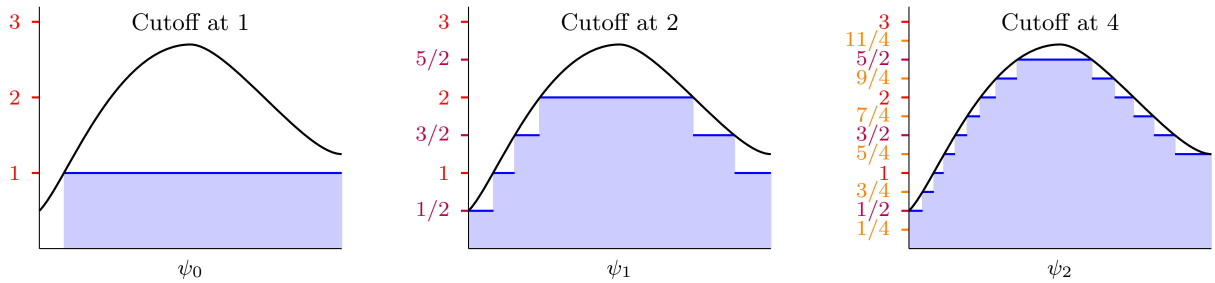
Proof. For $n \geq 0$ and $1 \leq k \leq 2^{2n}$, set

$$E_n^k := f^{-1}\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right] \qquad \text{and} \qquad F_n := f^{-1}(2^n, \infty].$$

Observe that $X = f^{-1}(0) \amalg F_n \amalg \coprod_{k=1}^{2^{2n}} E_n^k$. Define

$$\psi_n := 2^n \chi_{F_n} + \sum_{k=1}^{2^{2n}} \frac{k-1}{2^n} \chi_{E_n^k}.$$

Here is a cartoon of ψ_0, ψ_1, ψ_2:



Observe that $\psi_n \leq \psi_{n+1}$ for all $n \geq 0$, and $0 \leq f - \psi_n \leq 2^{-n}$ on $\{f \leq 2^n\}$. The result follows. $\square$

Exercise 3.2.5. Let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be the completion of the measure space $(X, \mathcal{M}, \mu)$.

(1) Show that if f is $\overline{\mathcal{M}}$-measurable and $g = f$ a.e., then g is $\overline{\mathcal{M}}$-measurable.
Optional: Does this hold with $\overline{\mathcal{M}}$ replaced by $\mathcal{M}$?

(2) Show that if f is $\overline{\mathcal{M}}$-measurable, there exists an $\mathcal{M}$-measurable g such that $f = g$ a.e.
Hint: First do the case f is $\mathbb{R}$-valued.

(3) Show that if (f_n) is a sequence of $\overline{\mathcal{M}}$-measurable functions and $f_n \to f$ a.e., then f is $\overline{\mathcal{M}}$-measurable.
Optional: Does this hold with $\overline{\mathcal{M}}$ replaced by $\mathcal{M}$?

(4) Show that if (f_n) is a sequence of $\mathcal{M}$-measurable functions and $f_n \to f$ a.e., then f is $\overline{\mathcal{M}}$-measurable. Deduce that there is an $\mathcal{M}$-measurable function g such that $f = g$ a.e., so $f_n \to g$ a.e.

For all parts, consider the cases of $\mathbb{R}$, $\overline{\mathbb{R}}$, and $\mathbb{C}$-valued functions.

3.3. **Integration of non-negative functions.**

For this section, fix a measure space $(X, \mathcal{M}, \mu)$. Define

$$L^+ := L^+(X, \mathcal{M}, \mu) = \{\mathcal{M}\text{-measurable } f : X \to [0,\infty]\}.$$

Definition 3.3.1. For $\psi = \sum_{k=1}^n c_k \chi_{E_k} \in \mathsf{SF}^+ \subset L^+$ in standard form, define

$$\int \psi := \int_X \psi \, d\mu := \int_X \psi(x)\, d\mu(x) := \sum_{k=1}^n c_k \mu(E_k).$$

For $E \in \mathcal{M}$, we define $\int_E \psi := \int \psi \cdot \chi_E$. Observe that to calculate $\int_E \psi$, we must write the simple function $\psi \cdot \chi_E$ in standard form.

We say that $\psi \in \mathsf{SF}^+$ is *integrable* if $\int \psi < \infty$. We write $\mathsf{ISF}^+ := \{\psi \in \mathsf{SF}^+ | \psi \text{ integrable}\}$.

Exercise 3.3.2. Suppose $f : (X, \mathcal{M}, \mu) \to [0, \infty]$ is $\mathcal{M}$-measurable and $\{f > 0\}$ is σ-finite. Show that there exists a sequence of $(\psi_n) \subset \mathsf{ISF}^+$ such that $\psi_n \nearrow f$ pointwise.
Optional: In what sense can you say $\psi_n \nearrow f$ uniformly?

Theorem 3.3.3. *The map $\int : \mathsf{SF}^+ \to [0, \infty]$ satisfies*

(1) *(homogeneous) for all $r \geq 0$, $\int r\psi = r \int \psi$.*
(2) *(monotone) if $\phi \leq \psi$ everywhere, then $\int \phi \leq \int \psi$.*
(3) *(additive) $\int \phi + \psi = \int \phi + \int \psi$.*

Hence $\int : \mathsf{SF}^+ \to [0, \infty]$ is an order-preserving $\mathbb{R}^+$-linear functional.

Proof.
(1) Observe if $r = 0$, then $\int r\psi = 0 = 0 \cdot \int \psi$. If $r > 0$ and $\psi = \sum^n c_k \chi_{E_k}$, then $r\psi = \sum^n r c_k \chi_{E_k}$ is in standard form, and

$$\int r\psi = \sum^n r c_k \mu(E_k) = r \sum^n c_k \mu(E_k) = r \int \psi.$$

(2) Suppose that $\phi = \sum^m a_j \chi_{E_j}$ and $\psi = \sum^n b_k \chi_{F_k}$ are in standard form. Here is the trick:

Trick. Since $X = \coprod^m E_j = \coprod^n F_k$, we have $E_j = \coprod_{k=1}^n E_j \cap F_k$ and $F_k = \coprod_{j=1}^m E_j \cap F_k$.

Since $\phi \leq \psi$ everywhere,

$$\phi = \sum_{j,k} a_j \chi_{E_j \cap F_k} \leq \sum_{j,k} b_k \chi_{E_j \cap F_k} = \psi,$$

and so $a_j \leq b_k$ whenever $E_j \cap F_k \neq \emptyset$. Thus

$$\int \phi = \sum_{j=1}^m a_j \mu(E_j) = \sum_{j=1}^m \sum_{k=1}^n a_j \mu(E_j \cap F_k) \leq \sum_{k=1}^n \sum_{j=1}^m b_k \mu(E_j \cap F_k) = \sum_{k=1}^n b_k \mu(F_k) = \int \psi.$$

(3) Suppose that $\phi = \sum^m a_j\chi_{E_j}$, $\psi = \sum^n b_k\chi_{F_k}$, and $\phi+\psi = \sum_{\ell=1}^p c_\ell\chi_{G_\ell}$ are in standard form. Similar to the argument in (2) above, $a_j + b_k = c_\ell$ whenever $E_j \cap F_k \cap G_\ell \neq \emptyset$. Then

$$\begin{aligned}\int \phi + \int \psi &= \sum_j a_j\mu(E_j) + \sum_k b_k\mu(F_k)\\ &= \sum_{j,k}(a_j+b_k)\mu(E_j\cap F_k)\\ &= \sum_{j,k,\ell}(a_j+b_k)\mu(E_j\cap F_k\cap G_\ell)\\ &= \sum_{j,k,\ell} c_\ell\mu(E_j\cap F_k\cap G_\ell)\\ &= \sum_\ell c_\ell\mu(G_\ell)\\ &= \int \phi+\psi.\end{aligned}$$

□

Remark 3.3.4. Observe that the map $\mathcal{M} \to [0,\infty]$ by $E \mapsto \int_E d\mu$ equals μ.

Lemma 3.3.5. *For $\psi \in \mathsf{SF}^+$, $\mu_\psi : \mathcal{M} \to [0,\infty]$ by $E \mapsto \int_E \psi$ is a measure.*

Proof.
(0) Observe that $\psi\chi_\emptyset = 0$, so

$$\mu_\psi(\emptyset) = \int_\emptyset \psi = \int \psi\chi_\emptyset = \int 0 = 0.$$

(1) Write $\psi = \sum_{j=1}^m a_j\chi_{E_j}$ in standard form. If $(F_n) \subset \mathcal{M}$ is a disjoint sequence, then observe $\psi\chi_{\coprod F_n} = \sum_{j=1}^m a_j\chi_{E_j\cap\coprod F_n}$ is also in standard form (up to a subset of $\{\psi\chi_{\coprod F_n} = 0\}$).

$$\begin{aligned}\mu_\psi\left(\coprod F_n\right) &= \int_{\coprod F_n}\psi\\ &= \int \psi\chi_{\coprod F_n}\\ &= \sum_j a_j\mu(E_j\cap \coprod F_n)\\ &= \sum_{j,n} a_j\mu(E_j\cap F_n)\\ &= \sum_n \int_{F_n}\psi.\end{aligned}$$

□

Definition 3.3.6. For $f \in L^+$, define

$$\int f := \int_X f\,d\mu := \int_X f(x)\,d\mu(x) := \sup\left\{\int \psi \middle| \psi \in \mathsf{SF}^+ \text{ such that } 0 \le \psi \le f\right\}.$$

Remarks 3.3.7.

(1) Observe that for $\psi \in \mathsf{SF}^+$, we have
$$\int \psi = \sup\left\{ \int \phi \middle| \phi \in \mathsf{SF}^+ \text{ such that } 0 \leq \phi \leq \psi \right\}.$$
Hence the above definition extends $\int \psi$ for $\psi \in \mathsf{SF}^+$ to $f \in L^+$.

(2) If $f, g \in L^+$ with $f \leq g$, then $\int f \leq \int g$ as we are taking sup over a larger set.

(3) If $f \in L^+$ and $r \in (0, \infty)$, then $\int rf = r \int f$, since if $S \subset [0, \infty]$, $\sup rS = r \cdot \sup S$. (Remember that $0 \cdot \infty = 0$.)

Proposition 3.3.8. *Suppose $f \in L^+$. The following are equivalent.*

(1) $\int f = 0$, and

(2) $f = 0$ a.e., i.e., there is a μ-null set $E \in \mathcal{M}$ such that $f|_{E^c} = 0$.

Proof.

(1) ⇒ (2): We'll prove the contrapositive. If f is not zero a.e., there is an $n > 0$ such that $\mu(\{\frac{1}{n} < f\}) > 0$. Then $f > \frac{1}{n}\chi_{\{\frac{1}{n} < f\}}$, so
$$0 < \frac{1}{n} \cdot \mu\left(\left\{\frac{1}{n} < f\right\}\right) = \int \frac{1}{n}\chi_{\{\frac{1}{n}<f\}} \leq \int f.$$
(2) ⇒ (1): First, if $f = \sum_{k=1}^n c_k \chi_{E_k} \in \mathsf{SF}^+$ is in standard form, then $\int f = 0$ if and only if $\mu(E_k) = 0$ for all k such that $c_k \neq 0$ if and only if $f = 0$ a.e. Second, if $f \in L^+$ with $f = 0$ a.e., then for all $\psi \in \mathsf{SF}^+$ with $0 \leq \psi \leq f$, $\psi = 0$ a.e., so $\int f = \sup_{0 \leq \psi \leq f} \int \psi = 0$. □

Theorem 3.3.9 (Monotone Convergence, a.k.a MCT)**.** *Suppose $(f_n) \subset L^+$ is an increasing sequence and $f = \lim f_n = \sup f_n$. Then*
$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof.

≤: Observe $(\int f_n)$ is increasing in $[0, \infty]$, and thus it converges. Moreover, $\int f_n \leq \int f$ for all n, so $\lim_{n\to\infty} \int f_n \leq \int f$.

≥: Pick a $\psi \in \mathsf{SF}^+$ with $0 \leq \psi \leq f$ and $0 < \varepsilon < 1$. Set $E_n := \{\varepsilon\psi < f_n\}$. Then observe $(E_n) \subset \mathcal{M}$ is an increasing sequence such that $\bigcup E_n = X$, so by continuity from below (μ3), $\int_{E_n} \psi \nearrow \int \psi$. Thus
$$\int f_n \geq \int_{E_n} f_n \geq \varepsilon \int_{E_n} \psi \xrightarrow{n\to\infty} \varepsilon \int \psi.$$
Hence $\lim \int f_n \geq \varepsilon \int \psi$ for all $0 < \varepsilon < 1$. Since ε was arbitrary, letting $\varepsilon \to 1$, we have $\lim \int f_n \geq \int \psi$. Taking sup over all $0 \leq \psi \leq f$ gives $\lim \int f_n \geq \int f$. □

Facts 3.3.10 (Corollaries of the MCT)**.**

(MCT1) If $f \in L^+$, then $\int f = \lim \int \psi_n$ for all sequences $(\psi_n) \subset \mathsf{SF}^+$ such that $\psi_n \nearrow f$.

(MCT2) For all $f, g \in L^+$, $\int f + g = \int f + \int g$.

Proof. If $\phi_n \nearrow f$ and $\psi_n \nearrow g$, then $\phi_n + \psi_n \nearrow f + g$, so
$$\int f + g \underset{\text{(MCT)}}{=} \lim \int \phi_n + \psi_n = \lim \int \phi_n + \lim \int \psi_n = \int f + \int g.$$ □

(MCT3) For $f, g \in L^+$, if $f = g$ a.e., then $\int f = \int g$.

Proof. Let $E \in \mathcal{M}$ such that $f\chi_E = g\chi_E$ and E^c is μ-null. Then

$$\int f \underset{\text{(MCT2)}}{=} \int f\chi_E + \int f\chi_{E^c} = \int f\chi_E = \int g\chi_E = \int g\chi_E + \int g\chi_{E^c} \underset{\text{(MCT2)}}{=} \int g.$$

□

(MCT4) For all $(f_n) \subset L^+$, $\sum \int f_n = \int \sum f_n$, where $\sum f_n$ is the sup of the sequence of partial sums (which is a measurable function).

Proof. Observe

$$\int \sum f_n = \int \lim_{N\to\infty} \sum^N f_n \underset{\text{(MCT)}}{=} \lim_{N\to\infty} \int \sum^N f_n \underset{\text{(MCT2)}}{=} \lim_{N\to\infty} \sum^N \int f_n = \sum \int f_n.$$

□

(MCT5) If $(f_n) \subset L^+$, $f_n \nearrow f$ a.e., and $f \in L^+$ (which is automatic if μ is complete), then $\int f = \lim \int f_n$.

Proof. Suppose $f_n \nearrow f$ on $E \in \mathcal{M}$ and E^c is μ-null. Then

$$\int f \underset{\text{(MCT3)}}{=} \int f\chi_E \underset{\text{(MCT)}}{=} \lim \int f_n\chi_E \underset{\text{(MCT3)}}{=} \lim \int f_n.$$

□

(MCT6) (Fatou's Lemma) If $(f_n) \subset L^+$, then $\int \liminf f_n \leq \liminf \int f_n$.

Proof. For all $j \geq k \in \mathbb{N}$, $\inf_{n\geq k} f_n \leq f_j$, so

$$\int \inf_{n\geq k} f_n \leq \int f_j \qquad \text{for all} \qquad j \geq k.$$

Thus $\int \inf_{n\geq k} f_n \leq \inf_{j\geq k} \int f_j$. Letting $k \to \infty$, we have

$$\int \liminf f_n \underset{\text{(MCT)}}{=} \lim_{k\to\infty} \int \inf_{n\geq k} f_n \leq \lim_{k\to\infty} \inf_{j\geq k} \int f_j = \liminf \int f_n.$$

□

(MCT7) If $(f_n) \subset L^+$, $f_n \to f$ a.e., and $f \in L^+$ (which is automatic if μ is complete), then $\int f \leq \liminf \int f_n$.

Proof. Let $E \in \mathcal{M}$ such that $f_n \to f$ on E and E^c is μ-null. Then

$$\int f \underset{(3)}{=} \int f\chi_E \underset{\text{(MCT6)}}{\leq} \liminf \int f_n\chi_E \underset{\text{(MCT3)}}{=} \liminf \int f_n.$$

□

Exercise 3.3.11. Assume Fatou's Lemma (MCT6) and prove the Monotone Convergence Theorem from it.

Exercise 3.3.12. If $f \in L^+$ and $\int f < \infty$, then $\{f = \infty\}$ is μ-null and $\{0 < f\}$ is σ-finite.

Exercise 3.3.13. Suppose $(X, \mathcal{M}, \mu)$ is a finite measure space and $f : X \to \mathbb{C}$ is measurable. Prove that $\mu(\{n \leq |f|\}) \to 0$ as $n \to \infty$.

Exercise 3.3.14. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f \in L^+$. For $E \in \mathcal{M}$, define

$$\nu(E) := \int_E f \, d\mu.$$

(1) Prove that ν is a measure on $\mathcal{M}$.
(2) Prove that $\int g \, d\nu = \int fg \, d\mu$ for all $g \in L^+$
Hint: First suppose g is simple.

3.4. **Integration of $\overline{\mathbb{R}}$-valued functions.** For this section, $(X, \mathcal{M}, \mu)$ is a fixed measure space.

Definition 3.4.1. An $\mathcal{M}$-measurable function $f : X \to \overline{\mathbb{R}}$ is called *integrable* if $\int f_\pm < \infty$ where $f = f_+ - f_-$ with $f_+ = 0 \vee f$ and $f_- = -(0 \wedge f)$. Since $|f| = f_+ + f_-$, observe that f is integrable if and only if $\int |f| < \infty$.

Define $L^1(\mu, \overline{\mathbb{R}}) := \{\text{integrable } f : X \to \overline{\mathbb{R}}\}$ and $L^1(\mu, \mathbb{R}) := \{\text{integrable } f : X \to \mathbb{R}\}$

Exercise 3.4.2. Show that a simple function $\psi = \sum_{k=1}^n c_k \chi_{E_k} \in \mathsf{SF}$ with $c_1, \ldots, c_n$ distinct and $E_1, \ldots, E_n$ disjoint is integrable if and only if $\mu(E_k) < \infty$ for all k such that $c_k \neq 0$.

Proposition 3.4.3. *The set $L^1(\mu, \overline{\mathbb{R}})$ is an $\mathbb{R}$-vector space, and $L^1(\mu, \mathbb{R})$ is a subspace. Moreover, $\int : L^1(\mu, \overline{\mathbb{R}}) \to \mathbb{R}$ given by $\int f := \int f_+ - \int f_-$ is a linear functional.*

Proof. If $r \in \mathbb{R}$ and $f, g \in L^1(\mu, \overline{\mathbb{R}})$, then $|rf + g| \leq |r| \cdot |f| + |g|$ which is integrable. Hence $L^1(\mu, \overline{\mathbb{R}})$ is an $\mathbb{R}$-vector space. Clearly $L^1(\mu, \mathbb{R})$ is a subspace.

If $r \in \mathbb{R}$ and $f \in L^1(\mu, \overline{\mathbb{R}})$, then there are three cases:

$$(rf)_\pm = \begin{cases} rf_\pm & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ -rf_\mp & \text{if } r < 0. \end{cases}$$

In all three cases, by Remarks 3.3.7(3), we have

$$\int rf = \int (rf)_+ - \int (rf)_- = \left\{ \begin{matrix} \displaystyle\int rf_+ - \int rf_- & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ \displaystyle\int (-r)f_- - \int (-r)f_+ & \text{if } r < 0 \end{matrix} \right\} = r \int f_+ - r \int f_-.$$

If $f, g \in L^1(\mu, \overline{\mathbb{R}})$, observe

$$(f+g)_+ - (f+g)_- = f + g = f_+ + g_+ - f_- - g_-$$

which implies

$$(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+.$$

By (MCT2),

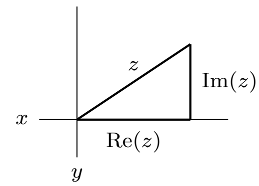
$$\int (f+g)_+ + \int f_- + \int g_- = \int (f+g)_- + \int f_+ + \int g_+,$$

and rearranging yields the result. $\square$

3.5. **Integration of $\mathbb{C}$-valued functions.** For this section, fix a measure space $(X, \mathcal{M}, \mu)$. Recall from Exercise 3.1.14(2) that $f : X \to \mathbb{C}$ is measurable if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable. By Exercise 3.1.20(2), $|f|$ is measurable.

Definition 3.5.1. A measurable function $f : X \to \mathbb{C}$ is *integrable* if $\int |f| < \infty$, i.e., $|f| \in L^1(\mu, \mathbb{R})$. Since

$$|f| \leq |\operatorname{Re}(f)| + |\operatorname{Im}(f)| \leq 2|f|$$



,

f is integrable if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are integrable. In this case, we define

$$\int f := \int \operatorname{Re}(f) + i \int \operatorname{Im}(f).$$

It follows from Proposition 3.4.3 that

$$L^1(\mu, \mathbb{C}) := \{\text{integrable } f : X \to \mathbb{C}\}$$

is a $\mathbb{C}$-vector space, and $\int : L^1(\mu, \mathbb{C}) \to \mathbb{C}$ is linear.

Proposition 3.5.2. *For all $f \in L^1(\mu, \mathbb{C})$, $\left|\int f\right| \leq \int |f|$.*

Proof.
Step 1: If f is $\mathbb{R}$-valued, then $\left|\int f\right| = \left|\int f_+ - \int f_-\right| \leq \int f_+ + \int f_- = \int |f|$.
Step 2: Suppose f is $\mathbb{C}$-valued. We may assume $\int f \neq 0$. We use the following trick:

Trick. Define $\operatorname{sgn}\left(\int f\right) := \frac{\int f}{\left|\int f\right|} \in \mathbb{T} := \{z \in \mathbb{C} \,|\, |z| = 1\}$. Then since $z^{-1} = \overline{z}$ for all $z \in \mathbb{T}$,

$$\left|\int f\right| = \overline{\operatorname{sgn}\left(\int f\right)} \int f = \underbrace{\int \overline{\operatorname{sgn}\left(\int f\right)} f}_{\in \mathbb{R}}.$$

We then calculate

$$\left|\int f\right| = \int \overline{\operatorname{sgn}\left(\int f\right)} f = \operatorname{Re} \int \overline{\operatorname{sgn}\left(\int f\right)} f = \int \operatorname{Re}\left(\overline{\operatorname{sgn}\left(\int f\right)} f\right)$$

$$\underset{(\text{Step 1})}{\leq} \int \left|\operatorname{Re}\left(\overline{\operatorname{sgn}\left(\int f\right)} f\right)\right| \leq \int \left|\underbrace{\overline{\operatorname{sgn}\left(\int f\right)}}_{\in \mathbb{T}} f\right| = \int |f|.$$

□

Corollary 3.5.3. *For all $f, g \in L^1(\mu, \mathbb{C})$, the following are equivalent:*

(1) $f = g$ a.e.
(2) $\int |f - g| = 0$
(3) for all $E \in \mathcal{M}$, $\int_E f = \int_E g$.

Proof.
(1) ⇔ (2) Observe $f = g$ a.e. if and only if $|f - g| = 0$ a.e. if and only if $\int |f - g| = 0$ by Proposition 3.3.8.
(2) ⇒ (3) By Proposition 3.5.2, for all $E \in \mathcal{M}$,

$$\left|\int_E f - \int_E g\right| = \left|\int (f-g)\chi_E\right| \le \int |f-g|\chi_E \le \int |f-g| = 0.$$

(3) ⇒ (1) Recall that $\int_E f - g = \int_E \operatorname{Re}(f-g) + i\int_E \operatorname{Im}(f-g)$. So by assumption,

$$\int_E \operatorname{Re}(f-g) = 0 \qquad \text{and} \qquad \int_E \operatorname{Im}(f-g) = 0 \qquad\qquad \forall\, E \in \mathcal{M}.$$

We now look at the following particular $E \in \mathcal{M}$:

$$\begin{aligned} E &= \{0 \le \operatorname{Re}(f-g)\} &\Rightarrow\quad& \operatorname{Re}(f-g)_+ = 0 \text{ a.e.}\\ E &= \{0 \ge \operatorname{Re}(f-g)\} &\Rightarrow\quad& \operatorname{Re}(f-g)_- = 0 \text{ a.e.}\\ E &= \{0 \le \operatorname{Im}(f-g)\} &\Rightarrow\quad& \operatorname{Im}(f-g)_+ = 0 \text{ a.e.}\\ E &= \{0 \ge \operatorname{Im}(f-g)\} &\Rightarrow\quad& \operatorname{Im}(f-g)_- = 0 \text{ a.e.} \end{aligned}$$

Hence $\operatorname{Re}(f-g) = 0$ and $\operatorname{Im}(f-g) = 0$ a.e., which is equivalent to $f = g$ a.e. □

Exercise 3.5.4. Suppose $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^1(\mu, \mathbb{C})$. Prove that $\{f \neq 0\}$ is σ-finite.

Exercise 3.5.5. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f \in L^1(\mu, \mathbb{C})$. Prove that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $E \in \mathcal{M}$ with $\mu(E) < \delta$, $\int_E |f| < \varepsilon$.

Definition 3.5.6. Define $\mathcal{L}^1(\mu, \mathbb{C}) := L^1(\mu, \mathbb{C})/\sim$ where $f \sim g$ if and only if $f = g$ a.e. We write $f \in \mathcal{L}^1(\mu, \mathbb{C})$ to mean $f \in L^1(\mu, \mathbb{C})$ representing its equivalence class in $\mathcal{L}^1(\mu, \mathbb{C})$.

Exercise 3.5.7. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Prove that $\|\cdot\|_1 : \mathcal{L}^1(\mu, \mathbb{C}) \to [0, \infty)$ given by $\|f\|_1 := \int |f|$ is a norm.
(2) Let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be the completion of $(X, \mathcal{M}, \mu)$. Find a canonical $\mathbb{C}$-vector space isomorphism $\mathcal{L}^1(\mu, \mathbb{C}) \cong \mathcal{L}^1(\overline{\mu}, \mathbb{C})$ which preserves $\|\cdot\|_1$.
Hint: Use Exercise 3.2.5.

Theorem 3.5.8 (Dominated Convergence, a.k.a. DCT)**.** *Suppose $(f_n) \subset \mathcal{L}^1(\mu, \mathbb{C})$ such that $f_n \to f$ a.e. If there is a $g \in L^1(\mu, \mathbb{C}) \cap L^+$ such that eventually $|f_n| \le g$ a.e., then $f \in \mathcal{L}^1(\mu, \mathbb{C})$ and $\int f = \lim \int f_n$.*

Proof. By redefining f on a μ-null set if necessary by Exercise 3.2.5, we may assume f is $\mathcal{M}$-measurable. Taking limits pointwise, $|f| \le g$, so $f \in L^1(\mu, \mathbb{C})$. Taking real and imaginary parts of f, we may assume $(f_n), f$ are all $\mathbb{R}$-valued. Then $-g \le f_n \le g$ a.e., so

$$g + f_n \ge 0 \qquad \text{and} \qquad g - f_n \ge 0 \qquad\qquad \text{a.e.}$$

By Fatou's Lemma (MCT6),

$$\begin{aligned} \int g + \int f = \int g + f \le \liminf \int g + f_n = \int g + \liminf \int f_n\\ \int g - \int f = \int g - f \le \liminf \int g - f_n = \int g - \limsup \int f_n. \end{aligned}$$

Combining these inequalities,

$$\limsup \int f_n \le \int f \le \liminf \int f_n. \qquad \square$$

Corollary 3.5.9. *Suppose $(f_n) \subset \mathcal{L}^1(\mu, \mathbb{C})$ such that $\sum \int |f_n| < \infty$. Then $\sum f_n$ converges a.e. to a function in $\mathcal{L}^1(\mu, \mathbb{C})$, and $\int \sum f_n = \sum \int f_n$.*

Exercise 3.5.10. Prove that the metric d_1 on $\mathcal{L}^1(\mu, \mathbb{C})$ induced by $\|\cdot\|_1$ is complete. That is, prove every Cauchy sequence converges in $\mathcal{L}^1$.
Note: This follows immediately from Corollary 3.5.9 if one shows that completeness of a normed vector space V is equivalent to the property that every absolutely convergent series converges in V.

Exercise 3.5.11. Let μ be a Lebesgue-Stieltjes Borel measure on $\mathbb{R}$. Show that $C_c(\mathbb{R})$, the continuous functions of compact support ($\overline{\{f \neq 0\}}$ compact) is dense in $\mathcal{L}^1(\mu, \mathbb{R})$. Does the same hold for $\overline{\mathbb{R}}$ and $\mathbb{C}$-valued functions?
Hint: You could proceed in this way:

(1) *Reduce to the case $f \in L^1 \cap L^+$.*
(2) *Reduce to the case $f \in L^1 \cap \mathsf{SF}^+$.*
(3) *Reduce to the case $f = \chi_E$ with $E \in \mathcal{B}_\mathbb{R}$ and $\mu(E) < \infty$.*
(4) *Reduce to the case $f = \chi_U$ with $U \subset \mathbb{R}$ open and $\mu(U) < \infty$.*
(5) *Reduce to the case $f = \chi_{(a,b)}$ with $a < b$ in $\mathbb{R}$.*

3.6. **Modes of convergence.** Let $(X, \mathcal{M}, \mu)$ be a measure space. For $(f_n), f$ all $\mathcal{M} - \mathcal{B}_\mathbb{C}$ measurable functions, $f_n \to f$ could mean many things:

- (pointwise) $f_n(x) \to f(x)$ for all $x \in X$.
- (a.e.) $f_n(x) \to f(x)$ for a.e. $x \in X$.
- (uniformly) for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > N$ implies $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$.
- (almost uniformly, a.k.a. a.u.) for all $\varepsilon > 0$, there is an $E \in \mathcal{M}$ with $\mu(E) < \varepsilon$ such that $f_n \chi_{E^c} \to f \chi_{E^c}$ uniformly.
- (in $\mathcal{L}^1$) $\int |f_n - f| \to 0$ as $n \to \infty$.
- (in measure) for all $\varepsilon > 0$, $\mu(\{\varepsilon \le |f - f_n|\}) \to 0$.

Observe that obviously uniform implies a.u., uniform implies pointwise, and pointwise implies a.e.

Proposition 3.6.1. *Almost uniform convergence implies almost everywhere convergence.*

Proof. Suppose $f_n \to f$ a.u. For $k \in \mathbb{N}$, let $E_k \in \mathcal{M}$ such that $\mu(E_k) < 1/k$ and $f_n \chi_{E_k^c} \to f \chi_{E_k^c}$ uniformly. Let $E := \bigcap E_k$. Then $\mu(E) = 0$ by continuity from above ($\mu 4$), and since $E^c = \bigcup E_k^c$, we have $f_n \chi_{E^c} \to f \chi_{E^c}$ pointwise. $\square$

Proposition 3.6.2. *Almost uniform convergence implies convergence in measure.*

Proof. Suppose $f_n \to f$ a.u. Let $\varepsilon > 0$. Show for all $\delta > 0$, there is an $N \in \mathbb{N}$ such that $n > N$ implies $\mu(\{\varepsilon \le |f - f_n|\}) < \delta$. Pick $E \in \mathcal{M}$ such that $\mu(E) < \delta$ and $f_n \chi_{E^c} \to f \chi_{E^c}$ uniformly. Then

$$\mu(\{\varepsilon \le |f - f_n|\}) = \underbrace{\mu(\{\varepsilon \le |f - f_n|\} \cap E)}_{\text{always } < \delta} + \underbrace{\mu(\{\varepsilon \le |f - f_n|\} \cap E^c)}_{= \emptyset \text{ for } n \text{ large}} < \delta$$

for n sufficiently large. $\square$

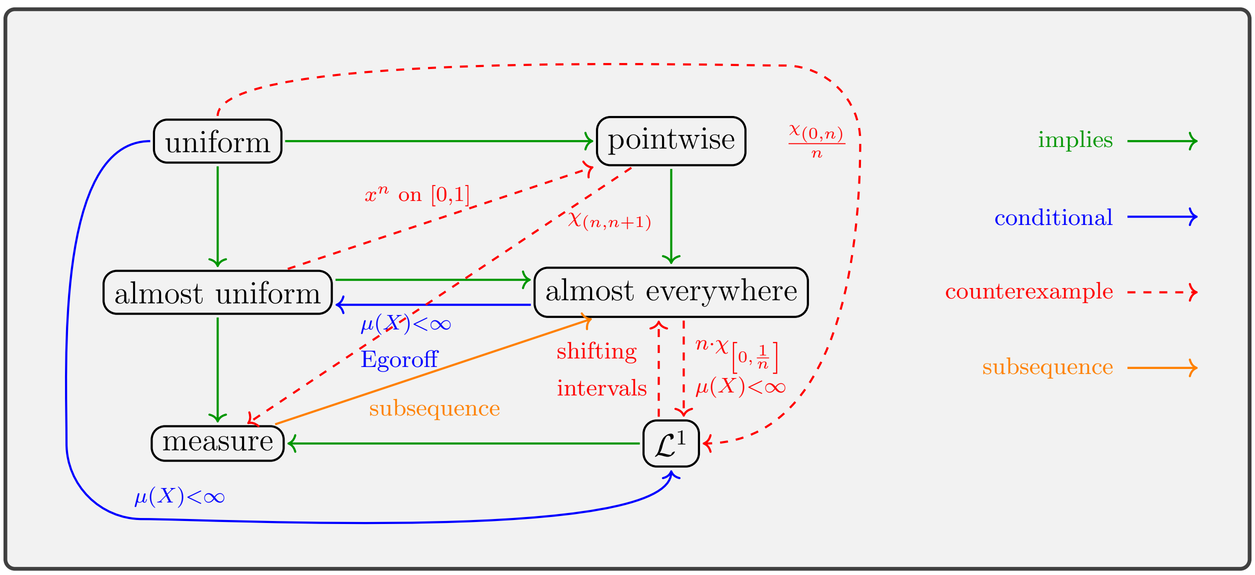
Proposition 3.6.3. *Convergence in $\mathcal{L}^1$ implies convergence in measure.*

Proof. Suppose $f_n \to f$ in $\mathcal{L}^1$. Let $\varepsilon > 0$, and set $E := \{\varepsilon \le |f - f_n|\}$. Then

$$\mu(E) = \int_E 1 = \frac{1}{\varepsilon} \int_E \varepsilon \le \frac{1}{\varepsilon} \int_E |f - f_n| \xrightarrow{n\to\infty} 0. \qquad \square$$

Facts 3.6.4 (Counterexamples)**.** We consider the following important counterexamples:

(1) $f_n = \frac{1}{n}\chi_{(0,n)}$ converges uniformly to zero, but not in $\mathcal{L}^1$.
(2) $f_n = \chi_{(n,n+1)}$ converges pointwise to zero, but not in measure.
(3) $f_n = n\chi_{[0,1/n]}$ converges a.e. to zero with $\mu(X) < \infty$, but not in $\mathcal{L}^1$.
(4) $f_n(x) := x^n$ on $[0,1]$ almost uniformly to zero, but not pointwise.
(5) (shifting intervals) $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[1,1/4]}$, $f_5 = \chi_{[1/4,1/2]}$, etc. converges in $\mathcal{L}^1$, but not a.e.



Lemma 3.6.5. *If $f_n \to f$ uniformly and $\mu(X) < \infty$, then $f_n \to f$ in $\mathcal{L}^1$.*

Proof. Observe that

$$\int |f_n - f| \le (\sup |f_n - f|) \cdot \int 1 = \underbrace{(\sup |f_n - f|)}_{\to 0 \text{ as } n\to\infty} \cdot \mu(X). \qquad \square$$

Theorem 3.6.6 (Egoroff)**.** *If $f_n \to f$ a.e. and $\mu(X) < \infty$, then $f_n \to f$ a.u.*

Proof. By replacing X with $X \setminus N$ for some μ-null set $N \in \mathcal{M}$, we may assume $f_n \to f$ pointwise. Now observe that for all $k \in \mathbb{N}$,

$$E_{n,k} := \bigcup_{j=n}^{\infty} \left\{ \frac{1}{k} \le |f - f_j| \right\} \searrow \emptyset \qquad \text{as} \qquad n \to \infty.$$

Since $\mu(X) < \infty$, by continuity from above (μ4), $\mu(E_{n,k}) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. For all $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that $\mu(E_{n_k,k}) < \varepsilon/2^k$. Setting $E := \bigcup_{k=1}^\infty E_{n_k,k}$, we have

$$\mu(E) \leq \sum_k \mu(E_{n_k,k}) < \varepsilon \sum 2^{-k} = \varepsilon.$$

Finally, observe that for all $n > n_k$, if $x \in E^c = \bigcap_{k=1}^\infty E_{n_k,k}^c$, then $|f(x) - f_n(x)| < 1/k$. Thus $f_n \to f$ uniformly on E^c. □

Definition 3.6.7. A sequence (f_n) of $\mathcal{M}$-measurable functions is *Cauchy in measure* if for all $\varepsilon > 0$,

$$\mu\left(\{\varepsilon \leq |f_m - f_n|\}\right) \xrightarrow{n,m\to\infty} 0.$$

Exercise 3.6.8. Prove that if $f_n \to f$ in measure, then (f_n) is Cauchy in measure.

Theorem 3.6.9. *If (f_n) is Cauchy in measure, then there exists a unique (up to μ-null set) $\mathcal{M}$-measurable function f such that $f_n \to f$ in measure. Moreover, there is a subsequence (f_{n_k}) such that $f_{n_k} \to f$ a.e.*

Proof.
Step 1: There is a subsequence (f_{n_k}) such that $\mu\left(\{2^{-k} \leq |f_{n_k} - f_{n_{k+1}}|\}\right) < 2^{-k}$.

> *Proof.* For all $k \in \mathbb{N}$, $\mu\left(\{2^{-k} \leq |f_n - f_m|\}\right) \to 0$ as $m, n \to \infty$. Pick n_k inductively so $n_{k+1} > n_k$ and $m, n \geq n_k$ implies $\mu\left(\{2^{-k} \leq |f_n - f_m|\}\right) < 2^{-k}$. □

Step 2: (f_{n_k}) is pointwise Cauchy off a μ-null set N.

> *Proof.* For $k \in \mathbb{N}$, set $E_k := \{2^{-k} \leq |f_{n_k} - f_{n_{k+1}}|\}$, and for $\ell \in \mathbb{N}$, set $N_\ell := \bigcup_{k=\ell} E_k$. Then $\mu(N_\ell) \leq \sum_{k=\ell} 2^{-k} = 2^{1-\ell}$. Setting $N = \bigcap N_\ell = \limsup E_k$, we have $\mu(N) = 0$ by continuity from above (μ4). If $x \in N^c$, then $x \notin N_\ell$ for some ℓ, and thus for all $\ell \leq i \leq j$,
>
> $$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{k=i}^{j-1} |f_{n_k}(x) - f_{n_{k+1}}(x)| \leq \sum_{k=i}^{j-1} 2^{-k} \leq 2^{1-i}. \tag{3.6.10}$$
>
> We conclude that (f_{n_k}) is pointwise Cauchy on N^c. □

Step 3: Define

$$f(x) := \begin{cases} 0 & \text{if } x \in N \text{ (which is } \mu\text{-null)} \\ \lim_k f_{n_k}(x) & \text{if } x \in N^c. \end{cases}$$

Then f is $\mathcal{M}$-measurable and $f_{n_k} \to f$ a.e.

> *Proof.* It remains to show f is measurable. Observe $f_{n_k} \cdot \chi_{N^c}$ is $\mathcal{M}$-measurable for all k, and thus so is $f = \lim f_{n_k} \cdot \chi_{N^c}$ by Exercise 3.2.5. □

Step 4: $f_{n_k} \to f$ in measure.

Proof. For all $x \in N_\ell^c$ and $k \geq \ell$, we have

$$|f_{n_k}(x) - f(x)| = \lim_{j\to\infty} |f_{n_k}(x) - f_{n_j}(x)| \underset{(3.6.10)}{\leq} 2^{1-k}.$$

Let $\varepsilon > 0$ and pick $\ell \in \mathbb{N}$ such that $0 < 2^{-\ell} < \varepsilon$. Then for all $k \geq \ell$,

$$\mu\left(\{\varepsilon \leq |f_{n_k} - f|\}\right) \leq \mu\left(\left\{\frac{1}{2^k} \leq |f_{n_k} - f|\right\}\right) < 2^{1-k} \xrightarrow{k\to\infty} 0.$$

□

Step 5: $f_n \to f$ in measure.

Proof. We use the following trick:

> **Trick.** For non-negative $\mathcal{M}$-measurable f, g, $\{a+b \leq f+g\} \subset \{a \leq f\} \cup \{b \leq g\}$.

Now observe that

$$\{\varepsilon \leq |f_n - f|\} \subseteq \underbrace{\left\{\frac{\varepsilon}{2} \leq |f_n - f_{n_k}|\right\}}_{\substack{\mu \to 0 \text{ as } (f_n) \\ \text{Cauchy in measure}}} \cup \underbrace{\left\{\frac{\varepsilon}{2} \leq |f_{n_k} - f|\right\}}_{\mu \to 0 \text{ by Step 4}}.$$

Hence $\mu(\{\varepsilon \leq |f_n - f|\}) \to 0$ as $n \to \infty$. □

Step 6: f is unique (up to a μ-null set) such that $f_n \to f$ in measure.

Proof. Suppose g is another such candidate. Then using the same trick as in Step 5,

$$\{\varepsilon \leq |f - g|\} \subseteq \underbrace{\left\{\frac{\varepsilon}{2} \leq |f - f_n|\right\}}_{\mu \to 0 \text{ as } n \to \infty} \cup \underbrace{\left\{\frac{\varepsilon}{2} \leq |g - f_n|\right\}}_{\mu \to 0 \text{ as } n \to \infty}.$$

Hence $\mu(\{\varepsilon \leq |f - g|\}) = 0$ for all $\varepsilon > 0$, and thus $f = g$ a.e. □

This concludes the proof. □

Exercise 3.6.11 (Lusin's Theorem)**.** Suppose $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\varepsilon > 0$. There is a compact set $E \subset [a, b]$ such that $\lambda(E^c) < \varepsilon$ and $f|_E$ is continuous.
Hint: Use Exercise 3.3.13 and Egoroff's Theorem 3.6.6.

Exercise 3.6.12. Suppose $f \in \mathcal{L}^1([0, 1], \lambda)$ is an integrable non-negative function.

(1) Show that for every $n \in \mathbb{N}$, $\sqrt[n]{f} \in \mathcal{L}^1([0, 1], \lambda)$.
(2) Show that $(\sqrt[n]{f})$ converges in $\mathcal{L}^1$ and compute its limit.

Hint for both parts: Consider $\{f \geq 1\}$ and $\{f < 1\}$ separately.

Exercise 3.6.13. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f_n \to f$ in measure and $g_n \to g$ in measure (these functions are assumed to be measurable). Show that

(1) $|f_n| \to |f|$ in measure.
(2) $f_n + g_n \to f + g$ in measure.

(3) $f_n g_n \to fg$ if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.
Hint: First show $f_n g \to fg$ in measure. To do so, one could follow the following steps.
  (a) Show that for any $\varepsilon > 0$, by Exercise 3.3.13, $X = E \amalg E^c$ where $|g|_E| < M$ and $\mu(E^c) < \varepsilon/2$.
  (b) For $\delta > 0$ and carefully chosen $M > 0$ and E,

$$\begin{aligned}\{|f_n g - fg| > \delta\} &= (\{|f_n g - fg| > \delta\} \cap E) \amalg (\{|f_n g - fg| > \delta\} \cap E^c)\\ &\subseteq \left\{|f_n - f| > \frac{\delta}{M}\right\} \cup E^c.\end{aligned}$$

Exercise 3.6.14 (Folland §2.4, #33 and 34)**.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f_n \to f$ in measure (these functions are assumed to be measurable).

(1) Show that if $f_n \geq 0$ everywhere, then $\int f \leq \liminf \int f_n$.
(2) Suppose $|f_n| \leq g \in \mathcal{L}^1$. Prove that $\int f = \lim \int f_n$ and $f_n \to f$ in $\mathcal{L}^1$.

Exercise 3.6.15. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $(E_n) \subset \mathcal{M}$ is a sequence of measurable sets with $\mu(E_n) < \infty$ for all n. Show that if $\chi_{E_n} \to f$ in $\mathcal{L}^1$ (this assumes f is $\mathcal{M}$-measurable), then there is an $E \in \mathcal{M}$ such that $f = \chi_E$ a.e.

3.7. **Comparison of the Lebesgue and Riemann integrals.** We now review the Riemann integral for a Reimann integrable function $f : [a, b] \to \mathbb{R}$.

Definition 3.7.1. A *partition* of $[a, b]$ is a set of points $P = \{a = s_0 < s_1 < \cdots < s_m = b\}$. We say an interval $J \in P$ if $J = [s_{i-1}, s_i]$ for some $i = 1, \ldots, m$. We write

$$m_J := \inf\{f(x) | x \in J\} \qquad\qquad M_J := \sup\{f(x) | x \in J\}.$$

We define the:

- Lower sum: $L(f, P) := \sum_{J \in P} m_J \lambda(J)$
- Upper sum: $U(f, P) := \sum_{J \in P} M_J \lambda(J)$

Here, $\lambda(J)$ is the length (Lebesgue measure) of the interval. Observe $L(f, P) \leq U(f, P)$.

A *refinement* of P is a partition $Q = \{a = t_0 < t_1 < \cdots < t_n = b\} \supset P$. Observe that if Q refines P, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Thus if P_1, P_2 are two partitions of $[a, b]$ and Q is a common refinement, then

$$\sup_{i=1,2} L(f, P_i) \leq L(f, Q) \leq U(f, Q) \leq \inf_{i=1,2} U(f, P_i).$$

We define the:

- Upper integral: $\overline{\int}_{[a,b]} f := \inf_P U(f, P)$
- Lower integral: $\underline{\int}_{[a,b]} f := \sup_P L(f, P)$

We say f is *Riemann integrable* on $[a, b]$ if $\overline{\int}_{[a,b]} f = \underline{\int}_{[a,b]} f$, and we denote this common value by $\int_a^b f(x)\, dx$.

Exercise 3.7.2. Suppose $f : [a, b] \to \mathbb{R}$. Prove the following are equivalent:

(1) f is Riemann integrable
(2) for all $\varepsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Theorem 3.7.3. *If f is Riemann integrable on $[a,b]$, then f is Lebesgue integrable and $\int_{[a,b]} f\,d\lambda = \int_a^b f(x)\,dx$.*

Proof. Let (P_n) be a sequence of partitions of $[a,b]$ such that P_{n+1} refines P_n and $U(f,P_n) - L(f,P_n) < 1/n$ for all $n \in \mathbb{N}$. Here's the trick:

Trick. Define the simple functions $\psi_n := \sum_{J\in P_n} m_J \chi_J$ and $\Psi_n := \sum_{J\in P_n} M_J \chi_J$.

Observe that $L(f,P_n) = \int \psi_n\,d\lambda$ and $U(f,P_n) = \int \Psi_n\,d\lambda$ and

$$\psi_n \le \psi_{n+1} \le f \le \Psi_{n+1} \le \Psi_n \qquad \forall n \in \mathbb{N}.$$

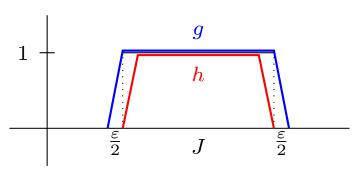
Define $\psi := \lim \psi_n$ and $\Psi := \lim \Psi_n$, which exists as (ψ_n) and (Ψ_n) are pointwise bounded and monotone. Then by (a slight modification of) the MCT 3.3.9, ψ, Ψ are integrable, and

$$\int \psi = \lim \int \psi_n = \int_a^b f(x)\,dx = \lim \int \Psi_n = \int \Psi.$$

But since $\Psi - \psi \ge 0$ everywhere, $\int \Psi - \psi = 0$ implies $\Psi = f = \psi$ a.e. So $f \in \mathcal{L}^1$ and $\int f = \int_a^b f(x)\,dx$. □

Lemma 3.7.4. *Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable and bounded. Then for all $\varepsilon > 0$, there are continuous functions $g, h : [a,b] \to \mathbb{R}$ such that $h \le f \le g$ and $\int_{[a,b]}(g-h)\,d\lambda \le \varepsilon$.*

Proof.
Step 1: If $f = \chi_J$ for some interval J, then we can find piecewise linear functions g, h such that $h \le f \le g$ such as in the following cartoon:



Then $\int_{[a,b]} g = \lambda(J) + \varepsilon/2$ and $\int_{[a,b]} h = \lambda(J) - \varepsilon/2$, so $\int g - h = \varepsilon$.
Step 2: Without loss of generality, we may assume $f \ge 0$. (Otherwise, treat $f_\pm$ separately.) Take a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon/2$. As in the trick in the previous theorem, define the simple functions

$$\psi_n := \sum_{J\in P} m_J \chi_J \le f \le \Psi_n := \sum_{J\in P} M_J \chi_J$$

so that $\int \psi = L(f,P)$ and $\int \Psi = U(f,P)$. Apply Step 1 to each χ_J to get continuous g_J, h_J with $h_J \le \chi_J \le g_J$ such that $\int g_J - h_J < \frac{\varepsilon}{2|P|M}$ where $|P|$ is the number of intervals of P and $M := \sup\{f(x) | a \le x \le b\}$. Setting $g := \sum_{J\in P} M_J g_J$ and $h := \sum_{J\in P} m_J h_J$, we have

$$h = \sum_{J\in P} m_J h_J \le \sum_{J\in P} m_J \chi_J = \psi \le f \le \psi = \sum_{J\in P} M_J \chi_J \le \sum_{J\in P} M_J g_J = g,$$

and thus

$$\begin{aligned}
\int g - h &= \sum_{J\in P} M_J \int g_J - m_J \int h_J \\
&= U(f,P) - L(f,P) + \sum_{J\in P} \underbrace{M_J}_{<M} \left(\int g_J - \lambda(J)\right) + \underbrace{m_J}_{<M} \left(\lambda(J) - \int h_J\right) \\
&< \underbrace{U(f,P) - L(f,P)}_{<\frac{\varepsilon}{2}} + M \sum_{J\in P} \underbrace{\int g_J - h_J}_{<\frac{\varepsilon}{2|P|M}} \\
&< \varepsilon. \qquad \square
\end{aligned}$$

Exercise 3.7.5. Let X be a topological space and let $g : X \to \mathbb{R}$. We say that g is *upper semicontinuous* at $x_0 \in X$ if for every $\varepsilon > 0$, there is an open neighborhood U of x_0 such that $x \in U$ implies $f(x) < f(x_0) + \varepsilon$. We say g is upper semicontinuous if g is upper semicontinuous at every $x \in X$.

(1) Show that g is upper semicontinuous if and only if $\{g < r\}$ is open in for all $r \in \mathbb{R}$.
(2) Define lower semicontinuity (both at $x_0 \in X$ and everywhere) and prove the analogous statement to (1).

Theorem 3.7.6 (Lebesgue)**.** *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous a.e.*

Proof.
$\underline{\Rightarrow}$: Suppose f is Riemann integrable. By Lemma 3.7.4, there are sequence of continuous functions (h_n) and (g_n) on $[a,b]$ with $h_n \leq f \leq g_n$ such that $\int g_n - h_n < 1/n$ for all $n \in \mathbb{N}$. Since

$$g_{n+1} \wedge g_n - h_{n+1} \vee h_n \leq g_{n+1} - h_{n+1} \qquad \forall n \in \mathbb{N},$$

we may assume that

$$h_n \leq h_{n+1} \leq f \leq g_{n+1} \leq g_n \qquad \forall n \in \mathbb{N}.$$

Setting $h := \lim h_n$ and $g := \lim g_n$, we have $h \leq f \leq g$ and $\int h = \int f = \int g$ by MCT 3.3.9. Since $g - h \geq 0$, we know $g = f = h$ a.e. on $[a,b]$.

Claim. *Since $g_n \searrow g$, g is upper semicontinuous. Similarly, h is lower semicontinuous*

Proof. Let $x_0 \in [a,b]$ and $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that $n \geq N$ implies $g_n(x_0) - g(x_0) < \varepsilon/2$. Pick $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$ implies $|g_N(x) - g_N(x_0)| < \varepsilon/2$. Then for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$,

$$g(x_0) > g_N(x_0) - \frac{\varepsilon}{2} > g_N(x) - \varepsilon \geq g(x) - \varepsilon. \qquad \square$$

Whenever $h(x_0) = f(x_0) = g(x_0)$, f is both upper semicontinuous and lower semicontinuous at x_0, i.e., f is continuous at x_0. This happens on $[a,b]$ a.e.
$\underline{\Leftarrow}$: Suppose f is continuous on $[a,b]$ a.e. Let E be the λ-null set of discontinuities, and let $\varepsilon > 0$. We'll construct a partition P such that $U(f,P) - L(f,P) < \varepsilon$. By outer regularity of λ, there is an open $U \supset E$ such that $\lambda(U) < \varepsilon'$ to be determined later. Let

$K := [a,b] \setminus U$, which is compact, and observe that f is continuous at all points of K (not $f|_K$!). For each $x \in K$, pick $\delta_x > 0$ such that $y \in [a,b]$ (not K!) and $|x-y| < \delta_x$ implies $|f(x) - f(y)| < \varepsilon'$. Then $\{B_{\delta_x/2}(x)\}_{x \in K}$ is an open cover of K, so there are $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n B_{\delta_{x_i}/2}(x_i)$. Set $\delta := \min\{\delta_{x_i}/2 | i = 1, \ldots, n\}$.

Claim. *If $x \in K$ and $y \in [a,b]$ and $|x-y| < \delta/2$, then $|f(x) - f(y)| < 2\varepsilon'$.*

Proof. Without loss of generality, $x \in B_{\delta_1/2}(x_1)$. Then $y \in B_{\delta_1}(x_1)$, and thus

$$|f(x) - f(y)| \le |f(x) - f(x_1)| + |f(x_1) - f(y)| < 2\varepsilon'. \qquad \square$$

Let P be any partition of $[a,b]$ whose intervals have length at most δ. Let P' consist of the intervals that intersect K and let P'' be the intervals that do not intersect K. By the claim, if $J \in P'$, then $M_J - m_j \le 2\varepsilon'$. Thus

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{J \in P} (M_J - m_J)\lambda(J) \\ &= \sum_{J \in P'} (M_J - m_J)\lambda(J) + \sum_{J \in P''} (M_J - m_J)\lambda(J) \\ &\le \sum_{J \in P'} 2\varepsilon' \lambda(J) + \sum_{J \in P''} (M - m)\lambda(J) \\ &\le 2\varepsilon'(b-a) + (M-m)\lambda(U) && \left(\bigcup_{J \in P''} J \subseteq U\right) \\ &< \varepsilon'(2(b-a) + (M-m)) \end{aligned}$$

where $M = \sup_{x \in [a,b]} f(x)$ and $m := \inf_{x \in [a,b]} f(x)$. Taking $\varepsilon' = \varepsilon/(2(b-a) + (M-m))$ works. $\square$

3.8. **Product measures.**

Definition 3.8.1. Given measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a *measurable rectangle* is a set of the form $E \times F \subset X \times Y$ where $E \in \mathcal{M}$ and $F \in \mathcal{N}$. The *product σ-algebra* $\mathcal{M} \times \mathcal{N} \subset P(X \times Y)$ is the σ-algebra generated by the measurable rectangles.

Exercise 3.8.2. Prove that $\mathcal{M} \times \mathcal{N}$ is the smallest σ-algebra such that the canonical projection maps $\pi_X : X \times Y \to Y$ and $\pi_Y : X \times Y \to X$ are measurable. Deduce that $\mathcal{M} \times \mathcal{N}$ is generated by $\pi_X^{-1}(\mathcal{E}_X) \cup \pi_Y^{-1}(\mathcal{E}_Y)$ for any generating sets $\mathcal{E}_X$ of $\mathcal{M}$ and $\mathcal{E}_Y$ of $\mathcal{N}$.

Warning 3.8.3. Recall that given topological spaces X, Y, the canonical projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are open maps. When $(X, \mathcal{M}), (Y, \mathcal{N})$ are measurable, however, π_X, π_Y need not map measurable sets to measurable sets. (Unfortunately, actually constructing a set in $\mathcal{M} \times \mathcal{N}$ whose projection to X is not measurable is quite difficult.)

Exercise 3.8.4. Show that the subset of $P(X \times Y)$ consisting of finite disjoint unions of measurable rectangles is an algebra which generates $\mathcal{M} \times \mathcal{N}$.
Hint: For $E, E_1, E_2 \in \mathcal{M}$ and $F, F_1, F_2 \in \mathcal{N}$,

- $(E_1 \times F_1) \cap (E_2 \times F_2) = (E_1 \cap E_2) \times (F_1 \cap F_2)$, *and*
- $(E \times F)^c = (E \times F^c) \amalg (E^c \times F) \amalg (E^c \times F^c)$.