Measurable fcts:

Suppose $f: X \to Y$. Then f induces fcts

$$f: P(X) \longrightarrow P(Y) \quad \text{by} \quad S \longmapsto \{f(s) \mid s \in S\}$$

$$f^{-1}: P(Y) \longrightarrow P(X) \quad \text{by} \quad T \longmapsto \{x \in X \mid f(x) \in T\}.$$

Exercise:

① Determine the relationship btwn $f^{-1}(f(S))$ and $S \subset X$

② - - - - - - - - - - - - - - $f(f^{-1}(T))$ and $T \subset Y$

Exercise: Show that $\forall S, T \subset Y$,

① $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

② $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

③ $f^{-1}(Y \setminus S) = X \setminus f^{-1}(S)$

what about $f: P(X) \to P(Y)$?

Def: If (X, τ) and (Y, θ) are topological spaces, a fct $f: X \to Y$ is *cts* if $\forall\, \mathcal{U} \in \theta$, $f^{-1}(\mathcal{U}) \in \tau$.

Exercise: Suppose $f: X \to Y$.

① Suppose $\theta \subset P(Y)$ is a topology on Y. Show that $f^{-1}(\theta) := \{f^{-1}(T) \mid T \in \theta\}$ is a topology on X. Deduce that $f^{-1}(\theta)$ is the *smallest* topology on X s.t. f is cts. Call $f^{-1}(\theta)$ the topology on X *induced* by f, θ.

② Suppose $\tau \subset P(X)$ is a topology on X. Show that $\{T \in Y \mid f^{-1}(T) \in \tau\}$ is a topology on Y. We call it the topology on Y *co-induced* by f, τ. Deduce that this topology is the *largest* topology on Y s.t. f is cts.

Def: If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, we call $f: X \to Y$ $(\mathcal{M}, \mathcal{N})$ *measurable* if $f^{-1}(E) \in \mathcal{M}$ $\forall E \in \mathcal{N}$.

Exercise: Suppose $f: X \to Y$.

① Suppose $\mathcal{N}$ is a σ-alg on Y. Show that $f^{-1}(\mathcal{N}) := \{f^{-1}(E) \mid E \in \mathcal{N}\}$ is a σ-alg on X. Deduce that $f^{-1}(\mathcal{N})$ is *smallest* σ-alg on X s.t. f is measurable. We call $f^{-1}(\mathcal{N})$ the σ-alg *induced* by $f, \mathcal{N}$.

② Suppose $\mathcal{M}$ is a σ-alg on X. Show that $\{E \subset Y \mid f^{-1}(E) \in \mathcal{M}\}$ is a σ-alg on Y. We call it the σ-alg *co-induced* by $f, \mathcal{M}$. Deduce it is the *largest* σ-alg on Y s.t. f is measurable.

Rem: Just as the composite of cts fcts is cts, the composite of measurable fcts is measurable.

Prop: Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, $f: X \to Y$, and $\mathcal{N} = \mathcal{M}(\mathcal{E})$ for some $\mathcal{E} \subset P(Y)$. Then f is measurable $\iff$ $f^{-1}(E) \in \mathcal{M}$ $\forall E \in \mathcal{E}$.

Pf: $\Rightarrow$: Trivial.

$\Leftarrow$: If $f^{-1}(E) \in \mathcal{M}$ $\forall E \in \mathcal{E}$, then $\mathcal{E}$ is contained in the σ-alg $\mathcal{N}_f$ on Y co-induced by $f, \mathcal{M}$. Since $\mathcal{N}_f$ is a σ-alg containing $\mathcal{E}$, $\mathcal{N} = \mathcal{M}(\mathcal{E}) \subset \mathcal{N}_f$. Since f is $(\mathcal{M}, \mathcal{N}_f)$-measurable, f is $(\mathcal{M}, \mathcal{N})$ measurable.

Def: Suppose (X,τ) and (Y,θ) are top. spaces. we call $f: X \to Y$ **Borel measurable** if f is $(\mathcal{B}_\tau, \mathcal{B}_\theta)$ measurable.

Cor: Every cts fct $f: X \to Y$ is Borel measurable.

Pf: f cts $\iff f^{-1}(U) \in \tau \ \forall U \in \theta$
$\implies f$ Borel measurable by the proposition.

Cor: If $(X, \mathcal{M})$ measurable and $f: X \to \mathbb{R}$, then TFAE:

① f is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable
② $f^{-1}(a,\infty) \in \mathcal{M} \ \forall a \in \mathbb{R}$
③ $f^{-1}[a,\infty) \in \mathcal{M} \ \forall a \in \mathbb{R}$
④ $f^{-1}(-\infty,a) \in \mathcal{M} \ \forall a \in \mathbb{R}$
⑤ $f^{-1}(-\infty,a] \in \mathcal{M} \ \forall a \in \mathbb{R}$

Can also use:
(a,b) $\forall a,b \in \mathbb{R}$
$[a,b)$ "
$(a,b]$ "
$[a,b]$ "

Cor: If $(X, \mathcal{M})$ measurable and $f: X \to \overline{\mathbb{R}} = [-\infty, \infty]$, then the above corollary holds replacing $\mathbb{R}$ w/ $\overline{\mathbb{R}}$ and intervals not including $\pm\infty$ w/ intervals including $\pm\infty$.

Rem: The Borel σ-alg on $\overline{\mathbb{R}}$ is generated by

Caution: The book says a fct $f: X \to \mathbb{R}$ is **$\mathcal{M}$-measurable** if f is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable. This has the following nasty consequence:

- If $f,g: \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable $(\mathcal{L}, \underbrace{\mathcal{B}_{\mathbb{R}}}_{(!)})$ then $f \circ g$ need <u>not</u> be Lebesgue measurable!

Notation: Given $f: X \to \overline{\mathbb{R}}$, write $\{f>a\} := f^{-1}(a,\infty]$.
Similarly, we'll define

Prop: Suppose $f,g: X \to \overline{\mathbb{R}}$ are $\mathcal{M}$-measurable.
The following fcts are all $\mathcal{M}$-measurable.

- $(f \vee g)(x) := \sup\{f(x), g(x)\}$

Pf: If $a \in \mathbb{R}$, $\{f \vee g > a\} = \{f>a\} \cup \{g>a\} \in \mathcal{M}$.

- $(f \wedge g)(x) := \inf\{f(x), g(x)\}$

Pf: If $a \in \mathbb{R}$, $\{f \wedge g > a\} = \{f>a\} \cap \{g>a\} \in \mathcal{M}$.

- any well-defined linear combination of f,g

[Recall: $0 \cdot (\pm\infty) = 0$ by convention, but $\pm\infty \mp \infty$ not defined.]

Pf:

① If $a \in \mathbb{R}$
$$\{cf > a\} = \left.\begin{cases} \emptyset & c=0 \le a \\ X & c=0>a \\ \{f > \frac{a}{c}\} & c>0 \\ \{f < \frac{a}{c}\} & c<0 \end{cases}\right\} \text{ all in } \mathcal{M}.$$

② If $f+g$ is well-defined, for $a \in \mathbb{R}$,
$$\{f+g>a\} = \bigcup_{\substack{r,s \in \mathbb{Q} \\ r+s>a}} [\{f>r\} \cap \{g>s\}] \in \mathcal{M}.$$

- fg

Pf: Step 1: Suppose f,g are non-negative. Then $\forall a \geq 0$,
$$\{fg>a\} = \bigcup_{\substack{r,s \in \mathbb{Q} \\ r,s>0 \\ rs>a}} [\{f>r\} \cap \{g>s\}] \in \mathcal{M}.$$

Also, $\forall a<0$, $\{fg>a\} = X \in \mathcal{M}$.

Step 2: For arbitrary f,g, write $f = f_+ - f_-$ where $f_+ := f \vee 0$
$f_- := -(f \wedge 0)$.
Same for g. Then:
$$fg = f_+g_+ - f_-g_+ - f_+g_- + f_-g_-,$$ well-defined! [disjoint support!]

Prop: Suppose (f_n) is a seq of fcts $X \to \overline{\mathbb{R}}$. The following fcts are $\mathcal{M}$-measurable:

- $\sup f_n$ and $\inf f_n$

Pf: $\forall\, a \in \mathbb{R}$, $\{\sup f_n > a\} = \bigcup_n \{f_n > a\} \in \mathcal{M}$.

$\{\inf f_n > a\} = \bigcap_n \{f_n > a\} \in \mathcal{M}$.

- $\limsup f_n$ and $\liminf f_n$

Pf:

$$\left.\begin{aligned} \limsup f_n &= \lim_{n\to\infty} \sup_{k\geq n} f_k = \inf_n \boxed{\sup_{k\geq n} f_k} \\ \liminf f_n &= \lim_{n\to\infty} \inf_{k\geq n} f_k = \sup_n \boxed{\inf_{k\geq n} f_k} \end{aligned}\right\} \text{measurable.}$$

Simple fcts:

Let $(X, \mathcal{M}, \mu)$ be a measure space.

Def: An $\mathcal{M}$-measurable fct $X \to \mathbb{R}$ is *simple* if it takes finitely many values. Observe if φ is simple, can write

$$\varphi = \sum_{k=1}^{n} c_k \chi_{E_k} \qquad c_1, \ldots, c_n \in \mathbb{R},\ E_1, \ldots, E_k \in \mathcal{M}.$$

Observe this expression is unique if

- $c_1, \ldots, c_n$ are *distinct* and *non-zero*, and
- $E_1, \ldots, E_n$ are *disjoint* and *nonempty*.

Remark: The simple fcts SF form a: finite!

- *$\mathbb{R}$-algebra* (closed under linear combinations + mult.) $\chi_E \chi_F = \chi_{E\cap F}$
- *lattice* (closed under inf + sup)

Define $SF^+ = \{\varphi \in SF \mid \varphi \geq 0\}$. Observe that SF^+ is:

- closed under addition, multiplication, $(r \geq 0)$-scalar mult.
- a *sublattice* of SF.

Prop: Suppose $f : X \longrightarrow [0,\infty]$ is $\mathcal{M}$-measurable. There is a seq. $(\varphi_n) \subset SF^+$ s.t.

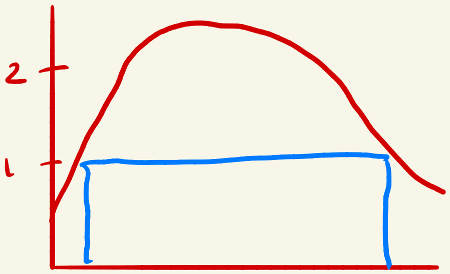
- $\varphi_n(x) \nearrow f(x) \ \forall x \in X$
- $\forall N \in \mathbb{N}$, $\varphi_n \longrightarrow f$ uniformly on $\{f \le N\}$.

Pf: For $n \geq 0$ and $1 \le k \le 2^{2n}$

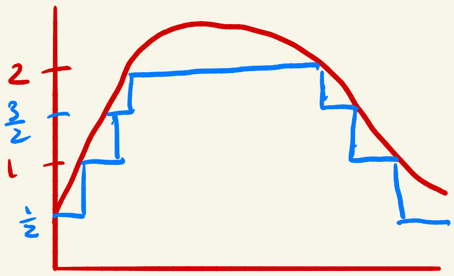
$E_n^k := f^{-1}\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ ← below 2^n, take steps $\frac{1}{2^n}$.

$F_n := f^{-1}(2^n, \infty]$ ← cut off at 2^n

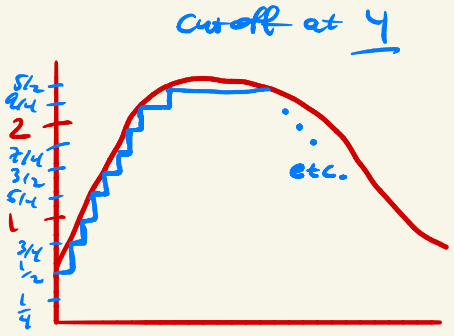
$$\varphi_n := 2^n \chi_{F_n} + \sum_{k=1}^{2^{2n}} \frac{k-1}{2^n} \chi_{E_n^k}.$$



φ_0



φ_1



φ_2

Then $\varphi_n \le \varphi_{n+1} \ \forall n \geq 0$ and $0 \le f - \varphi_n \le 2^{-n}$ on $\{f \le 2^n\}$. The result follows.

Exercise: Suppose $f : X \longrightarrow [0,\infty]$ is $\mathcal{M}$-measurable and $\{f > 0\}$ is σ-finite. Prove that we can find such a seq. (φ_n) as in the proposition which additionally satisfies the property that each

$$\left[\ \varphi_n = \sum_{j=1}^{k} c_j \chi_{E_j} \quad \text{where } \underline{\mu(E_j) < \infty} \ \ \forall j = 1, .., k. \right.$$

Such SF's will be called integrable.

Integration of non-negative fcts:

Fix a measure space $(X, \mathcal{M}, \mu)$. Define

$$L^+ := L^+(X, \mathcal{M}, \mu) = \{ \mathcal{M}\text{-measurable } f : X \to [0,\infty] \}.$$

Def: If $\psi = \sum_{k=1}^{n} c_k \chi_{E_k} \in SF^+ \subset L^+$, define

disjoint and $\neq \emptyset$

- $\int \psi := \int_X \psi \, d\mu := \int_X \psi(x) \, d\mu(x) := \sum_{k=1}^{n} c_k \mu(E_k)$.

distinct and $\neq 0$

- $\int_E \psi := \int \underbrace{\psi \cdot \chi_E}_{\text{write in its ! form.}} \quad \forall E \in \mathcal{M}$.

Thm: The map $\int : SF^+ \longrightarrow [0,\infty]$ satisfies:

- $\forall r \geq 0$, $\int r\psi = r \int \psi$

Pf: $\sum^{n} r c_k \mu(E_k) = r \sum^{n} c_k \mu(E_k)$ if $r > 0$. $r = 0$ easier.

- If $\phi \leq \psi$ everywhere, then $\int \phi \leq \int \psi$

Pf: Suppose $\phi = \sum^{m} a_j \chi_{E_j}$ and $\psi = \sum^{n} b_k \chi_{F_k}$.

Trick: we may assume $\coprod E_j = X = \coprod F_k$ by allowing exactly one term of the form $0 \cdot \chi_{E_j}$, $0 \cdot \chi_{F_k}$. Then $E_j = \coprod_k E_j \cap F_k$ and $F_k = \coprod_j E_j \cap F_k$.

Now $\phi = \sum_{j,k} a_j \chi_{E_j \cap F_k} \leq \psi = \sum_{j,k} b_k \chi_{E_j \cap F_k}$ implies $a_j \leq b_k$ whenever $E_j \cap F_k \neq \emptyset$. Thus:

$$\int \phi = \sum a_j \mu(E_j) = \sum_{j,k} a_j \mu(E_j \cap F_k)$$
$$\leq \sum_{j,k} b_k \mu(E_j \cap F_k) = \sum b_k \mu(F_k) = \int \psi.$$

- $\int \phi+\psi = \int\phi + \int\psi$

Pf: Suppose $\phi = \sum^{m} a_j \chi_{E_j}$, $\psi = \sum^{n} b_k \chi_{F_k}$, and $\phi+\psi = \sum^{p} c_\ell \chi_{G_\ell}$.

Trick: As before, assume $\amalg E_j = \amalg F_k = \amalg G_\ell = X$.

Since $\phi+\psi = (\phi+\psi)$, similar to before, $a_j + b_k = c_\ell$ whenever $E_j \cap F_k \cap G_\ell \neq \emptyset$. Then

$$\begin{aligned}\int\phi + \int\psi &= \sum_j a_j \mu(E_j) + \sum_k b_k \mu(F_k) \\ &= \sum_{j,k} (a_j+b_k)\, \mu(E_j \cap F_k) \\ &= \sum_{j,k,\ell} (a_j+b_k)\, \mu(E_j \cap F_k \cap G_\ell) \\ &= \sum_{j,k,\ell} c_\ell\, \mu(E_j \cap F_k \cap G_\ell) \\ &= \sum_\ell c_\ell\, \mu(G_\ell) = \int \phi+\psi.\end{aligned}$$

Hence $\int : SF^+ \to [0,\infty]$ is an order-preserving $\mathbb{R}^+$-linear map.

Remark: The map $\mathcal{M} \to [0,\infty]$ by $E \mapsto \int_E d\mu$ equals μ.

Lemma: For $\psi \in SF^+$, $\mu_\psi : \mathcal{M} \to [0,\infty]$ by $E \mapsto \int_E \psi$ is a measure.

Pf: $\psi\chi_\emptyset = 0$, so $\mu_\psi(\emptyset) = \int_E \chi_\emptyset = \int 0 = 0$.

Trick: write $\psi = \sum^{k} c_j \chi_{E_j}$ s.t. $\amalg^{n} E_j = X$ as before.

If $(F_n) \subset \mathcal{M}$ is a disjoint seq., then

$$\int_{\amalg F_n} \psi = \sum_j c_j \mu(E_j \cap \amalg F_n) = \sum_{j,n} c_j \mu(E_j \cap F_n) = \sum_n \int_{F_n} \psi.$$

Def: For $f \in L^+$, define $\int f := \sup\left\{ \int\psi \;\middle|\; \begin{array}{c} 0 \le \psi \le f, \\ \psi \in SF^+ \end{array} \right\}$.

Remarks: ① observe this extends $\int\psi$ for $\psi \in SF^+$ since
$\int\psi = \sup\{ \int\phi \mid 0 \le \phi \le \psi,\ \phi \in SF^+ \}$.

② $f,g \in L^+$ w/ $f \le g \Rightarrow \int f \le \int g$. ③ $f \in L^+$, $r \ge 0 \Rightarrow \int rf = r\int f$.

Prop: Suppose $f \in L^+$. Then $\int f = 0 \iff f = 0$ a.e.

Pf: $\Rightarrow$: Prove the contrapositive. If $\neg(f = 0 \text{ a.e.})$, $\exists n > 0$ s.t. $\mu(\{f > \frac{1}{n}\}) > 0$. Then $f \geq \frac{1}{n}\chi_{\{f > \frac{1}{n}\}}$, so

$$\int f \geq \frac{1}{n}\mu(\{f > \tfrac{1}{n}\}) > 0.$$

$\Leftarrow$: **Case 1:** If $f = \sum^n c_k \chi_{E_k} \in SF^+$ in its unique std. form, then $\int f = 0 \iff \mu(E_k) = 0 \;\forall k \iff f = 0$ a.e.

Case 2: If $f \in L^+$ w/ $f = 0$ a.e., then $\forall$ simple $\varphi \in SF^+$ w/ $0 \leq \varphi \leq f$, $\varphi = 0$ a.e. So $\int f = \sup_{\varphi \leq f} \int \varphi = 0$.

Monotone Convergence Thm: Suppose $(f_n) \subset L^+$ is an increasing seq. and $f = \lim f_n \;(= \sup f_n)$. Then

$$\int f = \lim_{n\to\infty} \int f_n .$$

Pf: Observe $(\int f_n)$ is an increasing seq in $[0,\infty]$, and thus converges. Moreover, $\int f_n \leq \int f \;\forall n$, so $\lim_{n\to\infty} \int f_n \leq \int f$.

$\geq$: Pick a simple $0 \leq \varphi \leq f$. Let $0 < \varepsilon < 1$. Set $E_n := \{f_n > \varepsilon\varphi\}$. Then $(E_n) \subset \mathcal{M}$ is an increasing seq. s.t. $\bigcup E_n = X$. Thus

$$\int f_n \geq \int_{E_n} f_n \geq \varepsilon \underbrace{\int_{E_n} \varphi \to \varepsilon \int \varphi}_{F \mapsto \int_F \varphi \text{ is a measure, cts from below.}} \quad \text{as } n \to \infty.$$

Hence $\lim \int f_n \geq \varepsilon \int \varphi \;\forall\, 0 < \varepsilon < 1$. Since ε was arbitrary, letting $\varepsilon \to 1$, we have $\lim \int f_n \geq \int \varphi$. Taking sup over all $\varphi \leq f$ gives $\lim \int f_n \geq \int f$.

Corollaries of MCT:

① $\int f = \lim \int \phi_n$ ∀ seq $(\phi_n) \subset L^+$ s.t. $\phi_n \nearrow f$.

② $\forall f, g \in L^+$, $\int f+g = \int f + \int g$

Pf: If $\phi_n \nearrow f$ and $\psi_n \nearrow g$, $\phi_n + \psi_n \nearrow f+g$, so

$$\int f+g = \lim \int \phi_n + \psi_n = \lim \int \phi_n + \lim \int \psi_n = \int f + \int g.$$

③ $\forall (f_n) \subset L^+$, $\sum \int f_n = \int \underbrace{\sum f_n}_{}$, sup of partial sums, meas.

Pf: By ② + induction, true ∀ finite sums. Then

$$\int \sum f_n = \int \lim_{N\to\infty} \sum^N f_n \overset{(MCT)}{=} \lim_{N\to\infty} \int \sum^N f_n = \lim_{N\to\infty} \sum^N \int f_n$$

④ If $(f_n) \subset L^+$, $f \in L^+$, and $\underbrace{f_n \nearrow f \text{ a.e.}}_{}$, then $\int f = \lim \int f_n$.

$\Rightarrow f \in L^+$ if μ complete.

Pf: Suppose $f_n \nearrow f$ on $E \in \mathcal{M}$ s.t. E^c is μ-null. Then $f - f\chi_E = 0$ a.e., and $f_n - f_n\chi_E = 0$ a.e. So

$$\int f \underset{④}{=} \int f\chi_E \underset{(MCT)}{=} \lim \int f_n \chi_E \underset{④}{=} \lim \int f_n.$$

⑤ [Fatou's Lemma] If $(f_n) \subset L^+$, $\int \liminf f_n \le \liminf \int f_n$.

Pf: $\forall j \ge k \in \mathbb{N}$, $\inf_{n \ge k} f_n \le f_j$, so $\int \inf_{n\ge k} f_n \le \int f_j$ $\forall j \ge k$

Thus $\int \inf_{n \ge k} f_n \le \inf_{j \ge k} \int f_j$. Letting $k \to \infty$, by MCT,

$$\int \liminf f_n \overset{(MCT)}{=} \lim_{k\to\infty} \int \inf_{n \ge k} f_n \le \lim_{k\to\infty} \inf_{j \ge k} \int f_j.$$

⑥ If $(f_n) \subset L^+$, $f \in L^+$, and $f_n \to f$ a.e., then $\int f \le \liminf \int f_n$

Pf: As in ④, suppose $f_n \to f$ on E w/ E^c μ-null.

Then $\int f = \int f \chi_E \underset{(Fatou)}{\le} \liminf \int f_n \chi_E = \liminf \int f_n$.

Exercise: If $f \in L^+$ and $\int f < \infty$, then $\{f = \infty\}$ is μ-null and $\{f > 0\}$ is σ-finite.

Integration of $\overline{\mathbb{R}}$-valued fcts: $(X, \mathcal{M}, \mu)$ fixed meas. space

Def: $f: X \to \overline{\mathbb{R}}$ $\mathcal{M}$-measurable is called integrable if

$\int f_\pm < \infty$ where $f = f_+ - f_-$, $f_+ = 0 \vee f$ and $f_- = -(0 \wedge f)$.

Since $|f| = f_+ + f_-$, f integrable $\iff \int |f| < \infty$.

Prop: $L^1(\mu, \overline{\mathbb{R}}) := \{$integrable $f: X \to \overline{\mathbb{R}}\}$ is a $\mathbb{R}$-vector space. Moreover, $\int : L^1(\mu, \overline{\mathbb{R}}) \longrightarrow \mathbb{R}$ given by $\int f := \int f_+ - \int f_-$ is linear.

Pf: If $r \in \mathbb{R}$ and $f, g \in L^1(\mu, \overline{\mathbb{R}})$, then $|rf + g| \le |r||f| + |g|$ which is integrable. Hence $L^1(\mu, \overline{\mathbb{R}})$ is an $\mathbb{R}$-vector space.

- If $r \in \mathbb{R}$, $f \in L^1(\mu, \overline{\mathbb{R}})$, $\int rf = \int r f_+ - \int r f_- = r \int f_+ - r \int f_- = r \int f$.
- If $f, g \in L^1(\mu, \overline{\mathbb{R}})$, $(f+g)_+ - (f+g)_- = f_+ + g_+ - f_- - g_-$, so $(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+$, all in $L^1(\mu, \overline{\mathbb{R}})$. By Cor ③ of MCT, $\int (f+g)_+ + \int f_- + \int g_- = \int (f+g)_- + \int f_+ + \int g_+$, and rearranging yields the result.

Exercise:

① Show that $\varphi = \sum^n \overset{\text{distinct}, \neq 0}{c_k} \chi_{\overset{\text{disjoint}, \neq \emptyset}{E_k}} \in SF$ integrable $\iff \mu(E_k) < \infty\ \forall k$.

② Show that if $\varphi \in SF$ is integrable, then whenever $\varphi = \sum^n c_k \chi_{E_k}$ [not assumed standard!], $\int \varphi = \sum^n c_k \mu(E_k)$.

Integration of $\mathbb{C}$-valued fcts: $(X, \mathcal{M}, \mu)$ fixed meas. space.

Exercise:

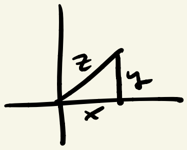
① Show $f: X \longrightarrow \mathbb{C}$ is $\mathcal{M}(\text{-}\mathcal{B}_{\mathbb{C}})$ measurable $\iff$ Re(f) and Im(f) are $\mathcal{M}(\text{-}\mathcal{B}_{\mathbb{R}})$ measurable.] Show directly from the definitions.

② $\{\mathcal{M}\text{-}\mathcal{B}$ measurable fcts $X \longrightarrow \mathbb{C}\}$ is a $\mathbb{C}$-vector space.

③ If $f: X \longrightarrow \mathbb{C}$ is $\mathcal{M}(\text{-}\mathcal{B}_{\mathbb{C}})$ measurable, $|f|: X \longrightarrow [0, \infty)$ is $\mathcal{M}(\text{-}\mathcal{B}_{\mathbb{R}})$ measurable.

Def: We call an $\mathcal{M}$ measurable fct $f: X \to \mathbb{C}$ integrable if $\int |f| < \infty$. Since

$$|f| \le |Re(f)| + |Im(f)| \le 2|f|,$$



f is integrable $\iff$ $Re(f)$ and $Im(f)$ integrable.

In this case, we define $\int f := \int Re(f) + i \int Im(f)$.

$$L^1 = L^1(\mu) = L^1(X, \mu) = L^1(X, \mathcal{M}, \mu) = L^1(\mu, \mathbb{C})$$
$$:= \{\text{integrable } f: X \to \mathbb{C}\}$$

is a $\mathbb{C}$-vector space, and $\int : L^1 \to \mathbb{C}$ is linear.

Prop: $\forall f \in L^1$, $|\int f| \le \int |f|$.

Step 1: If f is $\mathbb{R}$-valued, $|\int f| = |\int f_+ - \int f_-| \le \int f_+ + \int f_- = \int |f|$.

Step 2: We may assume $\int f \neq 0$.

Trick: Let $sgn(\int f) := \frac{\int f}{|\int f|}\Big|_{\in \mathbb{T} \subset \mathbb{C}}$. Then $|\int f| = \overline{sgn(\int f)} \int f = \underbrace{\int \overline{sgn(\int f)} f}_{\in \mathbb{R}}$.

Then $|\int f| = \int \overline{sgn(f)}\, f = Re \int \overline{sgn(f)}\, f \overset{\text{by def'n!}}{=} \int Re[\overline{sgn(f)} f]$

$$\underset{\text{Step 1}}{\le} \int |Re[\overline{sgn(f)} f]| \le \int |\underbrace{\overline{sgn(f)}}_{\in \mathbb{T}} f| = \int |f|.$$

Cor: For $f, g \in L^1$, TFAE:

① $f = g$ a.e.

② $\int |f - g| = 0$

③ $\forall E \in \mathcal{M}$, $\int_E f = \int_E g$.

Pf: ①$\iff$②: $f = g$ a.e. $\iff$ $|f-g| = 0$ a.e. $\iff$ $\int |f-g| = 0$

②$\Rightarrow$③: By the prop, $\forall E \in \mathcal{M}$,

$$\left|\int_E f - \int_E g\right| = \left|\int (f-g)\chi_E\right| \le \int |f-g|\chi_E \le \int |f-g| = 0.$$

③⇒①: Observe that $f\sim g$ a.e. $\iff \underbrace{\mathrm{Re}(f-g)=0 \text{ and } \mathrm{Im}(f-g)=0 \text{ a.e.}}_{\text{Prove this!}}$

Recall $\int_E f-g := \int \mathrm{Re}(f-g) + i\int \mathrm{Im}(f-g)$. So

$$\int_E \mathrm{Re}(f-g)=0 \quad \text{and} \quad \int_E \mathrm{Im}(f-g)=0 \qquad \forall E\in\mathcal{M}.$$

We now look at the following particular $E\in\mathcal{M}$:

$$\left.\begin{aligned} \{\mathrm{Re}(f-g)\geq 0\} &\Rightarrow \mathrm{Re}(f-g)_+=0 \text{ a.e.} \\ \{\mathrm{Re}(f-g)\leq 0\} &\Rightarrow \mathrm{Re}(f-g)_-=0 \text{ a.e.} \end{aligned}\right] \Rightarrow \mathrm{Re}(f-g)=0 \text{ a.e.}$$

$$\left.\begin{aligned} \{\mathrm{Im}(f-g)\geq 0\} &\Rightarrow \mathrm{Im}(f-g)_+=0 \text{ a.e.} \\ \{\mathrm{Im}(f-g)\leq 0\} &\Rightarrow \mathrm{Im}(f-g)_-=0 \text{ a.e.} \end{aligned}\right] \Rightarrow \mathrm{Im}(f-g)=0 \text{ a.e.}$$

Def: Define $\mathcal{L}^1 := L^1/\sim$ where $f\sim g \iff f=g$ a.e.

- We'll show later that $\|f\|_1 := \int|f|$ is a norm on $\mathcal{L}^1$, and moreover, $\mathcal{L}^1$ is complete wrt $\|\cdot\|_1$

  ↳ $\rho_1(f,g):=\|f-g\|_1$ is a metric s.t. $(\mathcal{L}^1,\rho_1)$ complete.

  $\|\cdot\|_1 : \mathcal{L}^1 \to [0,\infty)$ s.t.
  - $\|f\|_1=0 \iff f=0$
  - $\|\lambda f\|_1=|\lambda|\cdot\|f\|_1$
  - $\|f+g\|_1\leq\|f\|_1+\|g\|_1$
- write $f\in\mathcal{L}^1$ to mean $f\in L^1$ representing its eq. class in $\mathcal{L}^1$.
- say $(f_n)\to f$ in $\mathcal{L}^1$ if $\int|f_n-f|\to 0$ as $n\to\infty$.

Remarks: Let $(X,\overline{\mathcal{M}},\bar{\mu})$ be completion of $(X,\mathcal{M},\mu)$.

① If f is $\overline{\mathcal{M}}$-measurable and $g=f$ a.e., g is $\overline{\mathcal{M}}$-meas.

② If f is $\mathcal{M}$-measurable, $\exists$ $\overline{\mathcal{M}}$-measurable g s.t. $f=g$ a.e.

③ If (f_n) is a seq. of $\overline{\mathcal{M}}$-meas. fcts and $f_n\to f$ a.e., then f is $\overline{\mathcal{M}}$-meas.

④ If (f_n) is a seq. of $\mathcal{M}$-meas. fcts and $f_n\to f$ a.e., then f is $\overline{\mathcal{M}}$-measurable, so $\exists$ an $\mathcal{M}$-meas. fct g s.t. $f_n\to g$ a.e.

Pf: Homework!

Dominated Convergence Thm: Suppose $(f_n) \subset \mathcal{L}^1$ s.t. $f_n \to f$ a.e. and $\exists g \in L^1 \cap L^+$ s.t. $|f_n| \le g$ a.e. $\forall n$. Then $f \in \mathcal{L}^1$ and $\int f = \lim \int f_n$.

special case: $\mu(X) < \infty$, $|f_n| \le M$ a.e. $\forall n$.

Pf: we may assume f is $\mathcal{M}$-measurable by ⊕ above. Taking limits $|f| \le g$, so $f \in \mathcal{L}^1$. Taking Re and Im, we may assume f_n, f are $\mathbb{R}$-valued. Then $-g \le f_n \le g$ a.e. so

$$g + f_n \ge 0 \text{ a.e. } \underline{\text{and}} \quad g - f_n \ge 0 \text{ a.e.}$$

By Fatou's Lemma,

$$\int g + \int f = \int g + f \le \liminf \int g + f_n = \int g + \liminf \int f_n$$

$$\int g - \int f = \int g - f \le \liminf \int g - f_n = \int g - \limsup \int f_n$$

Combining these, $\limsup \int f_n \le \int f \le \liminf \int f_n$.

Cor: Suppose $(f_n) \subset \mathcal{L}^1$ s.t. $\sum \int |f_n| < \infty$. Then $\sum f_n$ converges a.e. to a fct in $\mathcal{L}^1$ and $\int \sum f_n = \sum \int f_n$.

Pf: By Cor to MCT, $\sum |f_n| \in \mathcal{L}^1$ w/ $\int \sum |f_n| = \sum \int |f_n|$. So $\sum |f_n(x)| < \infty$ a.e., so $\sum f_n(x)$ converges a.e. Now apply DCT for $\underbrace{|\overset{\wedge}{\sum} f_n|}_{\text{seq. in } \mathcal{L}^1} \le \underbrace{\sum |f_n|}_{g}$.

Modes of Convergence: Let $(X, \mathcal{M}, \mu)$ be a meas. space. For $(f_n), f$ $\mathcal{M}$-$\mathcal{B}_{\mathbb{C}}$ measurable fcts, "$f_n \to f$" could mean:

- (pointwise) $f_n(x) \to f(x)$ $\forall x \in X$

$\Downarrow$

- (pointwise a.e.) $f_n(x) \to f(x)$ a.e. $x \in X$
- (uniformly) $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$ $\underline{\forall x}$

$\Downarrow$

- ($\underbrace{\text{almost uniform}}_{\text{a.u.}}$) $\forall \varepsilon > 0$, $\exists E \subset X$ w/ $\mu(E) < \varepsilon$ s.t. $f_n \chi_{E^c} \to f \chi_{E^c}$ uniformly.

- (in $\mathcal{L}^1$) $\int |f_n - f| \to 0$ as $n \to \infty$.
- (in <u>measure</u>) $\forall \varepsilon > 0$, $\mu(\{|f - f_n| \geq \varepsilon\}) \to 0$.

<u>Question</u>: which modes of convergence imply each other?

① <u>(a.u.) $\Rightarrow$ (a.e.)</u>: Suppose $f_n \to f$ a.u. $\forall n \in \mathbb{N}$, let $E_n \in \mathcal{M}$ s.t. $\mu(E_n) < \frac{1}{n}$ and $f_n \chi_{E_n^c} \to f \chi_{E_n^c}$ unif. Let $E = \bigcap E_n$. Then $\mu(E) = 0$ by cts from above, and $f_n \chi_{E^c} \to f \chi_{E^c}$ ptwise.

② <u>(a.u.) $\Rightarrow$ (in measure)</u>: Suppose $f_n \to f$ a.u. Let $\varepsilon > 0$. Show $\forall \delta > 0$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow \mu(\{|f - f_n| \geq \varepsilon\}) < \delta$. Pick $E \in \mathcal{M}$ s.t. $\mu(E) < \delta$ and $f_n \chi_{E^c} \to f \chi_{E^c}$ unif. Then

$$\mu(\{|f-f_n| \geq \varepsilon\}) = \underbrace{\mu(\{|f-f_n| \geq \varepsilon\} \cap E)}_{\text{always } < \delta} + \underbrace{\mu(\{|f-f_n| \geq \varepsilon\} \cap E^c)}_{= \emptyset \text{ for } n \text{ large.}}$$

③ <u>($\mathcal{L}^1$) $\Rightarrow$ (in measure)</u>: Suppose $f_n \to f$ in $\mathcal{L}^1$. For $\varepsilon > 0$,

$$\mu(\underbrace{\{|f-f_n| \geq \varepsilon\}}_{E}) = \int_E 1 = \frac{1}{\varepsilon} \int_E \varepsilon \leq \frac{1}{\varepsilon} \int |f - f_n| \to 0 \text{ as } n \to \infty.$$

<u>Counterexamples to keep in mind:</u>

① $f_n = \frac{1}{n} \chi_{(0,n)}$ $f_n \to 0$ <u>uniformly</u>, <u>not</u> in $\mathcal{L}^1$.

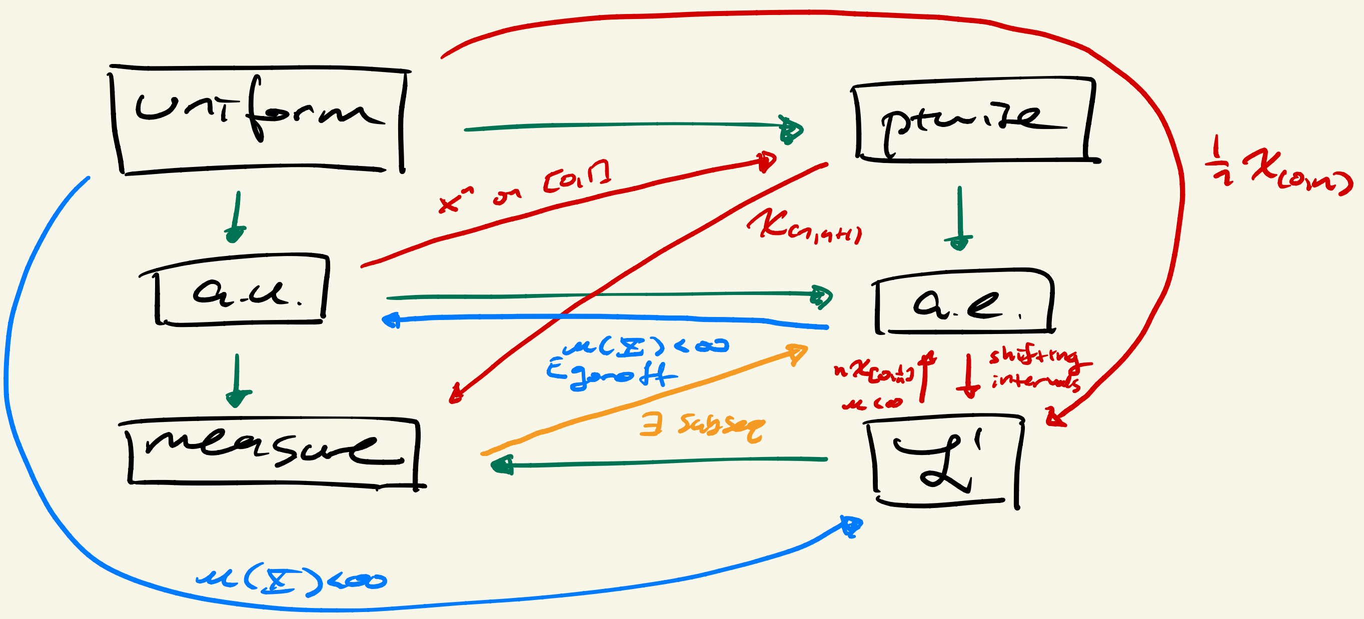
② $f_n = \chi_{(n,n+1)}$ $f_n \to 0$ <u>ptwise</u>, <u>not</u> in <u>measure</u> or $\mathcal{L}^1$.

③ $f_n = n \chi_{[0,\frac{1}{n}]}$ $f_n \to 0$ <u>a.e.</u> + <u>$\mu(X) < \infty$</u>, <u>not</u> in $\mathcal{L}^1$.

④ $f_n(x) = x^n$ on $[0,1]$ $f_n \to 0$ <u>almost unif.</u>, <u>not</u> <u>ptwise</u>.

⑤ $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,\frac{1}{2}]}$, $f_3 = \chi_{[\frac{1}{2},1]}$, $f_4 = \chi_{[0,\frac{1}{4}]}$, $f_5 = \chi_{[\frac{1}{4},\frac{1}{2}]}$, ...

$f_n \to 0$ in <u>$\mathcal{L}^1$</u>, <u>not</u> a.e.



Lemma: If $f_n \to f$ uniformly and $\mu(X)<\infty$, $f_n \to f$ in $\mathcal{L}^1$.

Pf:
$$\int |f_n - f| \le \left[\sup |f_n - f|\right] \cdot \int 1 = \underbrace{\left[\sup |f_n - f|\right]}_{\to 0 \text{ as } n\to\infty.} \cdot \mu(X)$$

Thm (Egoroff): If $f_n \to f$ a.e. and $\mu(X)<\infty$, $f_n \to f$ a.u.

Pf: We may assume $f_n \to f$ everywhere by replacing X with $X \setminus N$ for some μ-null N. Observe that $\forall k \in \mathbb{N}$,

$$E_{n,k} := \bigcup_{j=n}^{\infty} \{|f - f_j| \ge \tfrac{1}{k}\} \searrow \emptyset \quad \text{as } n \to \infty$$

By cts from above as $\mu(X)<\infty$, $\mu(E_{n,k}) \to 0$ as $n \to \infty$.

Let $\varepsilon>0$. For all $k\in\mathbb{N}$, choose $n_k \in \mathbb{N}$ s.t. $\mu(E_{n_k,k}) < \frac{\varepsilon}{2^k}$.

Setting $E := \bigcup_{k=1}^{\infty} E_{n_k,k}$, $\mu(E) \le \sum_k \mu(E_{n_k,k}) < \varepsilon \sum \frac{1}{2^k} = \varepsilon$, and $\forall n > n_k$, $x \notin E \Rightarrow x \notin E_{n_k,k} \Rightarrow |f(x) - f_n(x)| < \frac{1}{k}$. Thus $f_n \to f$ unif on $X \setminus E$.

Def: A seq. (f_n) of $\mathcal{M}$-meas. fcts is Cauchy in measure if $\forall \varepsilon>0$, $\mu(\{|f_m-f_n|\geq\varepsilon\})\to 0$ as $m,n\to\infty$.

Thm: If (f_n) is Cauchy in measure, $\exists!$ (up to μ-a.e.) $\mathcal{M}$-measurable f s.t. $f_n\to f$ in measure. Moreover, $\exists$ subseq. (f_{n_k}) s.t. $f_{n_k}\to f$ a.e.

Pf: **Step 1:** $\exists$ subseq. (f_{n_k}) s.t. $\mu(\{|f_{n_k}-f_{n_{k+1}}|\geq 2^{-k}\})<2^{-k}$.

Pf: $\forall k\in\mathbb{N}$, $\mu(\{|f_m-f_n|\geq 2^{-k}\})\to 0$ as $m,n\to\infty$. Pick n_k inductively so $n_{k+1}>n_k$ and $m,n\geq n_k \Rightarrow \mu(\{|f_m-f_n|\geq 2^{-k}\})<2^{-k}$.

Step 2: (f_{n_k}) ptwise Cauchy off a μ-null set N.

Pf: For $k\in\mathbb{N}$, set $E_k:=\{|f_{n_k}-f_{n_{k+1}}|\geq 2^{-k}\}$ and $N_\ell:=\bigcup_{k=\ell}^{\infty}E_k$. Then $\mu(N_\ell)\leq\sum_{k=\ell}^{\infty}2^{-k}=2^{1-\ell}$. Set $N:=\bigcap N_\ell=\limsup E_k$, so $\mu(N)=0$. If $x\notin N$, then $x\notin N_\ell$ for some ℓ, so $\forall\, \ell\leq i\leq j$,

(*) $$|f_{n_i}(x)-f_{n_j}(x)|\leq\sum_{k=i}^{j-1}|f_{n_k}(x)-f_{n_{k+1}}(x)|\leq\sum_{k=i}^{j-1}2^{-k}\leq 2^{1-i}.$$

Step 3: Define $$f(x):=\begin{cases}0 & x\in N \quad (\mu\text{-null})\\ \lim_k f_{n_k}(x) & x\notin N\end{cases}$$

Then f is measurable and $f_n\to f$ a.e.

Pf: Observe each $f_{n_k}|_{N^c}$ measurable wrt $\mathcal{M}|_{N^c}=\{E\cap N^c\mid E\in\mathcal{M}\}\subset\mathcal{M}$. By HW, $f|_{N^c}=\lim_k f_{n_k}$ is $\mathcal{M}|_{N^c}$-measurable. Observe $f=\chi_{N^c}\cdot f|_{N^c}$ and $\mathcal{M}|_{N^c}\subset\mathcal{M}$, so f is $\mathcal{M}$-measurable. Finally, $f_{n_k}\to f$ a.e. by construction.

Step 4: $f_{n_k}\to f$ in measure.

Pf: $\forall x \notin N_\ell$ and $k \geq \ell$, we have

$$|f_{n_k}(x) - f(x)| = \lim_{j\to\infty} |f_{n_k}(x) - f_{n_j}(x)| \leq 2^{1-k}.$$

Let $\varepsilon > 0$ and pick $\ell \in \mathbb{N}$ s.t. $0 < \frac{1}{2^\ell} < \varepsilon$. Then $\forall k \geq \ell$

$$\mu(\{|f_{n_k} - f| \geq \varepsilon\}) \leq \mu(\{|f_{n_k} - f| \geq \tfrac{1}{2^k}\}) < \tfrac{1}{2^k} \to 0.$$

Step 5: $f_n \to f$ in measure.

Pf: $$\{|f_n - f| \geq \varepsilon\} \subseteq \underbrace{\{|f_n - f_{n_k}| \geq \tfrac{\varepsilon}{2}\}}_{\substack{\mu \to 0 \text{ as } (f_n) \\ \text{Cauchy in measure}}} \cup \underbrace{\{|f_{n_k} - f| \geq \tfrac{\varepsilon}{2}\}}_{\mu \to 0 \text{ by Step 4.}}$$

Step 6: f is the unique (up to μ-a.e.) measurable fct s.t. $f_n \to f$ in measure.

Pf: If g is another such candidate,

$$\{|f-g| \geq \varepsilon\} \subseteq \underbrace{\{|f - f_n| \geq \tfrac{\varepsilon}{2}\}}_{\mu \to 0 \text{ as } n \to \infty} \cup \underbrace{\{|g - f_n| \geq \tfrac{\varepsilon}{2}\}}_{\mu \to 0 \text{ as } n \to \infty}.$$

Hence $\mu(\{|f-g| \geq \varepsilon\}) = 0$ $\forall \varepsilon > 0$, and $f = g$ a.e.

Review of Riemann Integral: $f : [a,b] \longrightarrow \mathbb{R}$

A partition of $[a,b]$ is a set of pts. $P = \{a = s_0 < s_1 < \cdots < s_m = b\}$.

Say an interval $J \in P$ if $J = [s_{i-1}, s_i]$ for $i = 1, \ldots, m$. Write

$$m_J = \inf\{f(x) \mid x \in J\} \quad \text{and} \quad M_J = \sup\{f(x) \mid x \in J\}$$

- Lower sum: $L(f,P) := \sum_{J \in P} m_J \lambda(J)$ ← length of interval.

  $\rotatebox{90}{$\leq$}$ $\qquad$ $\rotatebox{90}{$\leq$}$

- Upper Sum: $U(f,P) := \sum_{J \in P} M_J \lambda(J)$

A refinement of P is a $Q = \{a = t_0 < \cdots < t_n = b\} \supseteq P$.

Observe that if Q refines P, then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

If P_1, P_2 are two partitions of $[a,b]$ and Q is a common refinement,

$$\max_{i=1,2} L(f,P_i) \le L(f,Q) \le U(f,Q) \le \min_{i=1,2} U(f,P_i).$$

Define:

- upper integral $\quad \overline{\int_{[a,b]}} f := \inf_P U(f,P)$

$\geq \qquad\qquad \geq$

- lower integral $\quad \underline{\int_{[a,b]}} f := \sup_P L(f,P)$

Say f is Riemann integrable on $[a,b]$ if $\overline{\int_{[a,b]}} f = \underline{\int_{[a,b]}} f$.

Exercise: Suppose $f:[a,b] \to \mathbb{R}$. TFAE:

① f Riemann integrable

② $\forall \varepsilon > 0$, $\exists$ partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Thm: If f is Riemann integrable on $[a,b]$, then f is Lebesgue integrable and $\int_{[a,b]} f \, dx = \int_a^b f(x)\,dx$.

Pf: Let (P_n) be a seq. of partitions of $[a,b]$ s.t. P_{n+1} refines P_n and $U(f,P_n) - L(f,P_n) < \frac{1}{n}$.

Trick: Define $\varphi_n := \sum_{J \in P_n} m_J \chi_J$ and $\Phi_n := \sum_{J \in P_n} M_J \chi_J$

- $L(f,P) = \int \varphi_n$ and $U(f,P) = \int \Phi_n$ $\forall n$ and
- $\varphi_n \le \varphi_{n+1} \le f \le \Phi_{n+1} \le \Phi_n$ $\forall n$.

Define $\varphi := \lim \varphi_n$, $\Phi := \lim \Phi_n$. Then by MCT, φ, Φ integrable and $\int \varphi = \lim \int \varphi_n = \int_a^b f(x)\,dx = \lim \int \Phi_n = \int \Phi$.

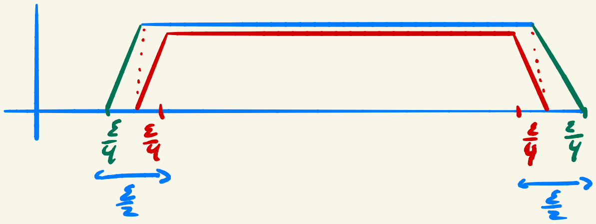
But $\int \underbrace{\Phi - \varphi}_{\geq 0!} = 0 \Rightarrow \Phi = \varphi = f$ a.e. So $f \in \mathcal{L}^1$ and $\int f = \int_a^b f(x)\,dx$.

Lemma: Suppose $f:[a,b]\to\mathbb{R}$ is Riemann integrable and bdd. $\forall\varepsilon>0$, $\exists$ cts fcts $g,h:[a,b]\to\mathbb{R}$ s.t. $h\le f\le g$ and

$$\int_{[a,b]} g-h \, d\lambda \le \varepsilon .$$

Pf:

Step 1: If $f=\chi_J$ for some interval J, then find piecewise linear fcts h,g s.t. $h\le f\le g$ such as:



Then $\int_{[a,b]} g=\lambda(J)+\frac{\varepsilon}{2}$ and $\int_{[a,b]} h=\lambda(J)-\frac{\varepsilon}{2}$, so $\int g-h=\varepsilon$.

Step 2: WLOG, $f\geq 0$. Take a partition P of $[a,b]$ s.t. $U(f,P)-L(f,P)<\frac{\varepsilon}{3}$. As in the trick in the previous theorem, define $\psi:=\sum_{J\in P} m_J\chi_J \le f \le \Psi:=\sum_{J\in P} M_J\chi_J$

$\int\psi=L(f,P)$ and $\int\Psi=U(f,P)$. Perform Step 1 to each χ_J to get cts g_J,h_J w/ $h_J\le\chi_J\le g_J$ s.t. $\int g_J-h_J<\frac{\varepsilon}{3|P|M}$ where $|P|:=$ # intervals in P and $M:=\sup\{f(x)\mid a\le x\le b\}$. Set $g:=\sum M_J g_J$ and $h:=\sum m_J h_J$ so that

$$h=\sum m_J h_J\le\sum m_J\chi_J=\psi\le f\le\Psi=\sum M_J\chi_J\le M_J g_J=g.$$

$$\int g-h=\sum_J M_J\int g_J-m_J\int h_J$$

$$=\underbrace{\sum_J \underbrace{M_J}_{<M}\underbrace{\left(\int g_J-\lambda(J)\right)}_{<\frac{\varepsilon}{3|P|M}}}_{<\frac{\varepsilon}{3}}+\underbrace{\sum_J \underbrace{m_J}_{<M}\underbrace{\left(\lambda(J)-\int h_J\right)}_{<\frac{\varepsilon}{3|P|M}}}_{<\frac{\varepsilon}{3}}+\underbrace{U(f,P)-L(f,P)}_{<\frac{\varepsilon}{3}}$$

$<\varepsilon$.

Lebesgue's Thm: A $\underline{\text{bdd}}$ fct $f: [a,b] \to \mathbb{R}$ is Riemann integrable $\iff$ f is continuous a.e.

Proof: $\Rightarrow$: Suppose f is Riemann integrable. By the lemma, $\exists$ seq. of cts fcts $h_n \le f \le g_n$ s.t. $\int g_n - h_n < \frac{1}{n}$. Since $\int g_{n+1} \wedge g_n - h_{n+1} \wedge h_n \le \int g_{n+1} - h_{n+1} < \frac{1}{n+1}$, we may assume $h_n \le h_{n+1} \le f \le g_{n+1} \le g_n$ $\forall n \in \mathbb{N}$. Let $h := \lim h_n$, $g := \lim g_n$. Then $h \le f \le g$ and $\int h = \int f = \int g$ by MCT, so $g - h \geq 0$ implies $g = h$ a.e. on $[a,b]$.

Claim: Since $g_n \searrow g$, g is upper semicts. Similarly, h lower semicts.

Pf: Let $x_0 \in [a,b]$ and $\varepsilon > 0$. Pick $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow f_n(x_0) - f(x_0) < \frac{\varepsilon}{2}$. Pick $\delta > 0$ s.t. $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b] \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{2}$. Then $\forall x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$,

$$f(x_0) \geq f_n(x_0) - \frac{\varepsilon}{2} \geq f_n(x) - \varepsilon \geq f(x) - \varepsilon.$$

whenever $h(x_0) = g(x_0)$ $(= f(x_0))$, f is both upper + lower semicts at x_0, i.e., f is $\underline{\text{cts}}$ at x_0. This happens a.e. on $[a,b]$.

$\Leftarrow$: Suppose f is cts a.e. on $[a,b]$. Let E be the set of discontinuities. Let $\varepsilon > 0$. We'll construct a partition P s.t. $U(f,P) - L(f,P) < \varepsilon$. Take an open $\mathcal{U} \supset E$ s.t. $\lambda(\mathcal{U}) < \underset{\text{TBD!}}{\underline{\varepsilon'}}$. Let $K = [a,b] \setminus \mathcal{U}$, cpt, and $f|_K$ is cts. $\forall x \in K$, $\exists \delta_x > 0$ s.t. $y \in \underset{(!)}{\underline{[a,b]}}$ and $|x - y| < \delta_x \Rightarrow |f(x) - f(y)| < \varepsilon'$. Then $\{B_{\delta_x/2}(x)\}_{x \in K}$ forms an open cover of K, so $\exists$ finite subcover centered at $x_1, \dots, x_n$. Let $\delta := \min\{\delta_{x_i}/2 \mid i = 1, \dots, n\}$.

Claim: If $x\in K$, $y\in[a,b]$ and $|x-y|<\delta$, $|f(x)-f(y)|<2\varepsilon'$.

Pf: WLOG, $x\in B_{\delta_1/2}(x_1)$. Then $y\in B_{\delta_1}(x_1)$, so

$$|f(x)-f(y)| \le |f(x)-f(x_1)| + |f(x_1)-f(y)| < 2\varepsilon'.$$

Let P be a partition of $[a,b]$ whose intervals have length at most δ. Let P' consist of the intervals that intersect K and let P'' be the intervals that do not. By the claim, if $J\in P'$, then $M_J - m_J \le 2\varepsilon'$. Thus

$$\begin{aligned}
U(f,P)-L(f,P) &= \sum_{J\in P}(M_J-m_J)\lambda(J)\\
&= \sum_{J\in P'}(M_J-m_J)\lambda(J) + \sum_{J\in P''}(M_J-m_J)\lambda(J)\\
&\le \sum_{J\in P'} 2\varepsilon'\lambda(J) + (M-m)\sum_{J\in P''}\lambda(J)\\
&\le 2\varepsilon'(b-a) + (M-m)\lambda(\mathcal{U}) \quad \Big]\ \bigcup_{J\in P''} J \subseteq \mathcal{U}\\
&< \varepsilon'\left[2(b-a)+(M-m)\right].
\end{aligned}$$

Taking $\varepsilon' = \dfrac{\varepsilon}{2(b-a)+(M-m)}$ does it.