2. Measures

We begin with an informal discussion.

Definition 2.0.1. Let X be a set. A *measure* on X is a function $\mu : \mathcal{M} \to [0, \infty]$ where $\mathcal{M} \subset P(X)$ is some collection of subsets (whose properties are to be determined) satisfying:

(1) $\mu(\emptyset) = 0$
(2) $\mu(\coprod E_n) = \sum \mu(E_n)$ when (E_n) is a collection of mutually disjoint subsets in $\mathcal{M}$, where $\coprod$ means *disjoint union*.

We now would like to discuss what kind of properties the subset $\mathcal{M} \subset P(X)$ should satisfy.

- $\emptyset, X \in \mathcal{M}$ ($\mathcal{M}$ is nonempty)
- closed under disjoint unions (finite? countable?)

Example 2.0.2 (Counting measure)**.** Let $\mathcal{M} = P(X)$ and $\mu(E) := |E|$.

Example 2.0.3 (Lebesgue measure)**.** There is a measure λ on some $\mathcal{M} \subset P(\mathbb{R})$ such that

- (normalized) $\lambda([0, 1)) = 1$, and
- (translation invariant) $\lambda(E + r) = \lambda(E)$ for all $E \in \mathcal{M}$ and $r \in \mathbb{R}$.

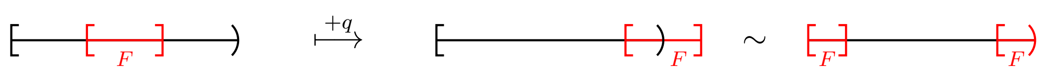
For this λ, we cannot have $\mathcal{M} = P(\mathbb{R})$! Indeed, define an equivalence relation on $[0, 1)$ by

$$x \sim y \qquad \Longleftrightarrow \qquad x - y \in \mathbb{Q}.$$

Using the Axiom of Choice, pick one representative from each equivalence class, and call this set E. For $q \in E \cap [0, 1)$, define

$$E_q := \{x + q | x \in E \cap [0, 1 - q)\} \cup \{x + q - 1 | x \in [1 - q, 1)\}.$$

Here is a cartoon of the basic idea:



Observe that there is some countable subset $Q \subset \mathbb{Q}$ such that $[0, 1) = \coprod_{q \in Q} E_q$.

Now if $\mathcal{M} = P(X)$, then we'd have

$$1 = \lambda([0,1)) = \lambda\left(\coprod_{q \in Q} E_q\right) = \sum_{q \in Q} \lambda(E_q) = \sum_{q \in Q} \lambda(E) = \lambda(E) \sum 1 \in \{0, \infty\},$$

a contradiction.

Exercise 2.0.4. Let X be a nonempty set and $\mathcal{E} \subset P(X)$ any collection of subsets which is closed under finite unions and intersections. Suppose $\nu : P(X) \to [0, \infty]$ be a function which satisfies

- (finite additivity) for any disjoint sets $E_1, \ldots, E_n \in P(X)$, $\nu\left(\coprod_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \nu(E_i)$.

Prove that ν also has the following properties.

(1) (monotonicity) Show that if $A, B \in \mathcal{E}$ with $A \subset B$, then $\nu(A) \leq \nu(B)$.
(2) (finite subadditivity) Show that for any (not necessarily disjoint) sets $E_1, \ldots, E_n \in \mathcal{E}$, $\nu\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \nu(E_i)$.
(3) Show that for all $A, B \in \mathcal{E}$, $\nu(A) + \nu(B) = \nu(A \cup B) + \nu(A \cap B)$.

Exercise 2.0.5. Suppose $\mathcal{E} \subset P(\mathbb{R})$ is any collection of subsets which contains the bounded open intervals and is closed under countable unions. Let $\nu : \mathcal{E} \to [0, \infty]$ be a function which satisfies

- (monotonicity) If $E, F \in \mathcal{E}$ with $E \subset F$, then $\nu(E) \leq \nu(F)$.
- (subadditivty) for any sequence of sets $(E_n)_{n=1}^{\infty} \subset \mathcal{E}$, $\nu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \nu(E_n)$.
- (extends length of open intervals) for all $a < b$ in $\mathbb{R}$, we have $\nu((a, b)) = b - a$.

Show that if $E \in \mathcal{E}$ is countable, then $\nu(E) = 0$.

2.1. σ-algebras.

Definition 2.1.1. A non-empty subset $\mathcal{M} \subset P(X)$ is called an *algebra* if

(1) $\mathcal{M}$ is closed under finite unions, and
(2) $\mathcal{M}$ is closed under complements.

Observe that every algebra

- contains $X = E \amalg E^c$ for some $E \in \mathcal{M}$, and thus $\emptyset = X^c$.
- is closed under finite intersections

$$\bigcap_1^k E_n = \left(\bigcap_1^k E_n\right)^{cc} = \left(\bigcup_1^k E_n^c\right)^c$$

If in addition an algebra $\mathcal{M}$ is closed under *countable* unions, then we call $\mathcal{M}$ a σ-algebra. Here, the 'σ' signifies 'countable'. We call the elements of a σ-algebra *measurable sets*.

Examples 2.1.2. Lex X be a set.

(1) $\{\emptyset, X\}$ is the *trivial* σ-algebra.
(2) $P(X)$ is the *discrete* σ-algebra.

Exercise 2.1.3. Define $\mathcal{M} := \{E \subset X | E \text{ or } E^c \text{ is countable}\}$. Show that $\mathcal{M}$ is a σ-algebra.

Exercise 2.1.4. Let X be a set. A *ring* $\mathcal{R} \subset P(X)$ is a collection of subsets of X which is closed under unions and set differences. That is, $E, F \in \mathcal{R}$ implies $E \cup F \in \mathcal{R}$ and $E \setminus F \in \mathcal{R}$.

(1) Let $\mathcal{R} \subset P(X)$ be a ring.
  (a) Prove that $\emptyset \in \mathcal{R}$.
  (b) Show that $E, F \in \mathcal{R}$ implies the symmetric difference $E \triangle F \in \mathcal{R}$.
  (c) Show that $E, F \in \mathcal{R}$ implies $E \cap F \in \mathcal{R}$.
(2) Show that any ring $\mathcal{R} \subset P(X)$ is an algebraic ring where the addition is symmetric difference and multiplication is intersection.
  (a) What is $0_{\mathcal{R}}$?
  (b) Show that this algebraic ring has *characteristic* 2, i.e., $E + E = 0_{\mathcal{R}}$ for all $E \in \mathcal{R}$.
  (c) When is the algebraic ring $\mathcal{R}$ unital? In this case, what is $1_{\mathcal{R}}$?
  (d) Determine the relationship (if any) between an algebra of sets in the sense of measure theory and an algebra in the algebraic sense.
  (e) Sometimes an algebra in measure theory is called a *field*. Why?

Trick. Suppose (E_n) is a sequence of subsets of X. Inductively define

$$F_1 := E_1 \qquad\qquad F_k := E_k \setminus \bigcup_1^{k-1} E_n = E_k \cap \left(\bigcup_1^{k-1} E_n\right)^c. \tag{II}$$

Then (F_n) is a sequence of pairwise disjoint subsets of X such that $\bigcup E_n = \coprod F_n$. Moreover, observe that if $(E_n) \subset \mathcal{M}$ for some algebra $\mathcal{M}$, then $(F_n) \subset \mathcal{M}$.

Definition 2.1.5. Observe that if $\mathcal{M}, \mathcal{N}$ are σ-algebras, then so is $\mathcal{M} \cap \mathcal{N}$. This means if $\mathcal{E} \subset P(X)$, there is a *smallest* σ-algebra $\mathcal{M}(\mathcal{E})$ which contains $\mathcal{E}$ called the σ-algebra *generated by* $\mathcal{E}$.

Exercise 2.1.6. Let $\mathcal{A} \subset P(X)$ be an algebra. Show that the following are equivalent:

(1) $\mathcal{A}$ is a σ-algebra,
(2) $\mathcal{A}$ is closed under countable disjoint unions, and
(3) $\mathcal{A}$ is closed under countable increasing unions.

Fact 2.1.7. Suppose $\mathcal{E}, \mathcal{F} \subset P(X)$ with $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$. Then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.

Example 2.1.8. Suppose $(X, \mathcal{T})$ is a topological space. We call $\mathcal{B}_{\mathcal{T}} := \mathcal{M}(\mathcal{T})$ the *Borel* σ-algebra.

Remark 2.1.9.

- A countable intersection of open sets is called a G_δ set.
- A countable union of closed sets is called an F_σ set.
- A countable union of G_δ sets is called a $G_{\delta\sigma}$ set.
- A countable intersection of F_σ sets is called an $F_{\sigma\delta}$ set.

And so on and so forth. Observe that $\mathcal{B}_{\mathcal{T}}$ contains all these types of sets, so $\mathcal{B}_{\mathcal{T}}$ is much larger than $\mathcal{T}$.

Proposition 2.1.10. *The Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ on $\mathbb{R}$ generated by the usual topology (which is induced by the metric $\rho(x, y) = |x - y|$) is also generated by the following collections of sets:*

($\mathcal{B}_{\mathbb{R}}1$) open intervals (a, b)
($\mathcal{B}_{\mathbb{R}}2$) closed intervals $[a, b]$
($\mathcal{B}_{\mathbb{R}}3$) half-open intervals $(a, b]$
($\mathcal{B}_{\mathbb{R}}4$) half-open intervals $[a, b)$
($\mathcal{B}_{\mathbb{R}}5$) open rays (a, ∞) and $(-\infty, a)$
($\mathcal{B}_{\mathbb{R}}6$) closed rays $[a, \infty)$ and $(-\infty, a]$

Proof. First, observe that each of ($\mathcal{B}_{\mathbb{R}}1$), ($\mathcal{B}_{\mathbb{R}}2$), ($\mathcal{B}_{\mathbb{R}}5$), ($\mathcal{B}_{\mathbb{R}}6$) are all open or closed, so they lie in $\mathcal{B}_{\mathbb{R}}$. Also, $(a, b] = (a, \infty) \cap (b, \infty)^c$, so each of the sets ($\mathcal{B}_{\mathbb{R}}3$) are contained in $\mathcal{B}_{\mathbb{R}}$. Similarly for ($\mathcal{B}_{\mathbb{R}}4$). Hence each of ($\mathcal{B}_{\mathbb{R}}1$)–($\mathcal{B}_{\mathbb{R}}6$) lie in $\mathcal{B}_{\mathbb{R}}$, so their generated σ-algebras are contained in $\mathcal{B}_{\mathbb{R}}$ by Fact 2.1.7.

For the other directions, observe all open sets in $\mathbb{R}$ are countable unions of open intervals. (You proved this on HW1.) Hence $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}((\mathcal{B}_{\mathbb{R}}1))$ by Fact 2.1.7. For $(j) =$ ($\mathcal{B}_{\mathbb{R}}2$)–($\mathcal{B}_{\mathbb{R}}6$), one shows that ($\mathcal{B}_{\mathbb{R}}1$) is contained in $\mathcal{M}((j))$:

$$(a,b) = \bigcup \left[a+\frac{1}{n}, b-\frac{1}{n}\right] \tag{$\mathcal{B}_{\mathbb{R}}2$}$$
$$= \bigcup \left(a, b-\frac{1}{n}\right] \tag{$\mathcal{B}_{\mathbb{R}}3$}$$
$$= \bigcup \left[a+\frac{1}{n}, b\right) \tag{$\mathcal{B}_{\mathbb{R}}4$}$$
$$= (a,\infty)\cap(-\infty,b) \tag{$\mathcal{B}_{\mathbb{R}}5$}$$
$$= ((-\infty,a]\cup[b,\infty))^c\,. \tag{$\mathcal{B}_{\mathbb{R}}6$}$$

Again by Fact 2.1.7, we have $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}((\mathcal{B}_{\mathbb{R}}1)) \subset \mathcal{M}((j)) \subset \mathcal{B}_{\mathbb{R}}$. □

Exercise 2.1.11. Define the *h-intervals*

$$\mathcal{H} := \{\emptyset\} \cup \{(-a,b] | -\infty \le a < b < \infty\} \cup \{(a,\infty) | a \in \mathbb{R}\}\,.$$

Let $\mathcal{A}$ be the collection of finite disjoint unions of elements of $\mathcal{H}$. Show *directly from the definitions* that $\mathcal{A}$ is an algebra. Deduce that the σ-algebra $\mathcal{M}(\mathcal{A})$ generated by $\mathcal{A}$ is equal to the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$.

Exercise 2.1.12. Denote by $\overline{\mathbb{R}}$ the extended real numbers $[-\infty,\infty]$ with its usual topology. Prove the following assertions.

(1) The Borel σ-algebra on $\overline{\mathbb{R}}$ is generated by the open rays $(a,\infty]$ for $a \in \mathbb{R}$.
(2) If $\mathcal{E} \subset P(\mathbb{R})$ generates the Borel σ-algebra on $\mathbb{R}$, then $\mathcal{E} \cup \{\{\infty\}\}$ generates the Borel σ-algebra on $\overline{\mathbb{R}}$.

Exercise 2.1.13. Let X be a set. A *π-system* on X is a collection of subsets $\Pi \subset P(X)$ which is closed under finite intersections. A *λ-system* on X is a collection of subsets $\Lambda \subset P(X)$ such that

- $X \in \Lambda$
- Λ is closed under taking complements, and
- for every sequence of disjoint subsets (E_i) in Λ, $\bigcup E_i \in \Lambda$.

(1) Show that $\mathcal{M}$ is a σ-algebra if and only if $\mathcal{M}$ is both a π-system and a λ-system.
(2) Suppose Λ is a λ-system. Show that for every $E \in \Lambda$, the set

$$\Lambda(E) := \{F \subset X | F \cap E \in \Lambda\}$$

is also a Λ-system.

Exercise 2.1.14 ($\pi - \lambda$ Theorem)**.** Let Π be a π-system, let Λ be the smallest λ-system containing Π, and let $\mathcal{M}$ be the smallest σ-algebra containing Π.

(1) Show that $\Lambda \subseteq \mathcal{M}$.
(2) Show that for every $E \in \Pi$, $\Pi \subset \Lambda(E)$ where $\Lambda(E)$ was defined in Exercise 2.1.13 above. Deduce that $\Lambda \subset \Lambda(E)$ for every $E \in \Pi$.
(3) Show that $\Pi \subset \Lambda(F)$ for every $F \in \Lambda$. Deduce that $\Lambda \subset \Lambda(F)$ for every $F \in \Lambda$.
(4) Deduce that Λ is a σ-algebra, and thus $\mathcal{M} = \Lambda$.

2.2. **Measures.**

Definition 2.2.1. A set X together with a σ-algebra $\mathcal{M}$ is called a *measurable space*. A *measure* on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

- (vacuum) $\mu(\emptyset) = 0$, and
- (countable additivity) for every sequence of disjoint sets $(E_n) \subset \mathcal{M}$, $\mu(\coprod E_n) = \sum \mu(E_n)$.

Observe that countable additivity implies finite additivity by taking all but finitely many of the E_n to be $\emptyset$.

We call the triple $(X, \mathcal{M}, \mu)$ a *measure space*. A measure space is called:

- *finite* if $\mu(X) < \infty$.
- *σ-finite* if $X = \bigcup E_n$ with $(E_n) \subset \mathcal{M}$ a sequence of measurable sets with $\mu(E_n) < \infty$. By disjointification (II), we may take such (E_n) to be disjoint.
- *semifinite* if for every $E \in \mathcal{M}$, $\mu(E) = \infty$, there is an $F \subset E$ with $F \in \mathcal{M}$ such that $0 < \mu(F) < \infty$.
- *complete* if $E \in \mathcal{M}$ with $\mu(E) = 0$ (E is *μ-null*) and $F \subset E$ implies $F \in \mathcal{M}$. *Note: We will see that $\mu(F) = 0$ by monotonicity below in (μ1) of Facts 2.2.4.*

Remark 2.2.2. In probability theory, a measure space is typically denoted $(\Omega, \mathcal{F}, P)$, and $P(\Omega) = 1$.

Examples 2.2.3.

(1) Counting measure on $P(X)$

(2) Pick $x_0 \in X$, and define μ_{x_0} on $P(X)$ by

$$\mu_{x_0}(E) = \delta_{x_0 \in E} := \begin{cases} 0 & \text{if } x_0 \notin E \\ 1 & \text{if } x_0 \in E. \end{cases}$$

We call μ_{x_0} the *point mass* or *Dirac measure* at x_0.

(3) Pick any $f : X \to [0, \infty]$. On $P(X)$, define

$$\mu_f(E) := \sum_{x \in E} f(x) := \sup \sum_{\substack{x \in F \\ F \text{ finite}}} f(x) = \lim_{\substack{\text{finite } F \\ \text{ordered by inclusion}}} \sum_{x \in F} f(x)$$

When $f = 1$, μ_f is counting measure. When $f = \delta_{x = x_0}$, we get the Dirac measure.

(4) On the σ-algebra of countable or co-countable sets, define

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } E \text{ is co-countable.} \end{cases}$$

Facts 2.2.4 (Basic properties of measures)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

(μ1) (Monotonicity) If $E, F \in \mathcal{M}$, then $F \subset E$ implies $\mu(F) \le \mu(E)$. In particular, if $\mu(E) = 0$, then $\mu(F) = 0$.

Proof. $\mu(E) = \mu(F \amalg (E \setminus F)) = \mu(F) + \mu(E \setminus F)$, and $\mu(E \setminus F) \ge 0$. $\square$

($\mu 2$) (Subadditivity) If $(E_n) \subset \mathcal{M}$, then $\mu\left(\bigcup E_n\right) \leq \sum \mu(E_n)$.

Proof. Use disjointification (II). That is, setting $F_1 := E_1$ and $F_k := E_k \setminus \bigcup_1^{k-1} E_n$, we have $F_k \subset E_k$ for all k, and
$$\mu\left(\bigcup E_n\right) = \mu\left(\coprod F_n\right) = \sum \mu(F_n) \leq \sum \mu(E_n). \qquad \square$$

($\mu 3$) (Continuity from below) If $E_1 \subset E_2 \subset E_3 \subset \cdots$ is an increasing sequence of elements of $\mathcal{M}$, then
$$\mu\left(\bigcup E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Set $E_0 = \emptyset$. In this setting, disjointification (II) is easy; just set $F_n := E_n \setminus E_{n-1}$ for all $n \geq 1$. Then
$$\begin{aligned}\mu\left(\bigcup E_n\right) &= \mu\left(\coprod F_n\right) = \sum \mu(F_n) = \sum \mu(E_n \setminus E_{n-1})\\ &= \lim_{k\to\infty} \sum^k \mu(E_n \setminus E_{n-1}) = \lim_{k\to\infty} \mu(E_k). \qquad \square\end{aligned}$$

($\mu 4$) (Continuity from above) If $E_1 \supset E_2 \supset E_3 \supset \cdots$ is a decreasing sequence of elements of $\mathcal{M}$ with $\mu(E_k) < \infty$ for some $k \in \mathbb{N}$, then
$$\mu\left(\bigcap E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. We may assume $\mu(E_1) < \infty$. Set $F_1 := E_1$ and $F_n := E_1 \setminus E_n$, so that $\mu(E_1) = \mu(E_n) + \mu(F_n)$ for all $n \geq 1$. Observe that
$$\bigcup F_n = \bigcup E_1 \cap E_n^c = E_1 \cap \left(\bigcup E_n^c\right) = E_1 \cap \left(\bigcap E_n\right)^c = E_1 \setminus \left(\bigcap E_n\right).$$
Hence
$$\begin{aligned}\mu\left(\bigcap E_n\right) &= \mu(E_1) - \mu\left(\bigcup F_n\right) \underset{(3)}{=} \mu(E_1) - \lim_{n\to\infty} \mu(F_n)\\ &= \mu(E_1) - \lim_{n\to\infty} (\mu(E_1) - \mu(E_n)) = \lim_{n\to\infty} \mu(E_n). \qquad \square\end{aligned}$$

Exercise 2.2.5. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $(E_n) \subset \mathcal{M}$. Recall that
$$\liminf E_n = \bigcup_k \bigcap_{n\geq k} E_n \qquad \text{and} \qquad \limsup E_n = \bigcap_k \bigcup_{n\geq k} E_n$$

(1) Prove that $\mu(\liminf E_n) \leq \liminf \mu(E_n)$.
(2) Suppose μ is finite. Prove that $\mu(\limsup E_n) \geq \limsup \mu(E_n)$.
(3) Does (2) above hold if μ is not finite? Give a proof or counterexample.

Theorem 2.2.6. *Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Define*
$$\overline{\mathcal{M}} := \{E \cup F | E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{M} \text{ with } \mu(N) = 0\}.$$

(1) $\overline{\mathcal{M}}$ is a σ-algebra containing $\mathcal{M}$.

(2) There is a unique complete measure $\overline{\mu}$ on $\overline{\mathcal{M}}$ such that $\overline{\mu}|_{\mathcal{M}} = \mu$. We call $\overline{\mu}$ the completion *of μ.*

Proof.
$\overline{\mathcal{M}}$ a σ-algebra:

(0) Observe that $\emptyset \in \mathcal{M} \subset \overline{\mathcal{M}}$, so $\overline{\mathcal{M}} \neq \emptyset$.

(1) If $(E_n \cup F_n) \subset \overline{\mathcal{M}}$, then

$$\bigcup E_n \cup F_n = \underbrace{\left(\bigcup E_n\right)}_{\in \mathcal{M}} \cup \underbrace{\left(\bigcup F_n\right)}_{\subset \bigcup N_n}.$$

Observe that each $F_n \subset N_n \in \mathcal{M}$ with $\mu(N_n) = 0$, so by countable subadditivity, we have $\mu\left(\bigcup N_n\right) \leq \sum \mu(N_n) = 0$. Hence $\overline{\mathcal{M}}$ is closed under countable unions.

(2) Suppose $E, N \in \mathcal{M}$ with $F \subset N$ μ-null. Observe that

$$\begin{aligned}(E \cup F)^c &= (E^c \cap F^c) = (E^c \cap F^c) \cap X = (E^c \cap F^c) \cap (N^c \amalg N) \\ &= (E^c \cap \underbrace{F^c \cap N^c}_{=N^c \in \mathcal{M}}) \amalg (E^c \cap F^c \cap N) = \underbrace{(E^c \cap N^c)}_{\in \mathcal{M}} \amalg \underbrace{(E^c \cap F^c \cap N)}_{\subset N}.\end{aligned}$$

Hence $\overline{\mathcal{M}}$ is closed under taking complements.

$\overline{\mu}$ unique: If $\overline{\mu}|_{\mathcal{M}} = \mu$, then for all $E \cup F \in \overline{\mathcal{M}}$ with $F \subset N$ μ-null, we have

$$\mu(E) = \overline{\mu}(E) \leq \overline{\mu}(E \cup F) \leq \overline{\mu}(E) + \overline{\mu}(F) \leq \overline{\mu}(E) + \overline{\mu}(N) = \mu(E) + \mu(N) = \mu(E).$$

Hence $\overline{\mu}(E \cup F) = \mu(E)$.

$\overline{\mu}$ exists: First, we show that $\overline{\mu}(E \cup F) := \mu(E)$ is a well-defined function on $\overline{\mathcal{M}}$. Suppose $E_1 \cup F_1 = E_2 \cup F_2$ with $F_i \subset N_i$ μ-null for $i = 1, 2$. Observe that

$$E_1 \subset E_1 \cup F_1 = E_2 \cup F_2 \subset E_2 \cup N_2 \qquad \Longrightarrow \qquad \mu(E_1) \leq \mu(E_2 \cup N_2) \leq \mu(E_2) + \mu(N_2) = \mu(E_2).$$

Swapping the roles of E_1, E_2, F_1, F_2, and N_1, N_2, we have $\mu(E_2) \leq \mu(E_1)$.

Next, we will show $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$:

(0) (Vacuum) Observe that $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$.

(1) (σ-additivity) Suppose $(E_n \cup F_n) \subset \overline{\mathcal{M}}$ is a sequence of disjoint sets with $F_n \subset N_n$ μ-null for each $n \in \mathbb{N}$. Then (E_n) and (F_n) are disjoint, and $\coprod F_n \subset \coprod N_n$ is μ-null. Hence

$$\overline{\mu}\left(\coprod E_n \cup F_n\right) = \overline{\mu}\left(\coprod E_n \cup \coprod F_n\right) = \mu\left(\coprod E_n\right) = \sum \mu(E_n) = \sum \overline{\mu}(E_n \cup F_n).$$

$\overline{\mu}$ complete: First, note that if $F \subset N$ with N μ-null, then $F = \emptyset \cup F \in \overline{\mathcal{M}}$. Suppose $G \subset E \cup F$ where $F \subset N$ is μ-null, and $\mu(E) = 0$. Then observe $G \subset E \cup N \in \mathcal{M}$, and $\mu(E \cup N) \leq \mu(E) + \mu(N) = 0$. Hence $G \in \overline{\mathcal{M}}$. $\square$

Exercise 2.2.7. Let Π be a π-system, and let $\mathcal{M}$ be the smallest σ-algebra containing Π. Suppose μ, ν are two measures on $\mathcal{M}$ whose restrictions to Π agree.

(1) Suppose that μ, ν are finite and $\mu(X) = \nu(X)$. Show $\mu = \nu$.
Hint: Consider $\Lambda := \{E \in \mathcal{M} | \nu(E) = \mu(E)\}$.

(2) Suppose that $X = \coprod_{j=1}^{\infty} X_j$ with $(X_j) \subset \Pi$ and $\mu(X_j) = \nu(X_j) < \infty$ for all $j \in \mathbb{N}$. (Observe that μ and ν are σ-finite.) Show $\mu = \nu$.

Exercise 2.2.8 (Folland §1.3, #14 and #15)**.** Given a measure μ on $(X, \mathcal{M})$, define ν on $\mathcal{M}$ by

$$\nu(E) := \sup \{\mu(F) | F \subset E \text{ and } \mu(F) < \infty\}.$$

(1) Show that ν is a semifinite measure. We call it the *semifinite part* of μ.
(2) Suppose $E \in \mathcal{M}$ with $\nu(E) = \infty$. Show that for any $n > 0$, there is an $F \subset E$ such that $n < \nu(F) < \infty$.
This is exactly Folland §1.3, #14 applied to ν.
(3) Show that if μ is semifinite, then $\mu = \nu$.
(4) Show there is a measure ρ on $\mathcal{M}$ (which is generally not unique) which assumes only the values 0 and ∞ such that $\mu = \nu + \rho$.

Exercise 2.2.9. Suppose μ, ν are two measures on a measurable space $(X, \mathcal{M})$. We say μ is *absolutely continuous* with respect to ν if $\nu(E) = 0$ implies $\mu(E) = 0$. Prove that when μ is finite, the following are equivalent:

(1) μ is absolutely continuous with respect to ν.
(2) For every $\varepsilon > 0$, there is a $\delta > 0$ such that $E \in \mathcal{M}$ with $\nu(E) < \delta$ implies $\mu(E) < \varepsilon$.

Which direction(s) still hold if μ is infinite?

2.3. **Outer measures.**

Definition 2.3.1. Let X be a set. A function $\mu^* : P(X) \to [0, \infty]$ is called an *outer measure* if

(0) (vacuum) $\mu^*(\emptyset) = 0$.
(1) (monotonicity) $E \subset F$ implies $\mu^*(E) \leq \mu^*(F)$.
(2) (countable subadditivity) $\mu^*\left(\bigcup E_n\right) \leq \sum \mu^*(E_n)$ for every sequence (E_n).

Exercise 2.3.2. Suppose $(\mu_i^*)_{i \in I}$ is a family of outer measures on X. Show that

$$\mu^*(E) := \sup_{i \in I} \mu_i^*(E)$$

is an outer measure on X.

Proposition 2.3.3. *Let $\mathcal{E} \subset P(X)$ be any collection of subsets of X satisfying*

- *$\emptyset \in E$, and*
- *for all $E \subset X$, there is a sequence $(E_n) \subset \mathcal{E}$ such that $E \subset \bigcup E_n$. (Observe that if $X \in \mathcal{E}$, this condition is automatic.)*

Suppose $\rho : \mathcal{E} \to [0, \infty]$ is any function such that $\rho(\emptyset) = 0$. Then

$$\mu^*(E) := \inf \left\{ \sum \rho(E_n) \middle| (E_n) \subset \mathcal{E} \text{ with } E \subset \bigcup E_n \right\} \tag{2.3.4}$$

is an outer measure, called the outer measure induced by $(\mathcal{E}, \rho)$.

Proof.

(0) Setting $E_n = \emptyset$ for all n gives $\mu^*(\emptyset) = 0$.
(1) Observe that whenever $F \subset \bigcup F_n$ with $F_n \in \mathcal{E}$ for all n, then $E \subset F \subset \bigcup F_n$. Hence the inf for E is less than or equal to the inf for F.

(2) We'll use the following two tricks:

> **Trick.** $\sum_1^\infty \frac{\varepsilon}{2^n} = \varepsilon$
>
> **Trick.** $r \leq s$ if and only if for all $\varepsilon > 0$, $r \leq s + \varepsilon$.

Suppose (E_n) is a sequence of sets and let $\varepsilon > 0$. For each n, there is a cover $(F_k^n)_k$ such that $E_n \subset \bigcup_k F_k^n$ such that

$$\sum_k \rho(F_k^n) \leq \mu^*(E_n) + \frac{\varepsilon}{2^n}.$$

Then $\bigcup E_n \subset \bigcup_n \bigcup_k F_k^n$, so

$$\mu^*\left(\bigcup E_n\right) \leq \sum_n \sum_k \rho(F_k^n) \leq \sum_n \mu^*(E_n) + \frac{\varepsilon}{2^n} = \sum \mu^*(E_n) + \sum \frac{\varepsilon}{2^n} = \sum \mu^*(E_n) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\mu^*\left(\bigcup E_n\right) \leq \sum \mu^*(E_n)$. $\square$

Exercise 2.3.5. Show that the second bullet point in Proposition 2.3.3 can be removed if we add the convention that $\inf \emptyset = \infty$.

Example 2.3.6. One can get an outer measure on $P(X)$ by taking *any* measure μ on a σ-algebra $\mathcal{M}$ and defining its induced outer measure μ^* as in (2.3.4).

We get a measure μ from an outer measure μ^* by restricting to the σ-algebra $\mathcal{M}^*$ of μ^*-measurable sets.

Definition 2.3.7. Given an outer measure μ^* on $P(X)$, we define the collection of *μ^*-measurable sets*

$$\mathcal{M}^* := \{E \subset X | \mu^*(E \cap F) + \mu^*(E^c \cap F) = \mu^*(F) \text{ for all } F \subset X\}.$$

That is, E is μ^*-measurable if it 'splits' every other set nicely with respect to μ^*.

Remarks 2.3.8.

(1) Clearly $\mu^*(F) \leq \mu^*(E \cap F) + \mu^*(E^c \cap F)$. So

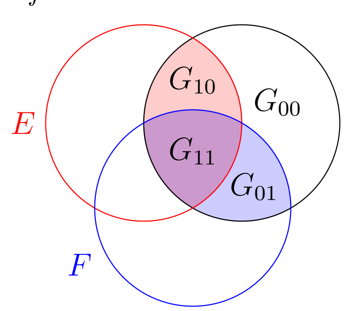
$$E \in \mathcal{M}^* \iff \mu^*(F) \geq \mu^*(E \cap F) + \mu^*(E^c \cap F) \qquad \forall F \subset X. \tag{2.3.9}$$

(2) All μ^*-null sets are in $\mathcal{M}^*$. That is, if $N \subset X$ with $\mu^*(N) = 0$, then for all $F \subset X$

$$\mu^*(\underbrace{F \cap N}_{\subset N}) + \mu^*(F \setminus N) = \mu^*(F \setminus N) \leq \mu^*(F).$$

Lemma 2.3.10. *For $G \subset X$ and $E, F \in \mathcal{M}^*$, define*

$$G_{00} := G \setminus (E \cup F)$$
$$G_{10} := G \cap (E \setminus F)$$
$$G_{01} := G \cap (F \setminus E)$$
$$G_{11} := G \cap E \cap F$$



.

Then we have

$$\mu^*(G) = \mu^*(G_{00}) + \mu^*(G_{01}) + \mu^*(G_{10}) + \mu^*(G_{11}). \tag{2.3.11}$$

Proof. Since $E \in \mathcal{M}^*$,

$$\mu^*(G) = \mu^*(G \cap E) + \mu^*(G \setminus E) = \mu^*(G_{11} \cup G_{10}) + \mu^*(G_{01} \cup G_{00}).$$

Since $F \in \mathcal{M}^*$,

$$\mu^*(G_{11} \cup G_{10}) = \mu^*(G_{11} \cup G_{10} \cap F) + \mu^*(G_{11} \cup G_{10} \setminus F) = \mu^*(G_{11}) + \mu^*(G_{10}).$$

Similarly, $\mu^*(G_{01} \cup G_{00}) = \mu^*(G_{01}) + \mu^*(G_{00})$. The result follows. $\square$

Theorem 2.3.12 (Carathéodory)**.** *Let μ^* be an outer measure on X. The collection of μ^*-measurable sets $\mathcal{M}^*$ is a σ-algebra, and $\mu := \mu^*|_{\mathcal{M}^*}$ is a (complete) measure.*

Proof.
Step 1: $\mathcal{M}^*$ is an algebra.

(0) Clearly $\emptyset \in \mathcal{M}^*$ since it is μ^*-null by Remarks 2.3.8(2).

(1) If $E, F \in \mathcal{M}^*$, then for all $G \subset X$, (2.3.11) holds above. By applying (2.3.11) to $G_{10} \cup G_{11} \cup G_{01}$, we have

$$\mu^*((E \cup F) \cap G) = \mu^*(G_{10} \cup G_{11} \cup G_{01}) \underset{(2.3.11)}{=} \mu^*(G_{10}) + \mu^*(G_{11}) + \mu^*(G_{01}).$$

Moreover, $\mu^*((E \cup F)^c \cap G) = \mu^*(G_{00})$. Again by (2.3.11), we have

$$\mu^*((E \cup F) \cap G) + \mu^*((E \cup F)^c \cap G) = (\mu^*(G_{10}) + \mu^*(G_{11}) + \mu^*(G_{01})) + \mu^*(G_{00}) \underset{(2.3.11)}{=} \mu^*(G).$$

(2) Observe that the Carathéodory Criterion (2.3.9) is preserved under taking complements.

Step 2: $\mathcal{M}^*$ is a σ-algebra.

Suppose $(E_n) \subset \mathcal{M}^*$ is a sequence of disjoint sets, and set $E := \coprod E_n$. By Step 1, for all $N \in \mathbb{N}$, $\coprod^N E_n \in \mathcal{M}^*$. Let $F \subset X$, and define $G := F \cap \coprod^N E_n$. Then since $E_N \in \mathcal{M}^*$, we have

$$\mu^*\left(F \cap \coprod^N E_n\right) = \mu^*(G) = \mu^*(E_N^c \cap G) + \mu^*(E_N \cap G) = \mu^*\left(F \cap \coprod^{N-1} E_n\right) + \mu^*(F \cap E_N).$$

By iterating as $E_n \in \mathcal{M}^*$ for all $n \in \mathbb{N}$, we have

$$\mu^*\left(F \cap \coprod^N E_n\right) = \sum^N \mu^*(F \cap E_n) \qquad \forall N \in \mathbb{N}.$$

It follows that for all $N \in \mathbb{N}$,

$$\mu^*(F) = \mu^*\left(F \cap \coprod^N E_n\right) + \mu^*\left(\underbrace{F \setminus \coprod^N E_n}_{\supset F \setminus E}\right) \geq \sum^N \mu^*(F \cap E_n) + \mu^*(F \setminus E).$$

Taking limits in $[0,\infty]$ as $N\to\infty$, we have

$$\begin{aligned}\mu^*(F) &\geq \sum^{\infty} \mu^*(F\cap E_n)+\mu^*(F\setminus E)\\ &\geq \mu^*\left(\coprod F\cap E_n\right)+\mu^*(F\setminus E)\\ &= \mu^*(F\cap E)+\mu^*(F\setminus E).\end{aligned} \tag{2.3.13}$$

Thus $E=\coprod E_n\in\mathcal{M}^*$.

Step 3: $\mu=\mu^*|_{\mathcal{M}^*}$ is a measure.

It remains to show μ is σ-additive on $\mathcal{M}^*$. Suppose $(E_n)\subset\mathcal{M}^*$ is a sequence of disjoint sets as in Step 2. Taking $F=E$ in (2.3.13) above shows us

$$\mu^*(E)\geq\sum\mu^*(E_n)\geq\mu^*(E),$$

so equality holds. $\square$

2.4. **Pre-measures.** In the last section, we gave a prescription for constructing a complete measure on X. Start with any collection of subsets $\mathcal{E}\subset P(X)$ with $\emptyset\in\mathcal{E}$ such that for every $E\subset X$, there is some sequence $(E_n)\subset\mathcal{E}$ with $E\subset\bigcup E_n$. Take any function $\rho:\mathcal{E}\to[0,\infty]$ such that $\rho(\emptyset)=0$. We get an induced outer measure μ^* by (2.3.4). Taking the μ^*-measurable sets $\mathcal{M}^*$, we get a σ-algebra, and $\mu:=\mu^*|_{\mathcal{M}^*}$ is a complete measure.

However, we get little control over $\mathcal{M}^*$ and μ. Consider the following two crucial questions:

(1) When is $\mathcal{E}\subset\mathcal{M}^*$?
(2) In this case, when does $\mu|_{\mathcal{E}}=\rho$?
Note: we always have $\mu^*\leq\rho$, since every $E\in\mathcal{E}$ is covered by itself. But there might be some cover $E\subset\bigcup E_n$ from $\mathcal{E}$ such that $\sum\rho(E_n)<\rho(E)$.

A sufficient condition to ensure a positive answer to both of these questions is that $\mathcal{E}$ is an algebra, and ρ is a *premeasure*.

Definition 2.4.1. Let $\mathcal{A}\subset P(X)$ be an algebra. A function $\mu_0:\mathcal{A}\to[0,\infty]$ is called a *premeasure* if

(0) (vacuum) $\mu_0(\emptyset)=0$, and
(1) (countable additivity) for every sequence $(E_n)\subset\mathcal{A}$ of disjoint sets such that $\coprod E_n\in\mathcal{A}$, we have $\mu_0\left(\coprod E_n\right)=\sum\mu_0(E_n)$.

The adjectives *finite*, *σ-finite*, and *semi-finite* for premeasures are defined analogously to those for measures.

Facts 2.4.2. The following are basic properties of a premeasure μ_0 on an algebra $\mathcal{A}\subset P(X)$.

(pre-μ1) (finite additivity) If $E_1,\dots,E_n\in\mathcal{A}$ are disjoint, then $\mu_0\left(\coprod E_n\right)=\sum\mu_0(E_n)$.

Proof. If $E_1,\dots,E_n\in\mathcal{A}$ are disjoint sets, then observe that $\coprod_{i=1}^n E_i\in\mathcal{A}$. So by setting $E_i=\emptyset$ for all $i>n$, we have

$$\mu_0\left(\coprod_{i=1}^n E_i\right)=\mu_0\left(\coprod E_i\right)=\sum\mu_0(E_i)=\sum_{i=1}^n\mu_0(E_i). \qquad\square$$

(pre-μ2) (monotonicity) If $E, F \in \mathcal{A}$ with $F \subset E$, then $\mu_0(F) \leq \mu_0(E)$.

Proof. Immediate by (pre-μ1) since $E = F \amalg (E \setminus F)$. $\square$

(pre-μ3) (countable subadditivity) If $(E_n) \subset \mathcal{A}$ such that $\bigcup E_n \in \mathcal{A}$, then $\mu_0(\bigcup E_n) \leq \sum \mu_0(E_n)$.

Proof. We use disjointification (II). Set $F_1 := E_1$ and inductively define $F_n := E_n \setminus \bigcup_{i=1}^{n-1} E_i$. Then $F_n \in \mathcal{A}$ for all n, and $\coprod F_n = \bigcup E_n$. Thus

$$\mu_0\left(\bigcup E_n\right) = \mu_0\left(\coprod F_n\right) = \sum \mu_0(F_n) \underset{\text{(pre-}\mu 2)}{\leq} \sum \mu_0(E_n). \qquad \square$$

(pre-μ4) (monotone countable subadditivity) Suppose $E \in \mathcal{A}$ and $(E_n) \subset \mathcal{A}$ such that $E \subset \bigcup E_n$. Then $\mu_0(E) \leq \sum \mu_0(E_n)$.
Warning: This does not follow immediately by monotonicity and countable subadditivity, since we are not assured that $\bigcup E_n \in \mathcal{A}$!

Proof. Let $F_1 := E \cap E_1$ and inductively set $F_n := E \cap \left(E_n \setminus \bigcup_{i=1}^{n-1} E_i\right)$. Then $F_n \in \mathcal{A}$ for all n, and $\coprod F_n = E \in \mathcal{A}$. Hence

$$\mu_0(E) = \mu_0\left(\coprod F_n\right) = \sum \mu_0(F_n) \underset{\text{(pre-}\mu 2)}{\leq} \sum \mu_0(E_n). \qquad \square$$

Remark 2.4.3. Recall that if μ_0 is only known to be finitely additive and not necessarily countably additive, then μ_0 still satisfies monotonicity and finite subadditivity (cf. Exercise 2.0.4).

Lemma 2.4.4. *Suppose μ_0 is a premeasure on $\mathcal{A}$. Let μ^* be the induced outer measure given by (2.3.4).*

(1) $\mu^*|_{\mathcal{A}} = \mu_0$, and
(2) $\mathcal{A} \subset \mathcal{M}^*$.

Proof.
(1) Suppose $E \in \mathcal{A}$.

$\underline{\mu^* \leq \mu_0}$: Setting $E_1 := E$ and $E_n := \emptyset$ for all $n > 1$, $\mu^*(E) \leq \sum \mu_0(E_n) = \mu_0(E)$.

$\underline{\mu^* \geq \mu_0}$: Let $\varepsilon > 0$. By definition of μ^* as an infimum, there is a sequence $(E_n) \subset A$ such that $E \subset \bigcup E_n$ and $\sum \mu_0(E_n) \leq \mu^*(E) + \varepsilon$. But by monotone countable subadditivity, $\mu_0(E) \leq \sum \mu_0(E_n)$, and thus $\mu_0(E) \leq \mu^*(E) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\mu_0 \leq \mu^*$ on $\mathcal{A}$.

(2) Suppose $E \in \mathcal{A}$ and $F \subset X$ and $\varepsilon > 0$. Pick $(F_n) \subset \mathcal{A}$ such that $F \subset \bigcup F_n$ and $\sum \mu_0(F_n) \leq \mu^*(F) + \varepsilon$. Since μ_0 is σ-additive on $\mathcal{A}$,

$$\begin{aligned}
\mu^*(F) + \varepsilon \geq \sum \mu_0(F_n) &= \sum \mu_0(F_n \cap E) + \mu_0(F_n \cap E^c) \\
&= \sum \mu_0(F_n \cap E) + \sum \mu_0(F_n \cap E^c) \\
&\geq \mu^*(F \cap E) + \mu^*(F \cap E^c).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$, and thus $E \in \mathcal{M}^*$. ☐

Construction 2.4.5. Starting with a premeasure μ_0 on an algebra $\mathcal{A}$, we get a σ-algebra $\mathcal{M}^*$ which contains $\mathcal{A}$, and a complete measure $\mu := \mu^*|_{\mathcal{M}^*}$ such that $\mu|_{\mathcal{A}} = \mu_0$.

Remark 2.4.6. Observe that by Fact 2.1.7, $\mathcal{M}^*$ contains $\mathcal{M} := \mathcal{M}(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$, and $\mu|_{\mathcal{M}}$ is a (possibly non-complete) measure.

Theorem 2.4.7. *Suppose μ_0 is a premeasure on an algebra $\mathcal{A}$, and μ is the measure on $\mathcal{M}^*$ from Construction 2.4.5. If ν is a measure on $\mathcal{M} = \mathcal{M}(\mathcal{A})$ such that $\nu|_{\mathcal{A}} = \mu_0$, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$.*

Proof. Suppose $E \in \mathcal{M}$.
Step 1: $\nu(E) \leq \mu(E)$.

Since $E \in \mathcal{M}$, for all sequences $(E_n) \subset \mathcal{A}$ such that $E \subset \bigcup E_n$,

$$\nu(E) \leq \sum \nu(E_n) = \sum \mu_0(E_n).$$

Hence $\nu(E) \leq \inf \{\sum \mu_0(E_n) | E \subset \bigcup E_n\} = \mu^*(E) = \mu(E)$.

Step 2: When $\mu(E) < \infty$, we show $\mu(E) \leq \nu(E)$, and thus $\mu(E) = \nu(E)$.

Let $\varepsilon > 0$. Then there exists a sequence $(E_n) \subset \mathcal{A}$ such that $E \subset \bigcup E_n$ and

$$\mu\left(\bigcup E_n\right) \leq \sum \mu_0(E_n) \leq \mu(E) + \varepsilon < \infty.$$

Since $E \subset \bigcup E_n$ and $\mu(E) < \infty$, we have

$$\mu\left(\left(\bigcup E_n\right) \setminus E\right) = \mu\left(\bigcup E_n\right) - \mu(E) \leq \varepsilon. \tag{2.4.8}$$

Now by continuity from below (μ3) for both μ and ν, we have

$$\begin{aligned} \mu\left(\bigcup E_n\right) &= \lim_{N\to\infty} \mu\left(\bigcup^N E_n\right) = \lim_{N\to\infty} \mu_0\left(\bigcup^N E_n\right) \\ &= \lim_{N\to\infty} \nu\left(\bigcup^N E_n\right) = \nu\left(\bigcup E_n\right). \end{aligned} \tag{2.4.9}$$

Putting these two equations together, we have

$$\begin{aligned} \mu(E) \leq \mu\left(\bigcup E_n\right) &\underset{(2.4.9)}{=} \nu\left(\bigcup E_n\right) = \nu(E) + \nu\left(\left(\bigcup E_n\right) \setminus E\right) \\ &\leq \nu(E) + \mu\left(\left(\bigcup E_n\right) \setminus E\right) \underset{(2.4.8)}{\leq} \nu(E) + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\mu(E) \leq \nu(E)$.

This concludes the proof. ☐

Corollary 2.4.10. *Suppose μ_0 is a premeasure on an algebra $\mathcal{A}$, and μ is the measure on $\mathcal{M}^*$ from Construction 2.4.5. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to $\mathcal{M} = \mathcal{M}(\mathcal{A})$.*

Proof. Recall that μ_0 is σ-finite if there exists a sequence $(E_n) \subset \mathcal{A}$ such that $\bigcup E_n = X$ and $\mu_0(E_n) < \infty$ for all n. Observe that by disjointification (II), we may assume that the E_n are disjoint.

Now for any other ν extending μ_0 and $E \in \mathcal{M}$, we have

$$\mu(E) = \mu\left(\coprod E \cap E_n\right) = \sum \underbrace{\mu(E \cap E_n)}_{<\infty} = \sum \nu(E \cap E_n) = \nu\left(\coprod E \cap E_n\right) = \nu(E). \quad \square$$

Exercise 2.4.11. Suppose $\mathcal{A}$ is an algebra on X, μ_0 a premeasure on $\mathcal{A}$, and μ^* the induced outer measure on $P(X)$ given by (2.3.4). Show that for every $E \subset X$, there is a μ^*-measurable set $F \supset E$ such that $\mu^*(F) = \mu^*(E)$.

Exercise 2.4.12 (Adapted from Folland §1.4, #18 and #22)**.** Suppose $\mathcal{A}$ is an algebra, and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Let μ_0 be a σ-finite premeasure on $\mathcal{A}$, μ^* the induced outer measure given by (2.3.4), and $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets. Show that the following are equivalent.

(1) $E \in \mathcal{M}^*$
(2) $E = F \setminus N$ where $F \in \mathcal{M}$ and $\mu^*(N) = 0$.
(3) $E = F \cup N$ where $F \in \mathcal{M}$ and $\mu^*(N) = 0$.

Deduce that if μ is a σ-finite measure on $\mathcal{M}$, then $\mu^*|_{\mathcal{M}^*}$ on $\mathcal{M}^*$ is the completion of μ on $\mathcal{M}$.

Exercise 2.4.13 (Folland §1.4, #20)**.** Let μ^* be an outer measure on $P(X)$, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, and $\mu := \mu^*|_{\mathcal{M}^*}$. Let μ^+ be the outer measure on $P(X)$ induced by the (pre)measure μ on the (σ-)algebra $\mathcal{M}^*$.

(1) Show that $\mu^*(E) \leq \mu^+(E)$ for all $E \subset X$ with equality if and only if there is an $F \in \mathcal{M}^*$ with $E \subset F$ and $\mu^*(E) = \mu^*(F)$.
(2) Show that if μ^* was induced from a premeasure μ_0 on an algebra $\mathcal{A}$, then $\mu^* = \mu^+$.
(3) Construct an outer measure μ^* on the two point set $X = \{0, 1\}$ such that $\mu^* \neq \mu^+$.

Exercise 2.4.14. Let X be a set, $\mathcal{A}$ an algebra on X, μ_0 a premeasure on $\mathcal{A}$, and μ^* the induced outer measure on $P(X)$ given by (2.3.4). Suppose that (E_n) is an *increasing* sequence of subsets of X, i.e., $E_1 \subset E_2 \subset E_3 \subset \cdots$. Prove that

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu^*(E_n).$$

Exercise 2.4.15 (Sarason)**.** Suppose μ_0 is a finite premeasure on the algebra $\mathcal{A} \subset P(X)$, and let $\mu^* : P(X) \to [0, \infty]$ be the outer measure induced by μ_0. Prove that the following are equivalent for $E \subset X$.

(1) $E \in \mathcal{M}^*$, the μ^*-measurable sets.
(2) $\mu^*(E) + \mu^*(X \setminus E) = \mu(X)$.

Hint: Use Exercise 2.4.12.

2.5. **Lebesgue-Stieltjes measures on $\mathbb{R}$.**

2.5.1. *Construction of Lebesgue-Stieltjes measures.* Recall from Exercise 2.1.11 that we define the collection of *h-intervals* by

$$\mathcal{H} := \{\emptyset\} \cup \{(a,b] | -\infty \leq a < b < \infty\} \cup \{(a,\infty) | a \in \mathbb{R}\}.$$

Let $\mathcal{A} = \mathcal{A}(\mathcal{H})$ be the collection of finite disjoint unions of elements of $\mathcal{H}$. By Exercise 2.1.11, $\mathcal{A}$ is an algebra, and the σ-algebra generated by $\mathcal{A}$ is $\mathcal{M}(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$, the Borel σ-algebra. Our goal is to construct a nice class of premeasures on $\mathcal{A}$.

Construction 2.5.1. Let $F : \mathbb{R} \to \mathbb{R}$ be any function which is

- (non-decreasing) $r \leq s$ implies $F(r) \leq F(s)$, and
- (right continuous) if $r_n \searrow a$, then $F(r_n) \searrow F(a)$

Extend F to a function $\overline{\mathbb{R}} = [-\infty, \infty] \to \overline{\mathbb{R}}$ by

$$F(-\infty) := \lim_{a \to -\infty} F(a) \qquad \text{and} \qquad F(\infty) := \lim_{b \to \infty} F(b).$$

Define $\mu_0 : \mathcal{H} \to [0, \infty]$ by

- $\mu_0(\emptyset) := 0$,
- $\mu_0((a,b]) := F(b) - F(a)$ for all $a \geq -\infty$, and
- $\mu_0((a,\infty)) := F(\infty) - F(a)$ for all $a \geq -\infty$.

In (LS4) below, we extend $\mu_0 : \mathcal{H} \to [0,\infty]$ to a well-defined function $\mathcal{A} = \mathcal{A}(\mathcal{H}) \to [0,\infty]$. In Theorem 2.5.7 below, we prove this extension to $\mathcal{A}$ is a *premeasure*. By Carathéodory's outer measure construction, we get an outer measure μ_F^* on $(\mathbb{R}, P(\mathbb{R}))$ by (2.3.4). By taking the σ-algebra of μ_F^*-measurable sets $\mathcal{M}_F := \mathcal{M}^*$, we get a complete measure $\mu_F := \mu_F^*|_{\mathcal{M}_F}$.

Definition 2.5.2. We call μ_F the *Lebesgue-Stieltjes measure* associated to F.

Remark 2.5.3. Since μ_F is σ-finite by construction, it follows from Exercise 2.4.12 that $\mathcal{M}_F$ is the completion $\overline{\mathcal{B}_{\mathbb{R}}}$ of the Borel σ-algebra for $\mu_F|_{\mathcal{B}_{\mathbb{R}}}$. Thus, sets in $\mathcal{M}_F$ are unions of Borel sets and subsets of Borel sets which are μ_F-null.

In the remainder of this section, we prove that μ_0 extends to a premeasure on $\mathcal{A} = \mathcal{A}(\mathcal{H})$.

Facts 2.5.4. We have the following facts about the function μ_0.

(LS1) Splitting $(a,\infty) = (a,b] \amalg (b,\infty)$, we have $\mu_0((a,\infty)) = \mu_0((a,b]) + \mu_0((b,\infty))$.

(LS2) If $(a,b] = \coprod_{i=1}^n (a_i, b_i]$, then $\mu_0((a,b]) = \sum_{i=1}^n \mu_0((a_i,b_i])$.

Proof. Re-indexing, we may assume $a = a_1 < b_1 = a_2 < b_2 = a_3 < \cdots < b_n$. Then

$$\mu_0((a,b]) = F(b) - F(a) = \sum_{i=1}^n F(b_i) - F(a_i) = \sum_{i=1}^n \mu_0((a_i,b_i]). \qquad \square$$

(LS3) If $E_1, \ldots, E_n \in \mathcal{H}$ are disjoint and $F \in \mathcal{H}$ such that $F \subset \coprod_{i=1}^n E_i$, then $\mu_0(F) = \sum_{i=1}^n \mu_0(F \cap E_i)$.

Proof. Removing elements of $(E_i)_{i=1}^n$ if necessary, we may assume that $F \cap E_i \neq \emptyset$ for all $i = 1, \ldots, n$. This means that $F \cap E_i \in \mathcal{H}$ for all i, and $F = \coprod_{i=1}^n F \cap E_i$. The result now follows by (LS1) and (LS2). $\square$

(LS4) If $(E_1,\dots,E_m)\subset\mathcal{H}$ and $(F_1,\dots,F_n)\subset\mathcal{H}$ are two collections of disjoint h-intervals with $\coprod_{i=1}^m E_i=\coprod_{j=1}^n F_j$, then $\sum_{i=1}^m \mu_0(E_i)=\sum_{j=1}^n\mu_0(F_j)$.

Proof. By applying (LS3) twice, we have
$$\sum_{i=1}^m \mu_0(E_i)\underset{(3)}{=}\sum_{i=1}^m\sum_{j=1}^n \mu_0(E_i\cap F_j)=\sum_{j=1}^n\sum_{i=1}^m\mu_0(E_i\cap F_j)\underset{(3)}{=}\sum_{j=1}^n\mu_0(F_j).\qquad\square$$

Hence μ_0 extends to a well-defined function still denoted $\mu_0:\mathcal{A}=\mathcal{A}(\mathcal{H})\to[0,\infty]$ by
$$\mu_0\left(\coprod_{i=1}^n E_i\right):=\sum_{i=1}^n\mu_0(E_i)\qquad\qquad \forall\text{ disjoint } E_1,\dots,E_n\in\mathcal{H}.$$

Corollary 2.5.5. *The extension $\mu_0:\mathcal{A}\to[0,\infty]$ afforded by (LS4) is finitely additive and thus monotone and finitely subadditive by Exercise 2.0.4.*

Proof. Suppose $E=\coprod_{i=1}^n E_i$ with $E,E_1,\dots,E_n\in\mathcal{A}$. Then we may write each $E_i=\coprod_{j=1}^{m_i}E_j^i$ where $E_j^i\in\mathcal{H}$ for all $j=1,\dots,m_i$, and thus $E=\coprod_{i=1}^n\coprod_{j=1}^{m_i}E_j^i$. Then by countable additivity of μ_0 on $\mathcal{H}$ from (LS4), we have
$$\mu_0(E)=\sum_{i=1}^n\sum_{j=1}^{m_i}\mu_0(E_j^i)=\sum_{i=1}^n\mu_0(E_i).\qquad\square$$

Exercise 2.5.6. Describe to the best of your ability the set of accumulation points of right endpoints (b_j) for a disjoint collection of bounded h-intervals $((a_n,b_n])_{n=1}^\infty$ such that $\coprod(a_n,b_n]=(a,b]$ for some $a<b$ in $\mathbb{R}$.

Theorem 2.5.7. *The extension $\mu_0:\mathcal{A}\to[0,\infty]$ afforded by (LS4) is a premeasure on $\mathcal{A}$.*

Proof. It is clear that $\mu_0(\emptyset)=0$ by construction.

Suppose $(E_n)\subset\mathcal{A}$ is a disjoint sequence such that $\coprod E_n\in\mathcal{A}$. Then there are disjoint h-intervals $F_1,\dots,F_k\in\mathcal{H}$ such that $\coprod E_n=\coprod_{j=1}^k F_j$. We may assume that $E_n\cap F_j\neq\emptyset$ for at most one j. Thus we may partition the (E_n) into (E_n^j) such that $\coprod E_n^j=F_j$ for $j=1,\dots,k$. We make the following claim.

Claim. *Suppose $H\in\mathcal{H}$ is a single h-interval such that $H=\coprod H_n$ where $(H_n)\subset\mathcal{H}$ is a sequence of disjoint h-intervals. Then $\mu_0(H)=\sum\mu_0(H_n)$.*

Then by applying (LS4), we have
$$\mu_0\left(\coprod E_n\right)=\mu_0\left(\coprod_{j=1}^k F_j\right)=\sum_{j=1}^k\mu_0(F_j)\underset{\text{(Claim)}}{=}\sum_{j=1}^k\sum\mu_0(E_n^j)=\sum\mu_0(E_n).$$

Thus it remains to prove the claim.

Proof of claim for $H = (a, b]$, $a, b \in \mathbb{R}$. Suppose $(a, b] = \coprod (a_j, b_j]$. Then for all $n \in \mathbb{N}$, $\coprod_{j=1}^{n}(a_j, b_j] \subset (a, b]$. By (LS4) and monotonicity, we have

$$\sum_{j=1}^{n} \mu_0((a_j, b]) = \mu_0\left(\coprod_{j=1}^{n}(a_j, b_j]\right) \leq \mu_0((a, b]).$$

Taking $n \to \infty$, we have $\sum \mu_0((a_j, b_j]) \leq \mu_0((a, b])$.
To show the reverse inequality, let $\varepsilon > 0$. Since F is right continuous,

- there is $\delta > 0$ such that $F(a + \delta) - F(a) < \frac{\varepsilon}{2}$, and
- for all $j \geq 1$, there is $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^{j+1}}$.

Observe now that $\{(a_j, b_j + \delta_j)\}_{j=1}^{\infty}$ is an open cover of the compact interval $[a + \delta, b]$. Hence there is a finite subcover, i.e., there is an $N \in \mathbb{N}$ such that $[a + \delta, b] \subset \bigcup_{j=1}^{N}(a_j, b_j + \delta_j)$. Then we calculate

$$\begin{aligned}
\mu_0((a, b]) &= F(b) - F(a) \\
&< F(b) - F(a + \delta) + \frac{\varepsilon}{2} \\
&= \mu_0((a + \delta, b]) + \frac{\varepsilon}{2} \\
&\leq \mu_0\left(\bigcup_{j=1}^{N}(a_j, b_j + \delta_j]\right) + \frac{\varepsilon}{2} \\
&\leq \sum_{j=1}^{N} \mu_0((a_j, b_j + \delta_j]) + \frac{\varepsilon}{2} \\
&= \sum_{j=1}^{N} (F(b_j + \delta_j) - F(a_j)) + \frac{\varepsilon}{2} \\
&\leq \sum_{j=1}^{N} \left(F(b_j) + \frac{\varepsilon}{2^{j+1}} - F(a_j)\right) + \frac{\varepsilon}{2} \\
&= \sum_{j=1}^{N} \mu_0((a_j, b_j]) + \sum_{j=1}^{N} \frac{\varepsilon}{2^{j+1}} + \frac{\varepsilon}{2} \\
&\leq \sum_{j=1}^{\infty} \mu_0((a_j, b_j]) + \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+1}} + \frac{\varepsilon}{2} \\
&= \sum_{j=1}^{\infty} \mu_0((a_j, b_j]) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\mu_0((a, b]) \leq \sum_{j=1}^{\infty} \mu_0((a_j, b_j])$. $\square$

The cases $H = (-\infty, b]$ for some $b < \infty$ and $H = (a, \infty)$ for $-\infty \leq a$ are left as the following exercise. $\square$

Exercise 2.5.8. Consider the extension $\mu_0 : \mathcal{A} \to [0, \infty]$ afforded by (LS4). Suppose H is $(-\infty, b]$ for some $b < \infty$ or (a, ∞) for $-\infty \leq a$. If $H = \coprod H_n$ where $(H_n) \subset \mathcal{H}$ is a sequence of disjoint h-intervals, then $\mu_0(H) = \sum \mu_0(H_n)$.

Exercise 2.5.9 (Folland, §1.5, #28)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous, and let μ_F be the associated Lebesgue-Stieltjes Borel measure on $\mathcal{B}_{\mathbb{R}}$. For $a \in \mathbb{R}$, define

$$F(a-) := \lim_{r \nearrow a} F(r).$$

Prove that:

(1) $\mu_F(\{a\}) = F(a) - F(a-)$,
(2) $\mu_F([a, b)) = F(b-) - F(a-)$,
(3) $\mu_F([a, b]) = F(b) - F(a-)$, and
(4) $\mu_F((a, b)) = F(b-) - F(a)$.

2.5.2. *Lebesgue measure.*

Definition 2.5.10. *Lebesgue measure* λ is the Lebesgue-Stieltjes measure μ_{id} where id : $\mathbb{R} \to \mathbb{R}$ is the identity function $\text{id}(r) = r$. The *Lebesgue σ-algebra* is $\mathcal{L} := \mathcal{M}^* = \overline{\mathcal{B}_{\mathbb{R}}}$ for $\lambda|_{\mathcal{B}_{\mathbb{R}}}$.

Definition 2.5.11. For $E \subset \mathbb{R}$ and $r, s \in \mathbb{R}$, define $rE := \{rx | x \in E\}$ and $s + E := \{s + x | x \in E\}$.

Theorem 2.5.12. *Suppose $E \in \mathcal{L}$.*

(1) (dilation homogeneity) If $r \in \mathbb{R}$, then $rE \in \mathcal{L}$ and $\lambda(rE) = |r| \cdot \lambda(E)$.
(2) (translation invariance) If $s \in \mathbb{R}$, then $s + E \in \mathcal{L}$ and $\lambda(s + E) = \lambda(E)$.

Proof. We will prove dilation homogeneity and leave translation invariance to the reader.

Step 1: $\mathcal{B}_{\mathbb{R}}$ is closed under $E \mapsto rE$. This is trivial if $r = 0$, so assume $r \neq 0$. Then multiplication by r is a bijection on $P(\mathbb{R})$ mapping open intervals to open intervals. Thus multiplication by r maps $\mathcal{B}_{\mathbb{R}}$ onto itself.

Step 2: It is a straightforward exercise to prove that $|r| \cdot \lambda$ is a measure on $\mathcal{L}$ and $\lambda^r(E) := \lambda(rE)$ is a measure on $\mathcal{B}_{\mathbb{R}}$.

Step 3: If $E \in \mathcal{H}$, then $\lambda^r(E) = |r| \cdot \lambda(E)$, so $\lambda^r = |r| \cdot \lambda$ on $\mathcal{A}(\mathcal{H})$ and thus all of $\mathcal{B}_{\mathbb{R}}$ by Corollary 2.4.10 (or Exercise 2.2.7) as λ^r and $|r| \cdot \lambda$ are both σ-finite.

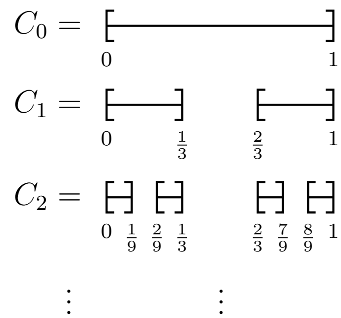
Step 4: If $E \in \mathcal{L}$ is λ-null, then $rE \in \mathcal{L}$ is λ-null. Indeed, by Remark 2.5.3, $E \in \mathcal{L}$ is λ-null if and only if there is an $N \in \mathcal{B}_{\mathbb{R}}$ such that $E \subset N$ and $\lambda(N) = 0$. Now $rE \subset rN$, and $\lambda(rN) = |r| \cdot \lambda(N) = 0$ by Step 3.

Step 5: Finally, as $\mathcal{L} = \overline{\mathcal{B}_{\mathbb{R}}}$ for λ, we see λ^r and $|r| \cdot \lambda$ are both defined on $\mathcal{L}$ and agree. □

Exercise 2.5.13. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra of $\mathbb{R}$. Suppose μ is a translation invariant measure on $\mathcal{B}_{\mathbb{R}}$ such that $\mu((0, 1]) = 1$. Prove that $\mu = \lambda|_{\mathcal{B}_{\mathbb{R}}}$, the restriction of Lebesgue measure on $\mathcal{L}$ to $\mathcal{B}_{\mathbb{R}}$.

Remark 2.5.14. By Exercise 2.5.9(1), $\lambda(\{r\}) = 0$ for all $r \in \mathbb{R}$, and thus $\lambda(E) = 0$ for all countable $E \subset \mathbb{R}$.

Example 2.5.15. The Cantor set C is defined as $\bigcap C_n$ where we define C_n inductively by 'removing middle thirds' of $[0, 1]$.



By continuity from above ($\mu 4$) for λ, we have $\lambda(C) = \lim_{n\to\infty} \lambda(C_n)$. By Exercise 2.5.9,

$$\begin{aligned}
&\lambda(C_0) = 1 \\
&\lambda(C_1) = 1 - \frac{1}{3} \\
&\lambda(C_2) = 1 - \frac{1}{3} - \frac{2}{9} \\
&\lambda(C_3) = 1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} \qquad\qquad \text{etc.} \\
\implies \qquad &\lambda(C) = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1 - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 1 - \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 0.
\end{aligned}$$

It is well known that C is uncountable; indeed it is in bijection with $\{0, 1\}^{\mathbb{N}}$ via base 3 decimal expansions where only the digits 0 and 2 occur. (Recall that decimal expansion is not unique; one must pick a particular convention here.)

Exercise 2.5.16. Show that the function $f : \{0, 1\}^{\mathbb{N}} \to C$ given by

$$f(x) := \sum_{n=1}^{\infty} \frac{2x_n}{3^n}$$

is a homeomorphism of $\{0, 1\}^{\mathbb{N}}$ onto the Cantor set.

Exercise 2.5.17. Suppose $E \in \mathcal{L}$ with $\lambda(E) > 0$. Show there is an $F \subset E$ such that $F \notin \mathcal{L}$. That is, show any Lebesgue measurable set with positive measure contains a non-measurable subset.

Exercise 2.5.18 (Sarason)**.** Suppose $E \in \mathcal{L}$ is Lebesgue null, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^1 function (continuous with continuous derivative). Prove that $\varphi(E)$ is also Lebesgue null.

Exercise 2.5.19. Let (X, ρ) be a metric (or simply a topological) space. A subset $S \subset X$ is called *nowhere dense* if $\overline{S}$ does not contain any open set in X. A subset $T \subset X$ is called *meager* if it is a countable union of nowhere dense sets.

Construct a meager subset of $\mathbb{R}$ whose complement is Lebesgue null.

Exercise 2.5.20. Suppose $F : \mathbb{R} \to \mathbb{R}$ is a bounded, non-decreasing, right continuous function, and let μ_F be the corresponding Lebesgue-Stieltjes measure. (Observe μ_F is finite.) Prove the following are equivalent:

(1) μ_F is absolutely continuous (see Exercise 2.2.9) with respect to Lebesgue measure λ.
(2) F is *absolutely continuous*, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any disjoint set of open intervals $(a_1, b_1), \ldots, (a_N, b_N)$,

$$\sum_{i=1}^{n}(b_i - a_i) < \delta \qquad \Longrightarrow \qquad \sum_{i=1}^{N}(F(b_i) - F(a_i)) < \varepsilon.$$

2.5.3. *Regularity properties of Lebesgue-Stieltjes measures.*

Definition 2.5.21. Suppose $(X, \mathcal{T})$ is a Hausdorff topological space and $\mathcal{M} \subset P(X)$ is any σ-algebra containing the Borel σ-algebra $\mathcal{B}(\mathcal{T})$, i.e., $\mathcal{T} \subset \mathcal{M}$. A measure μ on $\mathcal{M}$ is called:

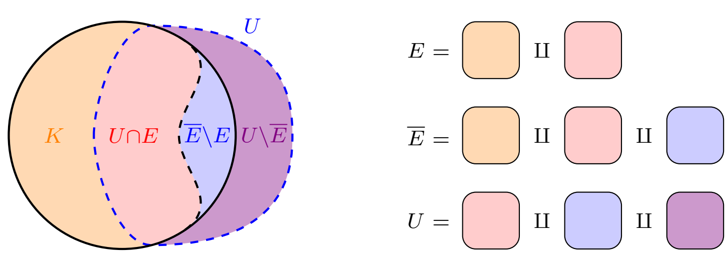
- *outer regular* if $\mu(E) = \inf\{\mu(U) | E \subset U \text{ open}\}$
- *inner regular* if $\mu(E) = \sup\{\mu(K) | \text{compact } K \subset E\}$
- *regular* if μ is both outer and inner regular.

Proposition 2.5.22. *Suppose $(X, \mathcal{T})$ is a Hausdorff topological space and μ is a Borel measure on $\mathcal{B}_{\mathcal{T}}$. If $(X, \mathcal{T})$ is σ-compact and μ is outer regular and finite on compact sets, then μ is inner regular and thus regular (and thus Radon; see Exercise 2.5.24 below).*

Proof.
Step 1: Suppose X is compact and $E \in \mathcal{B}_{\mathcal{T}}$. Then $\overline{E}$ is compact. Let $\varepsilon > 0$. By outer regularity, there is an open $U \supset \overline{E} \setminus E$ such that $\mu(U) \leq \mu(\overline{E} \setminus E) + \varepsilon$. Observe that:

- $\overline{E} \setminus E \subset U \setminus E$,
- $K := \overline{E} \setminus U$ is compact and contained in E, and
- since $\overline{E} = K \amalg (U \cap \overline{E})$ and $E \subset \overline{E}$, $E = (K \cap E) \amalg (U \cap E)$, and thus $U \cap E = K^c \cap E$.

Here is a cartoon of $K, E, \overline{E}, U$:



We now calculate

$$\begin{aligned}
\mu(K) &= \mu(E) - \mu(K^c \cap E) && (E = K \amalg (K^c \cap E)) \\
&= \mu(E) - \mu(U \cap E) && (E \cap U = E \cap K^c) \\
&= \mu(E) - (\mu(U) - \mu(U \setminus E)) && (U = (E \cap U) \amalg (U \setminus E)) \\
&\geq \mu(E) \underbrace{-\mu(U) + \mu(\overline{E} \setminus E)}_{\geq -\varepsilon} && (\overline{E} \setminus E \subset U \setminus E) \\
&\geq \mu(E) - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, μ is inner regular.
Step 2: Since X is σ-compact, by disjointification, we may write $X = \coprod X_n$ where each X_n has compact closure in X. In particular, $\mu(X_n) < \infty$ for all n. Let $E \in \mathcal{B}_{\mathcal{T}}$, and write $E = \coprod E_n$

where $E_n := E \cap X_n$. By Step 1, for each n, there is a compact set $K_n \subset E_n \subset X_n \subset \overline{X_n}$ such that $\mu(K_n) \geq \mu(E_n) - \frac{\varepsilon}{2^{n+1}}$. Set $F_n := \coprod_{i=1}^n K_i$, which is still compact. Observe that

$$\mu(F_n) \geq \mu\left(\coprod_{i=1}^n E_i\right) - \frac{\varepsilon}{2}.$$

There are two cases to consider now.

If $\mu(E) = \infty$, since $\mu(\coprod_{i=1}^n E_i) \nearrow \infty$, eventually $\mu(F_n) > M$ for every $M > 0$. Hence $\sup\{\mu(F_n)|n \in \mathbb{N}\} = \infty = \mu(E)$. Otherwise, $\mu(E) < \infty$, and there is an $N \in \mathbb{N}$ such that

$$\mu(E) \leq \mu\left(\coprod_{i=1}^N E_i\right) + \frac{\varepsilon}{2} \leq \mu(F_N) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mu(F_N) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude μ is inner regular. $\square$

Exercise 2.5.23. Suppose $(X, \mathcal{T})$ is a topological space, μ is a σ-finite regular Borel measure, and $E \in \mathcal{B}_\mathcal{T}$ is a Borel set. Prove the following assertions.

(1) For every $\varepsilon > 0$, there exist an open U and a closed F with $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon$.
(2) There exist an F_σ-set A and a G_δ-set B such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$.

Exercise 2.5.24. Suppose $(X, \mathcal{T})$ is a topological space, μ is a Borel measure on $\mathcal{B}_\mathcal{T}$. We call μ a *Radon measure* if μ is outer regular, finite on compact sets, and inner regular on all open sets.

(1) Show that if μ is a σ-finite Radon measure, then μ is inner regular and thus regular. Deduce that the finite Radon measures are exactly the finite regular Borel measures.
(2) Suppose μ is a σ-finite regular Borel measure. Is μ Radon? That is, is μ finite on all compact sets? Give a proof or a counterexample.

Exercise 2.5.25 (Folland, §7.2, #7)**.** Suppose μ is a σ-finite Radon measure on $(X, \mathcal{T})$ and $E \in \mathcal{B}_\mathcal{T}$ is a Borel set. Show that $\mu_E(F) := \mu(E \cap F)$ defines another (σ-finite) Radon measure.

Remark 2.5.26. Once we have developed the theory of integration, we will be able to upgrade Proposition 2.5.22 considerably. In Corollary 5.6.10, we will show that if X is LCH such that every open set is σ-compact, then every Borel measure which is finite on compact sets is regular and thus Radon.

Exercise 2.5.27. Suppose X is a metric space (not necessarily locally compact) and let μ be a finite Borel measure. Show that the collection $\mathcal{M} \subset \mathcal{B}_X$ of sets such that

$$\begin{aligned}\mu(E) &= \inf\{\mu(U)|E \subseteq U \text{ open}\}\\ &= \sup\{\mu(F)|E \supseteq F \text{ closed}\}\end{aligned}$$

is a σ-algebra containing all closed (or open) sets and is thus equal to $\mathcal{B}_X$. Deduce that μ is outer regular.

Exercise 2.5.28. Suppose X is a compact Hausdorff topological space, $\mathcal{B}_X$ is the Borel σ-algebra, and μ is a regular measure on $\mathcal{B}_X$ such that $\mu(X) = 1$. Prove there is a compact $K \subset X$ such that $\mu(K) = 1$ and $\mu(F) < 1$ for every proper compact subset $F \subsetneq K$.
Remark: One strategy uses Zorn's Lemma, but it is not necessary.

We now analyze the regularity of the Lebesgue-Stieltjes measure μ_F on $\mathcal{M}_F$ where $F : \mathbb{R} \to \mathbb{R}$ is any non-decreasing right continuous function.

Exercise 2.5.29. For every $E \subset \mathbb{R}$, show that

$$\mu_F(E) = \inf\left\{\sum_{n=1}^{\infty} \mu_F((a_n, b_n]) \middle| E \subset \bigcup_{n=1}^{\infty} (a_n, b_n] \text{ with } a_n, b_n \in \mathbb{R}, \forall\, n \in \mathbb{N}\right\}.$$

Lemma 2.5.30. *For all $E \in \mathcal{M}_F$, $\mu_F(E) = \inf\{\sum_{n=1}^{\infty} \mu_F((a_n, b_n)) | E \subset \bigcup_{n=1}^{\infty}(a_n, b_n)\}$.*

Proof. Denote the inf on the right hand side by $\nu(E)$.
Step 1: $\mu_F(E) \leq \nu(E)$.

Suppose $E \subset \bigcup(a_n, b_n)$. We can write each $(a_n, b_n) = \coprod_{i=1}^{\infty}(a_i^n, b_i^n]$. Then $E \subset \bigcup_{n=1}^{\infty} \coprod_{i=1}^{\infty}(a_i^n, b_i^n]$, and

$$\mu_F(E) \leq \sum_{n,i} \mu_F((a_i^n, b_i^n]) = \sum \mu_F((a_n, b_n)).$$

Step 2: $\mu_F(E) \geq \nu(E)$.

Let $\varepsilon > 0$. There exists $((a_n, b_n])$ such that $E \subset \bigcup(a_n, b_n]$ and $\sum \mu_F((a_n, b_n]) \leq \mu_F(E) + \frac{\varepsilon}{2}$. For each n, by right continuity of F, pick $\delta_n > 0$ such that $F(b_n + \delta_n) - F(b_n) < \frac{\varepsilon}{2^{n+1}}$. Then $E \subset \bigcup(a_n, b_n + \delta_n)$ and

$$\begin{aligned}
\sum \mu_F((a_n, b_n + \delta_n)) &\leq \sum F(b_n + \delta_n) - F(a_n) \\
&< \sum F(b_n) - F(a_n) + \frac{\varepsilon}{2^{n+1}} \\
&= \sum \mu_F((a_n, b_n]) + \sum \frac{\varepsilon}{2^{n+1}} \\
&\leq \mu_F(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \mu_F(E) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the result follows.

This concludes the proof. □

Theorem 2.5.31. *The Lebesgue-Stieltjes measure μ_F on $\mathcal{M}_F$ is regular.*

Proof. Since $\mathbb{R}$ is σ-compact and μ_F is finite on all compact intervals by Exercise 2.5.9, by Proposition 2.5.22, it remains to show μ_F is outer regular. Let $E \in \mathcal{M}_F$. By Lemma 2.5.30, given $\varepsilon > 0$, there is a sequence $((a_n, b_n))$ of open intervals such that $E \subset \bigcup(a_n, b_n)$ and $\sum \mu_F((a_n, b_n)) \leq \mu(E) + \varepsilon$. Setting $U = \bigcup(a_n, b_n)$, we have $E \subset U$ and

$$\mu_F(E) \leq \mu_F(U) \leq \sum \mu_F((a_n, b_n)) \leq \mu(E) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have $\mu_F(E) = \inf\{\mu_F(U) | E \subset U \text{ open}\}$. □

Exercise 2.5.32 (Steinhaus Theorem, Folland §1.5, #30 and #31)**.** Suppose $E \in \mathcal{L}$ and $\lambda(E) > 0$.

(1) Show that for any $0 \le \alpha < 1$, there is an open interval $I \subset \mathbb{R}$ such that $\lambda(E \cap I) > \alpha\lambda(I)$.
(2) Apply (1) with $\alpha = 3/4$ to show that the set

$$E - E := \{x - y | x, y \in E\}$$

contains the interval $(-\lambda(I)/2, \lambda(I)/2)$.

2.6. **Hausdorff measure.** Let (X, d) be a metric space. For $A, B \subset X$ nonempty, define

$$\begin{aligned} d(a, B) &:= \inf\{d(a, b) | b \in B\} \qquad && \forall\, a \in A \\ d(A, B) &:= \inf\{d(a, b) | a \in A,\, b \in B\}. \end{aligned}$$

For a set $Y \subset X$, define

$$\operatorname{diam}(Y) := \sup\{d(x, y) | x, y \in Y\}.$$

Definition 2.6.1. An outer measure μ^* on $P(X)$ is called a (Carathéodory) *metric outer measure* if

- (metric finite additivity) $d(A, B) > 0$ (which implies $A \cap B = \emptyset$) implies $\mu^*(A \coprod B) = \mu^*(A) + \mu^*(B)$.

Proposition 2.6.2. *If μ^* is a metric outer measure on $P(X)$, then the Borel σ-algebra $\mathcal{B}_d$ is contained in $\mathcal{M}^*$, the μ^*-measurable sets.*

Proof. Since $\mathcal{B}_d$ is generated by the open sets, it suffices to show all open sets are in $\mathcal{M}^*$. Let $U \subset X$ be open.
Step 1: We may assume $d(U, U^c) = 0$. Otherwise, for all $F \subset X$, $d(F \cap U, F \setminus U) > 0$, so $\mu^*(F) = \mu^*(F \cap U) + \mu^*(F \setminus U)$, and thus $U \in \mathcal{M}^*$.
Step 2: For $n \in \mathbb{N}$, define $A_n := \{x \in U | d(x, U^c) > 1/n\}$. Then (A_n) is increasing and $\bigcup A_n = \mathcal{U}$. Setting $A_0 = \emptyset$, define $B_n := A_n \setminus A_{n-1}$ for all $n \in \mathbb{N}$. Then $\coprod B_n = U$, and $B_n \neq \emptyset$ frequently. Indeed, observe $B_n = \emptyset$ for all $n > k$ if and only if $A_k = U$, which implies $d(U, U^c) \ge 1/k$.
Step 3: If $|m - n| > 1$ and $B_m \neq \emptyset \neq B_n$, then $d(B_m, B_n) > 0$.

Proof. Suppose $1 \le m < n - 1$. Let $x \in B_m$ and $y \in B_n$. Then $y \notin A_{n-1} \supset A_{m+1}$, so there is a $z \in U^c$ such that $d(y, z) \le \frac{1}{m+1}$. But $x \in B_m$, so $d(x, z) > \frac{1}{m}$. By the triangle inequality,

$$d(x, y) \ge d(x, z) - d(y, z) > \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}.$$

Taking sup over x, y, we have $d(B_m, B_n) \ge \frac{1}{m(m+1)} > 0$. $\square$

Step 4: Let $F \subset X$. If $\mu^*(F) = \infty$, then $\mu^*(F) \ge \mu^*(F \cap U) + \mu^*(F \setminus U)$. Assume $\mu^*(F) < \infty$. Then $\sum_{n=k}^{\infty} \mu^*(F \cap B_n) \to 0$ as $k \to \infty$.

Proof. By Step 3, for all $k \in \mathbb{N}$, we have

$$\sum^k \mu^*(F \cap B_{2n-1}) = \mu^*\left(\coprod^k F \cap B_{2n-1}\right) \le \mu^*(F)$$

$$\sum^k \mu^*(F \cap B_{2n}) = \mu^*\left(\coprod^k F \cap B_{2n}\right) \le \mu^*(F).$$

Taking $k \to \infty$, we have $\sum \mu^*(F \cap B_n) \le 2\mu^*(F) < \infty$. Hence the tail of the sum must converge to zero. $\square$

Step 5: We now calculate for all $n \in \mathbb{N}$ and $F \subset X$:

$$\begin{aligned}
\mu^*(F \cap U) + \mu^*(F \setminus U) &\le \mu^*(F \cap A_n) + \mu^*(F \cap (\underbrace{U \setminus A_n}_{\coprod_{k=n+1}^\infty B_k})) + \mu^*(F \setminus U) \\
&= \underbrace{\mu^*(F \cap A_n) + \mu^*(F \setminus U)}_{d(F\cap A_n, F\setminus U) \ge d(A_n, U^c) \ge \frac{1}{n}} + \mu^*\left(\coprod_{k=n+1}^\infty B_k\right) \\
&= \mu^*(F \cap (A_n \amalg F \setminus U)) + \mu^*\left(\coprod_{k=n+1}^\infty B_k\right) \\
&\le \mu^*(F) + \underbrace{\sum_{k=n+1}^\infty \mu^*(F \cap B_k)}_{\to 0 \text{ as } n \to \infty \text{ by Step 4.}}.
\end{aligned}$$

We conclude that $U \in \mathcal{M}^*$. $\square$

Definition 2.6.3. Suppose (X, d) is a metric space, $p \ge 0$, and $\varepsilon > 0$. For $E \subset X$, define

$$\eta^*_{p,\varepsilon}(E) := \inf\left\{ \sum_1^\infty (\operatorname{diam}(B_n))^p \,\middle|\, \begin{array}{l} (B_n) \text{ a } \le \varepsilon\textit{-diameter cover}, \text{ i.e., a sequence of open} \\ \text{balls with } \operatorname{diam}(B_n) \le \varepsilon \text{ for all } n \text{ and } E \subset \bigcup B_n \end{array} \right\},$$

where we use the convention that $\inf \emptyset = \infty$. By Exercise 2.3.5, $\eta^*_{p,\varepsilon}$ is the outer measure induced by

$$\begin{aligned}
\rho_{p,\varepsilon} : \{\emptyset\} \cup \{B_r(x) | x \in X \text{ and } r \le \varepsilon\} &\longrightarrow [0, \infty] \\
\emptyset &\longmapsto 0 \\
B_r(x) &\longmapsto (\operatorname{diam}(B_r(x)))^p.
\end{aligned}$$

Moreover, if $\varepsilon < \varepsilon'$, then $\eta^*_{p,\varepsilon}(E) \ge \eta^*_{p,\varepsilon'}(E)$ as we are taking an infimum over a smaller set (every $\le \varepsilon$-diameter cover is a $\le \varepsilon'$-diameter cover). Hence

$$\eta^*_p(E) := \lim_{\varepsilon \to 0} \eta^*_{p,\varepsilon}(E) = \sup_{\varepsilon > 0} \eta^*_{p,\varepsilon}(E)$$

gives an outer measure by Exercise 2.3.2.

Proposition 2.6.4. *η^*_p is a metric outer measure.*

Proof. Suppose $d(E,F) > \varepsilon > 0$. If there is no ε-diameter cover of $E \amalg F$, then there is no ε-diameter cover of one of E, F, and thus

$$\eta_p^*(E) + \eta_p^*(F) = \infty = \eta_p^*(E \amalg F).$$

Now suppose there exists an ε-diameter cover (B_n) of $E \amalg F$. Then for all $n \in \mathbb{N}$, B_n intersects at most one of E, F. So we may partition (B_n) into (B_n^E) and (B_n^F) such that

- $E \subset \bigcup B_n^E$ and $B_n^E \cap E \neq \emptyset$, and
- $F \subset \bigcup B_n^F$ and $B_n^F \cap F \neq \emptyset$.

Thus

$$\eta_{p,\varepsilon}^*(E) + \eta_{p,\varepsilon}^*(F) \leq \sum \operatorname{diam}(B_n^E)^p + \operatorname{diam}(B_n^F)^p \leq \sum \operatorname{diam}(B_n)^p$$

for any ε-diameter cover. Hence for all $\varepsilon < d(E,F)$,

$$\eta_{p,\varepsilon}^*(E) + \eta_{p,\varepsilon}^*(F) \leq \eta_{p,\varepsilon}^*(E \amalg F).$$

Taking $\varepsilon \to 0$, we get

$$\eta_p^*(E \amalg F) \leq \eta_p^*(E) + \eta_p^*(F) \leq \eta_p^*(E \amalg F),$$

and thus equality holds. □

Definition 2.6.5. Since the Borel σ-algebra $\mathcal{B}_d$ for (X,d) is contained in the η_p^*-measurable sets $\mathcal{M}_p^*$ by Propositions 2.6.2 and 2.6.4, we get a Borel measure $\eta_p := \eta_p^*|_{\mathcal{B}_d}$ called *p-dimensional Hausdorff measure.*

Facts 2.6.6. Here are some elementary properties about Hausdorff measures.

(Hμ1) If $f : X \to X$ is an *isometry* ($d(f(x), f(y)) = d(x,y)$ for all $x, y \in X$), then for all $E \in \mathcal{B}_d$, $\eta_p(E) = \eta_p(f(E))$.

> *Proof.* For all $\varepsilon > 0$, $\eta_{p,\varepsilon}^*(E) = \eta_{p,\varepsilon}^*(f(E))$ since $E \subset \bigcup B_n$ if and only if $f(E) \subset \bigcup f(B_n)$ as isometries are injective. □

(Hμ2) $\eta_1 = \lambda|_{\mathcal{B}_\mathbb{R}}$ on $\mathbb{R}$ with the usual metric.

> *Proof.* Since $\eta_1((0,1]) = 1$ (observe $\operatorname{diam}(B) = \lambda(B)$ for any open ball B and apply Lemma 2.5.30), this follows by uniqueness of the translation invariant Borel measure on $\mathbb{R}$ from Exercise 2.5.13. □

(Hμ3) If $\eta_p(E) < \infty$, then $\eta_q(E) = 0$ for all $q > p$.

Proof. Let $\varepsilon > 0$. Since $\eta_p(E) < \infty$, there is a sequence (B_n) of open balls with $\operatorname{diam}(B_n) \le \varepsilon$ such that $\sum \operatorname{diam}(B_n)^p \le \eta_p(E) + 1$. But if $q > p$, then

$$\begin{aligned}\eta^*_{q,\varepsilon}(E) &\le \sum \operatorname{diam}(B_n)^q \\ &= \sum \underbrace{\operatorname{diam}(B_n)^{q-p}}_{\le \varepsilon^{q-p}} \operatorname{diam}(B_n)^p \\ &\le \varepsilon^{q-p} \sum \operatorname{diam}(B_n)^p \\ &\le \varepsilon^{q-p}(\eta_p(E) + 1).\end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$\eta_q(E) = \eta^*_q(E) = \lim_{\varepsilon\to 0} \eta^*_{q,\varepsilon}(E) \le \lim_{\varepsilon \to 0} \varepsilon^{q-p}(\eta_p(E) + 1) = 0. \qquad \square$$

(Hμ4) If $\eta_p(E) > 0$, then $\eta_q(E) = \infty$ for all $q < p$.

Proof. This follows as the contrapositive of (Hμ3). $\square$

Definition 2.6.7. The *Hausdorff dimension* of $E \in \mathcal{B}_d$ is

$$\inf \{p \ge 0 | \eta_p(E) = 0\} = \sup \{p \ge 0 | \eta_p(E) = \infty\}.$$

Remark 2.6.8. If $E \in \mathcal{B}_d$ and $p \ge 0$ such that $0 < \eta_p(E) < \infty$, then the Hausdorff dimension of E is necessarily p by Lemma 2.6.6(3,4).

Exercise 2.6.9. Prove that the Cantor set from Example 2.5.15 has Hausdorff dimension $\ln(2)/\ln(3)$.

Exercise 2.6.10. Find an uncountable subset of $\mathbb{R}$ with Hausdorff dimension zero.