# Part I: Measures, integration, differentiation

Ch 1 — Measures; Ch 2 — integration; Ch 3 — differentiation

We want measures to study area/volume and to integrate fcts.

## Informal Discussion:

Let $X$ be a set. A <u>measure</u> on $X$ is a fct

$$\mu : \underbrace{\mathcal{M}}_{\subseteq P(X),\ \text{power set of } X} \longrightarrow [0,\infty] \quad \text{satisfying:}$$

⓪ $\mu(\emptyset) = 0$

① $\mu(\underbrace{\amalg}_{\text{disjoint union}} E_i) = \sum \mu(E_i)$ when $(E_i)_i$ are <u>disjoint</u> ($E_i \cap E_j = \emptyset,\ i \neq j$)

Call $\mu$ <u>finite</u> if $\mu(X) < \infty$

<u>Q:</u> What properties should $\mathcal{M} \subset P(X)$ satisfy?

- $\emptyset, X \in \mathcal{M}$ (nonempty!)
- closed under disjoint unions (finite? countable?)

<u>Example:</u> Let $\mathcal{M} = P(X)$, $\mu(E) = |E|$ "counting measure"

<u>Example:</u> There is a measure $\lambda$ on some $\mathcal{M} \subset P(\mathbb{R})$ s.t.

- $\lambda([0,1)) = 1$
- $\lambda(E + r) = \lambda(E)\ \forall E \in \mathcal{M}$.

For this $\lambda$, <u>can not</u> have $\mathcal{M} = P(\mathbb{R})$!

Define an equivalence rel'n on $[0,1)$ by

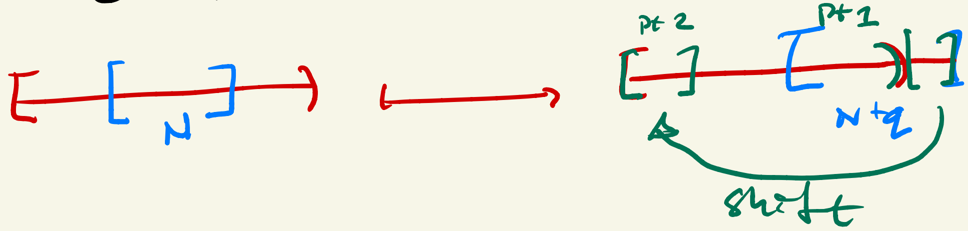
$$x \sim y \iff x-y \in \mathbb{Q}$$

[Exercise: Check it's an equiv. rel'n]

Using the Axiom of Choice, pick one representative from each equiv. class & call this set $E$. For $q \in \mathbb{Q} \cap [0,1)$, define

$$E_q := \{x+q \mid x \in E \cap [0,1-q)\} \cup \{x+q-1 \mid x \in [1-q, 1)\}$$

Idea:




Now if all sets were in $\mathcal{M}$, then we'd have

$$1 = \lambda([0,1)) = \lambda\Big(\underbrace{\coprod E_q}_{\text{ctble sum}}\Big) = \sum_q \underbrace{\lambda(E_q)}_{=\lambda(E)\ \forall q!} = \lambda(E) \sum_q 1 \in \{0, \infty\}$$

a contradiction.

Formal Discussion:

Def: We call $\mathcal{M} \subset \mathcal{P}(X)$ an algebra if

⓪ $\mathcal{M} \neq \emptyset$,

① $\mathcal{M}$ is closed under finite unions, and

② $\mathcal{M}$ is closed under complements

Observe: Every algebra:

- contains $X = E \amalg E^c$ ($\exists E \in \mathcal{M}$; $^c$ denotes complement)
- contains $\emptyset = X^c$
- is closed under finite intersections

$$\bigcap_i^n E_i = \Big[\bigcap_i^n E_i\Big]^{cc} = \Big[\bigcup_i^n E_i^c\Big]^c$$

(de Morgan)

If in addition an algebra $\mathcal{M}$ is closed under <u>ctble unions</u>, we call $\mathcal{M}$ a <u>$\sigma$-algebra</u>

'ctble'

<u>Examples</u>:

⓪ $\{\emptyset, X\}$ is <u>trivial</u> $\sigma$-alg

① $\mathcal{P}(X)$ is <u>discrete</u> $\sigma$-alg

② If $X$ is uncountable, can define

$$\mathcal{M} = \{E \subset X \mid E \text{ or } E^c \text{ is countable}\}$$

"countable union of countable sets is countable"

<u>Q</u>: What about $\bigcup E_i$ where one $E_j$ unctble?

$\hat{\mathcal{M}}$

<u>Exercises</u>:

- [Disjointification] Suppose we have a ctble collection of subsets $(E_i)$ of $X$. Show that

$$F_1 := E_1 \quad , \quad F_k := \underbrace{E_k \setminus \bigcup_1^{k-1} E_i}_{= E_k \cap \left(\bigcup_1^{k-1} E_i\right)^c} \quad \text{(inductively)}$$

gives a countable collection of <u>disjoint</u> subsets $(F_i)$ of $X$.

- If $\mathcal{M}, \mathcal{N}$ are $\sigma$-alg's, so is $\mathcal{M} \cap \mathcal{N}$.

This means if $\mathcal{E} \subset \mathcal{P}(X)$, there is a <u>smallest</u> $\sigma$-alg $\mathcal{M}(\mathcal{E})$ which contains $\mathcal{E}$. Call $\mathcal{M}(\mathcal{E})$ the $\sigma$-alg <u>generated</u> by $\mathcal{E}$.

③ Suppose $(X, \tau)$ a <u>topological space</u>. Call $\mathcal{M}(\tau)$ the <u>Borel</u> $\sigma$-alg.

<u>topology</u>: collection of subsets of $X$ s.t.

⓪ $\emptyset, X \in \tau$

① closed under <u>arbitrary</u> unions

② closed under <u>finite</u> intersection.

**Def:** A ctble intersection of open sets is called a $G_\delta$ set.
- - - - union - - closed - - - - - - - $F_\sigma$ set.
- - - - union - - $G_\delta$ - - - - - - - $G_{\delta\sigma}$ set.
- - - - intersection - - $F_\sigma$ - - - - - - - $F_{\sigma\delta}$ set.

- Get $\mathcal{M}(\tau)$ by iteratively adding all these in.

**When $X=\mathbb{R}$:** Look at topology induced by $\rho(x,y) := |x-y|$.
Let $\mathcal{B}_{\mathbb{R}}$ be the Borel $\sigma$-alg.

**Prop:** $\mathcal{B}_{\mathbb{R}}$ is gen by each of:

① open intervals $(a,b)$
② closed intervals $[a,b]$
③ half-open intervals $(a,b]$
③' - - - - - - - - $[a,b)$
④ open rays $(a,\infty)$ and $(-\infty,a)$
⑤ closed rays $[a,\infty)$ and $(-\infty,a]$

(*) Observe:

$$(a,b) = \bigcap (a, b+\tfrac{1}{n}]$$
$$= \bigcap [b-\tfrac{1}{n}, a)$$
$$= (a,\infty) \cap (-\infty, b)$$
$$= \left((-\infty,a] \cup [b,\infty)\right)^c$$

To show this, we'll use:

**Observation:** If $\mathcal{E}, \mathcal{F} \subset P(X)$ with $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$

- follows by minimality of $\mathcal{M}(\mathcal{E})$.

**Pf of prop:** First, ①, ②, ④, ⑤ are all open or closed, so they lie in $\mathcal{B}_{\mathbb{R}}$

③ $(a,b] = (a,\infty) \cap (b,\infty)^c \in \mathcal{B}_{\mathbb{R}}$  ③' Similar

$\Rightarrow$ all ①–⑤ lie in $\mathcal{B}_{\mathbb{R}}$, so their generated $\sigma$-alg's do too.

For other direction: All open sets in $\mathbb{R}$ are countable unions of open intervals [This is Prop. 0.21]. Hence $\mathcal{M}(①) \supset \mathcal{B}_{\mathbb{R}}$.

For ②–⑤, show $\underbrace{\mathcal{M}(Ⓢ) \supset ①}_{(*)}$, $\Rightarrow \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(①) \subset \mathcal{M}(Ⓢ) \subset \mathcal{B}_{\mathbb{R}}$ ✓

**Def:** A set $X$ together w/ a $\sigma$-alg $\mathcal{M}$ is called a measurable space. A measure on $(X, \mathcal{M})$ is a fct $\mu : \mathcal{M} \longrightarrow [0, \infty]$ s.t.

⓪ $\mu(\emptyset) = 0$

① $\forall$ seq. of disjoint sets $(E_i)$, $\mu(\amalg E_i) = \sum \mu(E_i)$

← ctble additivity. $\Rightarrow$ finite additivity by taking $E_i = \emptyset$ for large $i$.

We call $(X, \mathcal{M}, \mu)$ a measure space. A meas. space is called:

- finite if $\mu(X) < \infty$
- $\sigma$-finite if $X = \cup E_i$, $E_i \in \mathcal{M}$ $\forall i$, and $\mu(E_i) < \infty$ $\forall i$
- (ex) semifinite if $\forall E \in \mathcal{M}$ with $\mu(E) = \infty$, $\exists F \subset E$ s.t. $0 < \mu(F) < \infty$.
- complete if $E \in \mathcal{M}$ with $\underbrace{\mu(E) = 0}_{E \text{ is } (\mu\text{-})\text{null}}$ and $F \subset E \Rightarrow F \in \mathcal{M}$.

(ex) we'll see that $\mu(F) = 0$ by monotonicity

**Examples:** Recall

① Counting measure on $P(X)$

② Pick $x_0 \in X$. On $P(X)$, define $\mu(E) = \begin{cases} 0 & x_0 \notin E \\ 1 & x_0 \in E \end{cases}$

Called point mass or Dirac measure at $x_0$

③ Pick any $f : X \to [0, \infty]$. On $P(X)$, define $\mu(E) = \sum_{x \in E} f(x)$

Counting: $f = 1$ ; Dirac: $f = \delta_{x_0}$.

④ On $\sigma$-alg of ctble or co-ctble sets, $\mu(E) := \begin{cases} 0 & \text{ctble} \\ 1 & \text{co-ctble} \end{cases}$

Basic properties of measures: $(X, \mathcal{M}, \mu)$ a meas. space.

① (Monotonicity) $E, F \in \mathcal{M}$, $E \subset F \Rightarrow \mu(E) \le \mu(F)$.

Pf: $\mu(F) = \mu(E \sqcup (F \setminus E)) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\ge 0}$

② (Subadditivity) $(E_i)_{i=1}^{\infty} \subset \mathcal{M}$, $\mu(\bigcup E_i) \le \sum \mu(E_i)$

Pf: write $E_1 = F_1$, $F_k = E_k \setminus \bigcup_1^{k-1} E_i$, so $F_k \subset E_k$ $\forall k$.

Then $\mu(\bigcup E_i) = \mu(\bigsqcup F_k) = \sum \mu(F_k) \le \sum \mu(E_k)$.

③ (Continuity from below) If $E_1 \subset E_2 \subset E_3 \subset \cdots$, then

$$\mu(\bigcup E_i) = \lim_{i \to \infty} \mu(E_i)$$

Pf: Set $E_0 := \emptyset$. Then

$$\mu(\bigcup E_i) = \mu(\bigsqcup (E_i \setminus E_{i-1})) = \sum \mu(E_i \setminus E_{i-1})$$
$$= \lim_n \sum^n \mu(E_i \setminus E_{i-1}) = \lim_n \mu(E_n).$$

④ (Continuity from above) If $E_1 \supset E_2 \supset E_3 \supset \cdots$ and $\mu(E_1) < \infty$, then $\mu(\bigcap E_i) = \lim_{i \to \infty} \mu(E_i)$.

Pf: Let $F_i := E_1 \setminus E_i$. Then $F_1 \subset F_2 \subset F_3 \subset \cdots$ and $\mu(E_1) = \mu(E_i) + \mu(F_i)$ $\forall i$. Observe

$$\bigcup F_i = \bigcup E_1 \cap E_i^c = E_1 \cap [\bigcup E_i^c] = E_1 \cap [\bigcap E_i]^c = E_1 \setminus \bigcap E_i.$$

Hence

$$\mu(\bigcap E_i) = \mu(E_1) - \mu(\bigcup F_i) \overset{③}{=} \mu(E_1) - \lim_n \mu(F_n)$$
$$= \mu(E_1) - \lim_n [\mu(E_1) - \mu(E_n)] = \lim_n \mu(E_n).$$

Cor: If $E \in \mathcal{M}$ with $\mu(E) = 0$ and $F \subset E$ w/ $F \in \mathcal{M}$, $\mu(F) = 0$.

Thm (Completion): Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Define $\overline{\mathcal{M}} := \{E \cup F \mid E \in \mathcal{M} \text{ and } F \subset N \in \mathcal{M} \text{ with } \mu(N) = 0\}$. Then $\overline{\mathcal{M}}$ is a $\sigma$-alg, and $\exists!$ measure $\overline{\mu}$ on $\overline{\mathcal{M}}$ s.t. $\overline{\mu}|_{\mathcal{M}} = \mu$.

Pf: ⓪ observe $\emptyset \in \mathcal{M} \subset \overline{\mathcal{M}}$.

① If $(E_i \cup F_i)$ is a seq. of sets in $\overline{\mathcal{M}}$, then

$$\bigcup (E_i \cup F_i) = \underbrace{\left(\bigcup E_i\right)}_{\in \mathcal{M}} \cup \underbrace{\left(\bigcup F_i\right)}_{\subset \bigcup N_i}.$$

Now observe $\bigcup F_i \subset \bigcup \underbrace{N_i}_{\in \text{null}}$, and $\mu(\bigcup N_i) \leq \sum \mu(N_i) = 0$.

Hence $\bigcup (E_i \cup F_i) \in \overline{\mathcal{M}}$.

② If $E, N \in \mathcal{M}$ with $F \subset N$ null, use:

Trick: Intersect w/ $X = N^c \sqcup N$.

$$(E \cup F)^c = [E^c \cap F^c] \cap [N^c \sqcup N]$$

$$= \underbrace{[E^c \cap \underbrace{F^c \cap N^c}_{\substack{= N^c \in \mathcal{M} \\ \text{as } F \subset N}}]}_{\in \mathcal{M}} \sqcup [\underbrace{E^c \cap F^c \cap N}_{\subset N}]$$

Hence $\overline{\mathcal{M}}$ is closed under taking complements.

$!\bar{\mu}$: If $\bar{\mu}|_{\mathcal{M}} = \mu$, then $\forall\, E \cup F \in \overline{\mathcal{M}}$ w/ $F \subset N$ null,

$$\mu(E) = \bar{\mu}(E) \leq \bar{\mu}(E \cup F) \leq \bar{\mu}(E) + \bar{\mu}(F)$$
$$\leq \bar{\mu}(E) + \bar{\mu}(N) = \mu(E) + \mu(N) = \mu(E).$$

Hence $\bar{\mu}(E \cup F) = \mu(E)$.

$\exists \bar{\mu}$: By the above, we must define $\bar{\mu}(E \cup F) := \mu(E)$.

well-defined: If $E_1 \cup F_1 = E_2 \cup F_2$ w/ $F_i \subset N_i$ $\mu$-null, then

$E_1 \subset E_1 \cup F_1 = E_2 \cup F_2 \subset E_2 \cup N_2$, so $\quad E_2 \subset E_1 \cup N_1$

$$\mu(E_1) \leq \mu(E_2 \cup N_2) \leq \mu(E_2) + \cancelto{0}{\mu(N_2)} = \mu(E_2) \leq \mu(E_1).$$

⓪ $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$

① If $(E_i \cup F_i)$ are disjoint, so are $(E_i)$, $(F_i)$. If $F_i \subset N_i$ $\mu$-null, then $\bigsqcup F_i \subset \bigsqcup N_i$, $\mu$-null. Hence

$$\bar{\mu}\left(\bigsqcup E_i \cup F_i\right) = \bar{\mu}\left(\left(\bigsqcup E_i\right) \cup \left(\bigsqcup F_i\right)\right) = \mu\left(\bigsqcup E_i\right) = \sum \mu(E_i) = \sum \bar{\mu}(E_i \cup F_i)$$

**Outer Measures:** $\mu^*: P(X) \to [0,\infty]$ s.t.

(0) $\mu^*(\phi) = 0$

(1) $E \subset F \implies \mu^*(E) \leq \mu^*(F)$

(2) $\mu^*(\bigcup E_n) \leq \sum \mu^*(E_n)$ $\forall$ seq. $(E_n)$

**Prop:** Suppose $\mathcal{E} \subset P(X)$ and $\rho: \mathcal{E} \to [0,\infty]$ s.t.

- $\phi \in \mathcal{E}$ and $\rho(\phi) = 0$, and
- $\forall E \subset X$, $\exists (E_n) \subset \mathcal{E}$ s.t. $E \subset \bigcup E_n$ [satisfied if $X \in \mathcal{E}$]

Define $\mu^*(E) := \inf \{ \sum_1^\infty \rho(E_n) \mid E_n \in \mathcal{E}$ and $\underline{E \subset \bigcup E_n}\}$

Then $\mu^*$ is an outer measure.

↳ observe such $(E_i)$ exists!

**Pf:** (0) Take $E_i = \phi$ $\forall i$.

(1) Observe whenever $F \subset \bigcup F_i$ w/ $F_i \in \mathcal{E}$, then $E \subset F \subset \bigcup F_i$. Hence inf for $E$ is $\leq$ inf for $F$.

(2) we'll use the following:

Trick: $\sum_1^\infty \frac{\varepsilon}{2^n} = \varepsilon$

Trick: $r \leq s \iff \forall \varepsilon > 0,\ r \leq s + \varepsilon$

Suppose $(E_i)_i$ is a seq. of sets. Let $\varepsilon > 0$. For each $n$, $\exists (F_j^n)_j$ s.t. $E_n \subset \bigcup_j F_j^n$ and $\sum_j \rho(F_j^n) \leq \mu^*(E_n) + \frac{\varepsilon}{2^n}$.

↑ $\exists$ by def of inf.

Then $\bigcup E_n \subset \bigcup_n \bigcup_j F_j^n$, so

$$\mu^*(\bigcup E_n) \leq \sum_n \sum_j \rho(F_j^n) \leq \sum_n \mu^*(E_n) + \frac{\varepsilon}{2^n} = \sum \mu^*(E_n) + \overbrace{\boxed{\sum \frac{\varepsilon}{2^n}}}^{\varepsilon}$$

Since $\varepsilon$ was arbitrary, $\mu^*(\bigcup E_n) \leq \sum \mu^*(E_n)$.

We can recover a measure from an outer measure by passing to the collection $\mathcal{M}^*$ of $\mu^*$-measurable sets:

**Def:** $\mathcal{M}^* := \{ E \subset X \mid \mu^*(E \cap F) + \mu^*(E^c \cap F) = \mu^*(F)\ \forall F \subset X \}$.

**Remarks:** ① Obviously $\mu^*(F) \le \mu^*(E\cap F) + \mu^*(E^c\cap F)$. So
$E \in \mathcal{M}^* \iff \mu^*(F) \ge \mu^*(E\cap F) + \mu^*(E^c \cap F)\ \forall F$.

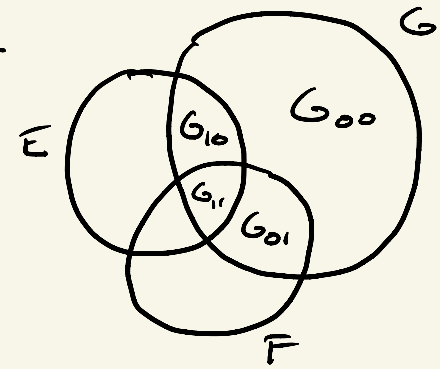
② All $\mu^*$-null sets are in $\mathcal{M}^*$.

Pf: $\forall F$, $\mu^*(\underbrace{F\cap N}_{\subset N}) + \mu^*(F\setminus N) = \mu^*(F\setminus N) \le \mu^*(F)$.

$\subset N$, $\mu^*(N) = 0$.

**Lemma:** Suppose $G \subset X$ and $E, F \in \mathcal{M}^*$

Define
$$G_{00} = G\setminus(E\cup F)$$
$$G_{10} = G\cap(E\setminus F)$$
$$G_{01} = G\cap(F\setminus E)$$
$$G_{11} = G\cap E\cap F.$$




Then $$\boxed{\mu^*(G) = \mu^*(G_{00}) + \mu^*(G_{10}) + \mu^*(G_{01}) + \mu^*(G_{11})} \quad (*)$$

Pf: Since $E\in\mathcal{M}^*$, $\mu^*(G) = \mu^*(G\cap E) + \mu^*(G\setminus E)$
$= \mu^*(G_{11}\cup G_{10}) + \mu^*(G_{01}\cup G_{00})$

Since $F \in \mathcal{M}^*$, $\mu^*(G_{11}\cup G_{10}) = \mu^*(\underbrace{(G_{11}\cup G_{10})\cap F}_{G_{11}}) + \mu^*(\underbrace{(G_{11}\cup G_{10})\setminus F}_{G_{10}})$
$= \mu^*(G_{11}) + \mu^*(G_{10})$.

Similarly, $\mu^*(G_{01}\cup G_{00}) = \mu^*(G_{01}) + \mu^*(G_{00})$.

The result follows.

**Thm (Carathéodory):** $\mathcal{M}^*$ is a $\sigma$-alg, and $\mu^*|_{\mathcal{M}^*}$ is a measure.

Pf: Step 1: $\mathcal{M}^*$ is an algebra

⓪ Clearly $\emptyset \in \mathcal{M}^*$ as $\mu^*(F) = \mu^*(F\cap X) + \mu^*(F\cap\emptyset)$.

① If $E, F\in\mathcal{M}^*$, then $\forall G\subset X$, $(*)$ holds above.

$\mu^*((E\cup F)\cap G) = \mu^*(G_{10}\cup G_{01}\cup G_{11}) = \mu^*(G_{10}) + \mu^*(G_{01}) + \mu^*(G_{11})$

(*) applied to $G_{10}\cup G_{01}\cup G_{11}$.

$\mu^*((E\cup F)^c\cap G) = \mu^*(G_{00})$

$\Rightarrow$ sum is $\mu^*(G)$ by $(*)$

② observe defining characteristic of $\mathcal{M}^*$ is preserved under taking complements.

**Step 2:** $\mathcal{M}^*$ is a $\sigma$-alg.

Suppose $(E_n)$ seq. of disjoint sets in $\mathcal{M}^*$, and $E := \coprod E_n$.

By Step 1, $\forall N \in \mathbb{N}$, $\coprod^N E_n \in \mathcal{M}^*$. Let $F \subset X$ and define

$G := F \cap \coprod^N E_n$. Then since $E_N \in \mathcal{M}^*$,

$$\mu^*(F \cap \coprod^N E_n) = \mu^*(G) = \mu^*(\underline{E_N^c \cap G}) + \mu^*(\underline{E_N \cap G})$$
$$= \mu^*(\underbrace{F \cap \coprod^{N-1} E_n}) + \mu^*(\underbrace{F \cap E_N})$$

By iterating as $E_n \in \mathcal{M}^*$ $\forall n$, we have

$$\mu^*(F \cap \coprod^N E_n) = \sum^N \mu^*(F \cap E_n) \quad \forall N \in \mathbb{N}.$$

Then $\forall N \in \mathbb{N}$,

$$\mu^*(F) = \mu^*(F \cap \coprod^N E_n) + \mu^*(\overbrace{F \setminus \coprod^N E_n}^{F \cap (\coprod^N E_n)^c})$$
$$= \sum^N \mu^*(F \cap E_n) + \mu^*(\underbrace{F \setminus \coprod^N E_n}_{\supset F \setminus E \text{ as } \coprod^N E_n \subset E})$$
$$\geq \sum^N \mu^*(F \cap E_n) + \mu^*(F \setminus E)$$

Taking limits in $[0, \infty]$ as $N \to \infty$, we have

$$(**) \left[ \begin{aligned} \mu^*(F) &\geq \sum^\infty \mu^*(F \cap E_n) + \mu^*(F \setminus E) \\ &\geq \mu^*(\coprod^\infty F \cap E_n) + \mu^*(F \setminus E) \\ &= \mu^*(F \cap E) + \mu^*(F \setminus E). \end{aligned} \right.$$

Hence $E \in \mathcal{M}^*$.

**Step 3:** $\mu^*|_{\mathcal{M}^*}$ is a measure.

Suppose $(E_n)$ are as in Step 2. Take $F = E$ in $(**)$

to see $\mu^*(E) \geq \sum \mu^*(E_n) \geq \mu^*(E)$, so equality holds.

We'll now use this thm to extend premeasures on algebras to measures on $\sigma$-alg's.

**Def:** Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra. A fct $\mu_0 : \mathcal{A} \longrightarrow [0,\infty]$ is called a <u>premeasure</u> if

⓪ $\mu_0(\emptyset) = 0$

① $\forall$ seq. of disjoint sets $(E_n) \subset \mathcal{A}$ s.t. $\coprod E_n \in \mathcal{A}$, we have $\mu_0(\coprod E_n) = \sum \mu_0(E_n)$.

We define finite, $\sigma$-finite, and semifinite for premeasures analogous to the def's for measures.

## Properties of premeasures:

⓪ (finite additivity) If $E_1, \ldots, E_n \in \mathcal{A}$ are disjoint, $\mu_0(\coprod^n E_i) = \sum^n \mu_0(E_i)$

**Pf:** Take $E_k = \emptyset$ $\forall k > n$. Then $\coprod E_i = \coprod^n E_i \in \mathcal{A}$, so
$\mu_0(\coprod^n E_i) = \mu_0(\coprod E_i) = \sum \mu_0(E_i) = \sum^n \mu_0(E_i)$.

① (monotonicity) If $E, F \in \mathcal{A}$ with $E \subset F$, then $\mu_0(E) \leq \mu_0(F)$.

**Pf:** $F = E \amalg (F \setminus E)$, both in $\mathcal{A}$. Apply finite additivity.

② (countable subadditivity) If $(E_n) \subset \mathcal{A}$ s.t. $\cup E_n \in \mathcal{A}$, then
$$\mu_0(\cup E_n) \leq \sum \mu_0(E_n).$$

**Pf:** Set $F_1 := E_1$ and $F_n := E_n \setminus \bigcup^{n-1} E_i$. Then $F_n \in \mathcal{A}$ $\forall n$, and $\coprod F_n = \cup E_n \in \mathcal{A}$, so $\mu_0(\cup E_n) = \mu_0(\coprod F_n) = \sum \mu_0(F_n)$
$\leq \sum \mu_0(E_n)$ by ①.

②' Suppose $E \in \mathcal{A}$ and $(E_n) \subset \mathcal{A}$ s.t. $E \subset \cup E_n$. Then $\mu_0(E) \leq \sum \mu_0(E_n)$.

**Pf:** Let $F_1 := E \cap E_1$ and $F_n := E \cap [E_n \setminus \bigcup^{n-1} E_i]$. Then $\coprod F_n = E \in \mathcal{A}$,
so $\mu_0(E) = \mu_0(\coprod F_n) = \sum \mu_0(F_n) \leq \sum \mu_0(E_n)$ by ①.

**Remark:** If $\mu_0$ merely satisfies finite additivity, we still have monotonicity! [finite additivity $\Rightarrow$ monotonicity]

Starting w/ a <u>premeasure</u> $\mu_0$ on $\mathcal{A}$, get an <u>outer measure</u> $\mu^*$ on $P(X)$ by $\mu^*(E) := \inf\{\sum \mu_0(E_n) \mid E \subset \bigcup E_n \text{ and } E_n \in \mathcal{A}\ \forall n\}$.

<u>Lemma:</u> $\mu^*|_{\mathcal{A}} = \mu_0$

<u>Pf:</u> Suppose $E \in \mathcal{A}$.

<u>$\mu^* \leq \mu_0$:</u> Setting $E_1 := E$ and $E_n := \emptyset$ $\forall n > 1$,

$$\mu^*(E) \leq \sum \mu_0(E_n) = \mu_0(E).$$

<u>$\mu_0 \leq \mu^*$:</u> Let $\varepsilon > 0$. By defn of $\mu^*$, $\exists$ seq. $(E_n) \subset \mathcal{A}$ s.t.

$E \subset \bigcup E_n$ and $\mu_0(E) \leq \sum \mu_0(E_n) \leq \mu^*(E) + \varepsilon$.

Since $\varepsilon$ was arbitrary, $\mu_0 \leq \mu^*$.

Can look at the $\mu^*$-measurable sets $\mathcal{M}^*$

<u>Lemma:</u> $\mathcal{A} \subset \mathcal{M}^*$

<u>Pf:</u> Suppose $E \in \mathcal{A}$ and $F \subset X$ and $\varepsilon > 0$. Pick $(F_n) \subset \mathcal{A}$ s.t. $F \subset \bigcup F_n$ and $\sum \mu_0(F_n) \leq \mu^*(F) + \varepsilon$. Since $\mu_0$ is additive on $\mathcal{A}$,

$$\begin{aligned}\mu^*(F) + \varepsilon &\geq \sum \mu_0(F_n) = \sum \mu_0(F_n \cap E) + \mu_0(F_n \cap E^c)\\ &= \sum \mu_0(F_n \cap E) + \sum \mu_0(F_n \cap E^c)\\ &\geq \mu^*(F \cap E) + \mu^*(F \cap E^c)\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$, and $E \in \mathcal{M}^*$.

Get a measure $\mu := \mu^*|_{\mathcal{M}^*}$ on $\sigma$-alg $\mathcal{M}^* \supset \mathcal{A}$. So if $\mathcal{M} := \mathcal{M}(\mathcal{A})$, $\mathcal{M} \subset \mathcal{M}^*$, $\mu|_{\mathcal{M}}$ is a measure, $\mathcal{A} \subset \mathcal{M}$, and $\mu|_{\mathcal{A}} = \mu_0$.

<u>Thm:</u> If $\nu$ is a measure on $\mathcal{M} = \mathcal{M}(\mathcal{A})$ s.t. $\nu|_{\mathcal{A}} = \mu_0$, then $\nu(E) \leq \mu(E)$ $\forall E \in \mathcal{M}$, with equality when $\mu(E) < \infty$.

Pf: Suppose $E \in \mathcal{M}$. Then $\forall (E_n) \subset \mathcal{A}$ s.t. $E \subset \bigcup E_n$,

$$\nu(E) \le \sum \nu(E_n) = \sum \mu_0(E_n).$$

Hence $\nu(E) \le \inf\{\sum \mu_0(E_n) \mid E \subset \bigcup E_n\} = \mu^*(E) = \mu(E)$.

If $\mu(E) < \infty$, can choose $(E_n) \subset \mathcal{A}$ s.t. $E \subset \bigcup E_n$ and

$$\mu\left(\bigcup^{\infty} E_n\right) \le \sum \mu_0(E_n) \le \mu(E) + \varepsilon.$$

Since $E \subset \bigcup E_n$, we have

① $\mu\left(\left(\bigcup^{\infty} E_n\right) \setminus E\right) \le \varepsilon$.

By cts from below for $\mu$ and $\nu$,

② $\mu\left(\bigcup^{\infty} E_n\right) = \lim_N \underline{\mu\left(\bigcup^{N} E_n\right)} = \lim_N \underline{\nu\left(\bigcup^{N} E_n\right)} = \nu\left(\bigcup^{\infty} E_n\right)$.

Putting ① + ② together, we get: $\boxed{\text{both} = \mu_0\left(\bigcup^{N} E_n\right)}$

$$\left.\begin{aligned} \mu(E) &\le \mu\left(\bigcup E_n\right) \overset{②}{=} \nu\left(\bigcup E_n\right) = \nu(E) + \nu\left(\left(\bigcup E_n\right) \setminus E\right) \\ &\underset{(\nu \le \mu)}{\le} \nu(E) + \mu\left(\left(\bigcup E_n\right) \setminus E\right) \underset{①}{\le} \nu(E) + \varepsilon \end{aligned}\right] \mu(E) \le \nu(E) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\mu(E) \le \nu(E)$, so $\mu(E) = \nu(E)$.

Cor: If $\boxed{\mu_0 \text{ is } \sigma\text{-finite}}$, $\mu$ is ! extension of $\mu_0$ to $\mathcal{M}$.

↳ for a premeasure on $\mathcal{A}$, means $X = \bigcup E_n$ w/ $E_n \in \mathcal{A}$ and $\mu_0(E_n) < \infty$ $\forall n$. WLOG, can take $(E_n)$ disjoint!

Pf: For any other $\nu$ extending $\mu_0$ and $E \subset \mathcal{M}$,

$$\nu(E) = \nu\left(\coprod E \cap E_n\right) = \sum \nu(E \cap E_n) = \sum \underbrace{\mu(E \cap E_n)}_{< \infty} = \mu\left(\coprod E \cap E_n\right) = \mu(E).$$

## Construction of Lebesgue-Stieltjes measures on $\mathbb{R}$

Def: Let $\mathcal{H} := \{\emptyset\} \cup \{(a,b] \mid -\infty \le a < b < \infty\} \cup \{(a,\infty) \mid -\infty \le a < \infty\}$ ← called h-intervals

Let $\mathcal{A}$ be the set of finite disjoint unions of elts of $\mathcal{H}$.

By HW, $\mathcal{A}$ is an algebra, and $\mathcal{M}(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$.

Let $F:\mathbb{R}\to\mathbb{R}$ be nondecreasing ($s\le t \Rightarrow F(s)\le F(t)$) and right cts ($a_n\searrow a \Rightarrow F(a_n)\to F(a)$). Extend $F$ to a fct $[-\infty,\infty]\to[-\infty,\infty]$ by $F(-\infty):=\lim_{a\to-\infty}F(a)$ and $F(b):=\lim_{b\to\infty}F(b)$.

Define $\mu_0:\mathcal{H}\to[0,\infty]$ by

- $\mu_0(\emptyset):=0$
- $\mu_0((a,b]):=F(b)-F(a)$
- $\mu_0((a,\infty)):=F(\infty)-F(a)$

Goal: Extend $\mu_0$ to $\mathcal{A}=\mathcal{A}(\mathcal{H})$.

Step 1: If $(a,b]=\coprod_{j=1}^n (a_j,b_j]$, then $\mu_0((a,b])=\sum^n\mu_0((a_j,b_j])$.

Here, $-\infty\le a<b<\infty$.

Pf: After re-indexing, can assume $a=a_1<b_1=a_2<b_2=a_3<\cdots<b_n$.

Then $\mu_0((a,b])=F(b)-F(a)=\sum_{j=1}^n F(b_j)-F(a_j)=\sum_{j=1}^n\mu_0((a_j,b_j])$.

Step 2: If $(a,\infty)=(a_0,\infty)\amalg\coprod_{j=1}^n(a_j,b_j]$, then

$$\mu_0((a,\infty))=\mu_0((a_0,\infty))+\sum_{j=1}^n\mu_0((a_j,b_j]).$$

Pf: Similar to Step 1 and omitted.

Step 3: If $E_1,\dots,E_n\in\mathcal{H}$ are disjoint and $F\in\mathcal{H}$ s.t. $F\subset\coprod_{i=1}^n E_i$, then $\mu_0(F)=\sum_{i=1}^n\mu_0(F\cap E_i)$

Pf: Removing elts of $(E_i)_{i=1}^n$ if necessary, we may assume that $F\cap E_i\neq\emptyset$ $\forall i=1,\dots,n$, so $F\cap E_i\in\mathcal{H}$ and $F=\coprod_{i=1}^n F\cap E_i$. The result now follows by Steps 1+2.

Step 4: If $(E_i)_{i=1}^m$ and $(F_j)_{j=1}^n$ are two sets of disjoint h-intervals s.t. $\coprod^m E_i=\coprod^n F_j$, then $\sum^m\mu_0(E_i)=\sum^n\mu_0(F_j)$

Hence $\mu_0$ extends to a well-defined fct.

$\mathcal{A}=\mathcal{A}(\mathcal{H})\longrightarrow[0,\infty]$ by $\mu_0(\coprod^n E_i):=\sum^n\mu_0(E_i)$.

Pf: By Step 3, $\sum_{i=1}^m\mu_0(E_i)=\sum_{i=1}^m\sum_{j=1}^n\mu_0(E_i\cap F_j)=\sum_{j=1}^n\sum_{i=1}^m\mu_0(E_i\cap F_j)=\sum_{j=1}^n\mu_0(F_j)$.

use for $E_i$ use for $F_j$

**Lemma:** $\mu_0$ is finitely additive $\quad E = \coprod_{i=1}^{n} E_i \Rightarrow \mu_0(E) = \sum_{i=1}^{n} \mu_0(E_i)$

$\Rightarrow$ monotone $\quad E \subseteq F \Rightarrow \mu_0(E) \leq \mu_0(F)$

$\Rightarrow$ finitely subadditive $\quad E \subset \bigcup_{i=1}^{n} E_i \Rightarrow \mu_0(E) \leq \sum_{i=1}^{n} \mu_0(E_i)$

**Pf:** Suppose $E \in \mathcal{A}$ and $E = \coprod_{i=1}^{n} E_i$ w/ $E_1, \dots, E_n \in \mathcal{A}$. Then each

① $E_i = \coprod_{j=1}^{n_i} E_j^i$ w/ $E_j^i \in \mathcal{H}$ $\forall j = 1, \dots, n_i$, and ② $E = \coprod_{i=1}^{n} \coprod_{j=1}^{n_i} E_j^i$.

By two instances of Step 4, we have

$$\mu_0(E) \underset{②}{=} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \mu_0(E_j^i) \underset{①}{=} \sum_{i=1}^{n} \mu_0(E_i).$$

**Thm:** $\mu_0$ is a premeasure on $\mathcal{A}$.

**Pf:** ⓪ Clear $\mu_0(\emptyset) = 0$ by definition.

① Suppose $(E_n) \subset \mathcal{A}$ is a disjoint seq. s.t. $\coprod E_n \in \mathcal{A}$. Then $\exists$ disjoint h-intervals $F_1, \dots, F_k \in \mathcal{H}$ s.t. $\coprod E_n = \coprod_{j=1}^{k} F_j$.

may assume $E_n \cap F_j \neq \emptyset$ for at most one $j$

Partition the $(E_n)$ into $(E_n^j)$ s.t. $\coprod_n E_n^j = F_j$ $\forall j = 1, \dots, k$.

We'll show $\boxed{\mu_0(F_j) = \sum \mu_0(E_n^j) \quad \forall j = 1, \dots, k.}$ Then by Step 4,

$$\mu_0(\coprod E_n) = \mu_0(\coprod_{j=1}^{k} F_j) := \sum_{j=1}^{k} \mu_0(F_j) = \sum_{j=1}^{k} \sum \mu_0(E_n^j) = \sum \mu_0(E_n).$$

**Case 1a:** Suppose $(a,b] = \coprod_{j=1}^{\infty} (a_j, b_j]$. ($-\infty < a < b < \infty$) $\forall n \in \mathbb{N}$, $\coprod_{j=1}^{n} (a_j, b_j] \subset (a,b]$,

By Step 4 and monotonicity,

$$\sum^{n} \mu_0((a_j, b_j]) = \mu_0(\coprod^{n} (a_j, b_j]) \leq \mu_0((a,b]).$$

taking $n \to \infty$, $\sum^{\infty} \mu_0((a_j, b_j]) \leq \mu_0((a,b])$.

To show the reverse inequality, let $\varepsilon > 0$. Since $F$ is right cts,

- $\exists \delta > 0$ s.t. $F(a+\delta) - F(a) < \frac{\varepsilon}{2}$
- $\forall j$, $\exists \delta_j > 0$ s.t. $F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^{j+1}}$.

Observe $\{(a_j, b_j+\delta_j)\}_{j=1}^{\infty}$ is an open cover of $[a+\delta, b]$. ← cpt!

By compactness, $\exists$ finite subcover, so $\exists N>0$ s.t.

$$[a+\delta, b] \subset \bigcup_{j=1}^{N} (a_j, b_j+\delta_j).$$

Then
$$\begin{aligned}
\mu_0((a,b]) &= F(b)-F(a) \\
&< F(b) - F(a+\delta) + \tfrac{\varepsilon}{2} && \Big] \ F(a+\delta)-F(a) < \tfrac{\varepsilon}{2} \\
&= \mu_0((a+\delta, b]) + \tfrac{\varepsilon}{2} \\
&\le \mu_0\Big(\bigcup_{j=1}^{N} (a_j, b_j+\delta_j]\Big) + \tfrac{\varepsilon}{2} && \Big] \ (a+\delta, b] \subset \bigcup_{j=1}^{N} (a_j, b_j+\delta_j] \\
&\le \sum_{j=1}^{N} \mu_0((a_j, b_j+\delta_j]) + \tfrac{\varepsilon}{2} && \Big] \ E \subset \bigcup^{N} E_i \Rightarrow \mu_0(E) \le \sum^{N} \mu_0(E_i) \\
&= \sum_{j=1}^{N} [F(b_j+\delta_j) - F(a_j)] + \tfrac{\varepsilon}{2} \\
&\le \sum_{j=1}^{N} \Big[F(b_j) + \tfrac{\varepsilon}{2^{j+1}} - F(a_j)\Big] + \tfrac{\varepsilon}{2} && \Big] \ F(b_j+\delta_j) - F(b_j) < \tfrac{\varepsilon}{2^{j+1}} \\
&= \sum_{j=1}^{N} F(b_j) - F(a_j) + \sum_{j=1}^{N} \tfrac{\varepsilon}{2^{j+1}} + \tfrac{\varepsilon}{2} \\
&\le \sum_{j=1}^{N} \mu_0((a_j, b_j]) + \varepsilon \\
&\le \sum^{\infty} \mu_0((a_j, b_j]) + \varepsilon
\end{aligned}$$

<u>Case 1b:</u> $(-\infty, b] = \coprod (a_j, b_j]$ w/ $-\infty \le a_j < b_j < \infty$ and $b < \infty$

<u>Case 1c:</u> $(a, \infty) = \coprod (a_j, b_j]$ or $(a_0, \infty) \sqcup \coprod (a_j, b_j]$ ] <u>HW!</u>

$\uparrow_{-\infty \le a}$

Now $(\mathbb{R}, \mathcal{A}, \mu_0) \rightsquigarrow (\mathbb{R}, \mathcal{P}(\mathbb{R}), \underbrace{\mu_F^*}_{\text{outer measure}}) \rightsquigarrow (\mathbb{R}, \underbrace{\mathcal{M}_F := \mathcal{M}^*}_{\mu^*\text{-measurable sets}}, \mu_F := \mu_F^*|_{\mathcal{M}^*})$

$\mathcal{H} \subset \mathcal{A} \subset \mathcal{M}_F \Rightarrow \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}_F$

<u>Def:</u> Call $\mu_F|_{\mathcal{B}_{\mathbb{R}}}$ the <u>Lebesgue-Stieltjes measure</u> associated to $F$.

**Remark:** Since $\mu_F$ is $\sigma$-finite, it follows from HW2 that $\mathcal{M}_F = \overline{\mathcal{B}_{\mathbb{R}}}$ for $\mu_F$, i.e., sets in $\mathcal{M}_F$ are unions of Borel sets and subsets of Borel sets which are $\mu_F$-null.

**Def:** Lebesgue measure is $\lambda := \mu_{id}$ where $id: \mathbb{R} \to \mathbb{R}$ is $x \mapsto x$.

$\mathcal{L} := \mathcal{M}^* = \overline{\mathcal{B}_{\mathbb{R}}}$ for $\lambda|_{\mathcal{B}_{\mathbb{R}}}$

**Translation + dilation properties of $\lambda$:**

**Def:** For $E \subset \mathbb{R}$, $r, s \in \mathbb{R}$, define $rE := \{rx \mid x \in E\}$ and $E+s := \{x+s \mid x \in E\}$.

**Thm:** Suppose $E \in \mathcal{L}$.

① If $r \in \mathbb{R}$, then $rE \in \mathcal{L}$ and $\lambda(rE) = |r| \cdot \lambda(E)$.

② If $s \in \mathbb{R}$, then $E+s \in \mathcal{L}$ and $\lambda(E+s) = \lambda(E)$.

**Pf:** We'll prove ① and ② is similar.

**Step 1:** $|r| \cdot \lambda$ is a measure on $\mathcal{L}$.

Pf: Exercise.

**Step 2:** Observe that $\mathcal{B}_{\mathbb{R}}$ is closed under $E \mapsto rE$. Hence $\lambda^r(E) := \lambda(rE)$ defines a measure on $\mathcal{B}_{\mathbb{R}}$ s.t. $\lambda^r = r \cdot \lambda|_{\mathcal{B}_{\mathbb{R}}}$.

Pf: That $\lambda^r$ is a measure on $\mathcal{B}_{\mathbb{R}}$ is an exercise left to the reader. By HW3, if $E \in \mathcal{A}$, then $\lambda^r(E) = |r| \cdot \lambda(E)$, so $\lambda^r = |r|\lambda$ on $\mathcal{A}$ and thus all of $\mathcal{B}_{\mathbb{R}}$ by the uniqueness corollary as $\lambda^r$ and $|r|\lambda$ are both $\sigma$-finite.

**Step 3:** If $E \in \mathcal{L}$ is $\lambda$-null, then $rE \in \mathcal{L}$ is $\lambda$-null.

Pf: Recall $E \in \mathcal{L}$ is $\lambda$-null $\iff$ $\exists N \in \mathcal{B}_{\mathbb{R}}$ s.t. $E \subset N$ and $\lambda(N) = 0$. Now $rE \subset rN$ and $\lambda(rN) \underset{\text{Step 2}}{=} |r|\lambda(N) = 0$, so $rE \in \mathcal{L}$ is $\lambda$-null.

Now as $\mathcal{L} = \overline{\mathcal{B}_{\mathbb{R}}}$ for $\lambda$, we see $\lambda^r$ and $|r| \cdot \lambda$ are both defined on $\mathcal{L}$ and agree. Hence $\lambda^r = |r| \cdot \lambda$ on $\mathcal{L}$.

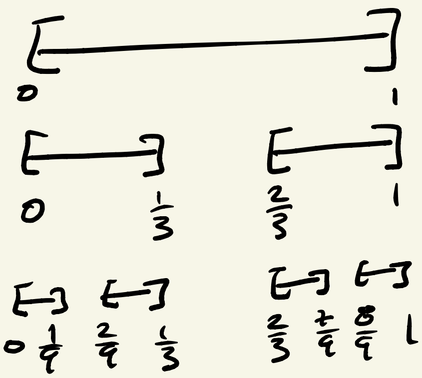
By HW3, $\lambda(\{x\}) = 0 \quad \forall x \in \mathbb{R}$. Hence if $E \subset \mathbb{R}$ is countable, then $E = \coprod_{n=1}^{\infty} \{x_n\}$, so $\lambda(E) = \sum \lambda(\{x_n\}) = 0$.

**Def:** The Cantor set $C$ is defined as $\bigcap C_n$ where we define $C_n$ inductively by "removing middle thirds":

$C_0 :=$

$C_1 :=$

$C_2 :=$




etc.

Then by continuity from above,

$$\lambda(C) = \lim_{n \to \infty} \lambda(C_n).$$

By HW 3,

$$\lambda(C_0) = 1$$

$$\lambda(C_1) = 1 - \frac{1}{3}$$

$$\lambda(C_2) = 1 - \frac{1}{3} - \frac{2}{9}$$

$$\lambda(C_3) = 1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} \quad \text{etc.}$$

$$\Rightarrow \quad \lambda(C) = 1 - \sum_{j=1}^{\infty} \frac{2^{j-1}}{3^j} = 1 - \frac{1}{3} \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j$$

$$= 1 - \frac{1}{3}\left[\frac{1}{1-\frac{2}{3}}\right] = 0.$$

It is well known that $C$ is uncountable. Indeed it is in bijection w/ $\{0,1\}^{\mathbb{N}}$ via base-3 decimal expansion.

## Regularity properties of Lebesgue–Stieltjes measures:

For $F:\mathbb{R}\to\mathbb{R}$ increasing and right cts, have $(\mathcal{M}_F, \mu_F)$ s.t.

$$\mu_F(E) = \inf\Big\{ \sum_{j=1}^{\infty} \underbrace{F(b_j)-F(a_j)}_{\mu_F((a_j,b_j])} \;\Big|\; E\subset \bigcup_{j=1}^{\infty}(a_j,b_j]\Big\}.$$

**Lemma:** $\forall E\in\mathcal{M}_F$, $\mu_F(E) = \inf\{\sum^{\infty}\mu_F((a_j,b_j)) \mid E\subset \bigcup^{\infty}(a_j,b_j)\}$.

**Pf:** Denote the inf on the RHS by $\nu(E)$.

**Step 1:** $\mu_F(E)\le \nu(E)$.

Suppose $E\subset \bigcup(a_j,b_j)$. Can write each $(a_j,b_j) = \coprod_{i=1}^{\infty}(a_i^j, b_i^j]$.

Then $E\subset \bigcup_j \coprod_i (a_i^j,b_i^j]$, and $\mu_F(E)\le \sum_{j,i}\mu_F((a_i^j,b_i^j]) = \sum_j \mu_F((a_j,b_j))$.

Hence $\mu_F(E)\le \nu(E)$.

**Step 2:** $\nu(E)\le\mu_F(E)$.

Let $\varepsilon>0$. $\exists ((a_j,b_j])_{j=1}^{\infty}$ s.t. $E\subset\bigcup(a_j,b_j]$ and

$$\sum^{\infty}\mu_F((a_j,b_j]) \le \mu_F(E)+\frac{\varepsilon}{2}.$$

For each $j$, pick $\delta_j>0$ s.t. $F(b_j+\delta_j)-F(b_j) < \frac{\varepsilon}{2^{j+1}}$. Then

$E\subset\bigcup(a_j,b_j+\delta_j)$ and

$$\begin{aligned}\sum\mu_F((a_j,b_j+\delta_j)) &= \sum F(b_j+\delta_j)-F(a_j) < \sum F(b_j)-F(a_j)+\frac{\varepsilon}{2^{j+1}}\\ &= \sum\mu_F((a_j,b_j]) + \sum\frac{\varepsilon}{2^{j+1}}\\ &\le \mu_F(E)+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\end{aligned}$$

Hence $\nu(E)\le\mu_F(E)+\varepsilon$. Since $\varepsilon>0$ was arbitrary, $\nu(E)\le\mu_F(E)$.

**Def:** Suppose $(X,\tau)$ is a Hausdorff topological space and $\mathcal{M}\subset P(X)$ is a σ-alg containing $\mathcal{B}(\tau)$ (i.e., $\tau\subset\mathcal{M}$).

A measure $\mu$ on $\mathcal{M}$ is called:

- <u>outer regular</u> if $\mu(E) = \inf\{\mu(U) \mid E\subset U \text{ open}\}$
- <u>inner regular</u> if $\mu(E) = \sup\{\mu(K) \mid \text{compact } K\subset E\}$.
- <u>regular</u> if $\mu$ is both <u>inner</u> + <u>outer</u> regular.

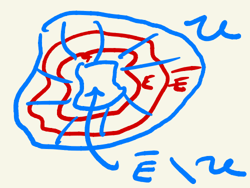
**Thm:** $\mu_F$ on $\mathcal{M}_F$ is regular.

**Step 1:** $\mu_F$ is outer regular

**Pf:** Let $E \in \mathcal{M}_F$. By the lemma, $\forall \varepsilon > 0$, $\exists ((a_j, b_j))_{j=1}^{\infty}$ s.t. $E \subset \bigcup (a_j, b_j)$ and $\sum \mu_F((a_j,b_j)) \le \mu(E) + \varepsilon$. If $\mathcal{U} := \bigcup (a_j, b_j)$, then $\mu_F(\mathcal{U}) \le \sum \mu_F((a_j,b_j)) \le \mu(E) + \varepsilon$. Hence $\forall \varepsilon > 0$, $\exists$ open $\mathcal{U} \supset E$ s.t. $\mu_F(\mathcal{U}) \le \mu_F(E) + \varepsilon$. Since $\mu_F(E) \le \mu_F(\mathcal{U})$, we have $\mu_F(E) = \inf\{\mu_F(\mathcal{U}) \mid E \subset \mathcal{U} \text{ open}\}$.

**Step 2:** $\mu_F$ is inner regular.

**Step 2a:** Suppose $E \in \mathcal{M}_F$ is bdd, so $\bar{E}$ is cpt and $\mu_F(\bar{E}) < \infty$. ($\bar{E} \subset [a,b]$, and $\mu_F(\bar{E}) \le F(b) - F(a) < \infty$.) Let $\varepsilon > 0$.
By Step 1, $\exists$ open $\mathcal{U} \supset (\bar{E} \setminus E)$ s.t. $\mu_F(\mathcal{U}) \le \mu_F(\bar{E} \setminus E) + \varepsilon$.
Then $K := \bar{E} \setminus \mathcal{U}$ is cpt and contained in $E$.
We have




$$
\begin{aligned}
\mu_F(K) &= \mu_F(E) - \mu_F(E \cap K^c) \\
&= \mu_F(E) - \mu_F(E \cap \mathcal{U}) \\
&= \mu_F(E) - [\mu_F(\mathcal{U}) - \mu_F(\mathcal{U} \setminus E)] \\
&\ge \mu_F(E) \underbrace{- \mu_F(\mathcal{U}) + \mu_F(\bar{E} \setminus E)}_{\ge -\varepsilon} \qquad [\mathcal{U} \setminus E \supset \bar{E} \setminus E] \\
&\ge \mu_F(E) - \varepsilon.
\end{aligned}
$$

**Step 2b:** If $E \in \mathcal{M}_F$ unbdd, $E = \coprod E_j$ where $E_j := E \cap (j, j+1]$.
Let $\varepsilon > 0$. By Step 2a, $\forall j$, $\exists$ cpt $K_j \subset E_j$ s.t. $\mu_F(K_j) \ge \mu_F(E_j) - \frac{\varepsilon}{2^{|j|+1}}$.
For $n \in \mathbb{N}$, let $F_n := \coprod_{-n}^{n} K_j$, cpt. $\forall n$, $\mu(F_n) \ge \mu(\bigcup_{-n}^{n} E_j) - \frac{\varepsilon}{2}$.
If $\mu(E) = \infty$, since $\mu(\bigcup_{-n}^{n} E_j) \uparrow \mu(E)$, eventually $\mu(F_n) > M$, so $\sup\{\mu(F_n) \mid n \in \mathbb{N}\} = \infty = \mu(E)$. Otherwise, $\mu(E) < \infty$, and $\exists$ $N \in \mathbb{N}$ s.t. $\mu(E) \le \mu(\bigcup_{-N}^{N} E_j) + \frac{\varepsilon}{2} \le \mu(F_N) + \underbrace{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}}_{\varepsilon}$. Hence $\forall \varepsilon > 0$, $\exists$ cpt $F \subset E$ s.t. $\mu(F) \le \mu(E) \le \mu(F) + \varepsilon$. Hence $\mu_F$ is inner regular.

## Hausdorff Measure:

Let $(X, \rho)$ be a metric space. For $A, B \subset X$, $A \neq \emptyset \neq B$, define

$$\rho(a, B) := \inf\{\rho(a,b) \mid b \in B\} \quad [a \in A]$$

$$\rho(A, B) := \inf\{\rho(a,b) \mid a \in A, b \in B\}$$

<u>Def:</u> An outer measure $\mu^*$ on $P(X)$ is called a (Carathéodory) <u>metric outer measure</u> if

- $\rho(A,B) > 0$ $[\Rightarrow A \cap B = \emptyset] \Longrightarrow \mu^*(A \amalg B) = \mu^*(A) + \mu^*(B)$.

<u>Prop:</u> If $\mu^*$ is a metric outer measure on $P(X)$, then

$\mathcal{B}_\rho$ (Borel $\sigma$-alg) $\subset \mathcal{M}^*$ ($\mu^*$-measurable sets)

<u>Pf:</u> Since $\mathcal{B}_\rho$ is generated by the open sets, it suffices to show all open sets are in $\mathcal{M}^*$. Let $\mathcal{U} \subset X$ be open.

<u>Step 1:</u> We may assume $\rho(\mathcal{U}, X \setminus \mathcal{U}) = 0$.

Otherwise, $\forall F \subset X$, $\rho(F \cap \mathcal{U}, F \setminus \mathcal{U}) > 0$, so $\mu^*(F) = \mu^*(F \cap \mathcal{U}) + \mu^*(F \setminus \mathcal{U})$, and $\mathcal{U} \in \mathcal{M}^*$.

<u>Step 2:</u> For $n \in \mathbb{N}$, define $A_n := \{x \in \mathcal{U} \mid \rho(x, X \setminus \mathcal{U}) > \frac{1}{n}\}$. Then $(A_n)$ is increasing and $\bigcup A_n = \mathcal{U}$. Setting $A_0 = \emptyset$, define $B_n := A_n \setminus A_{n-1}$ $\forall n \in \mathbb{N}$. Then $\coprod B_n = \mathcal{U}$, and $B_n \neq \emptyset$ <u>frequently</u>.

Observe $B_n = \emptyset$ $\forall n > k \iff A_k = \mathcal{U} \Rightarrow \rho(\mathcal{U}, X \setminus \mathcal{U}) \geq \frac{1}{k}$.

<u>Step 3:</u> If $|m-n| > 1$ and $B_m \neq \emptyset \neq B_n$, then $\rho(B_m, B_n) > 0$.

<u>Pf:</u> Suppose $1 \leq m < n-1$. Let $x \in B_m$ and $y \in B_n$. Then $y \notin A_{n-1} \supseteq A_{m+1}$ so $\exists z \in X \setminus \mathcal{U}$ s.t. $\rho(y, z) \leq \frac{1}{m+1}$. But $x \in B_m$, so $\rho(x, z) > \frac{1}{m}$.

By the $\Delta$ ineq., $\rho(x,y) \geq \rho(x,z) - \rho(y,z) > \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}$.

Hence $\rho(B_m, B_n) \geq \frac{1}{m(m+1)} > 0$.

**Step 4:** Let $F \subset X$. If $\mu^*(F) = \infty$, then

$$\mu^*(F) \geq \mu^*(F \cap \mathcal{U}) + \mu^*(F \setminus \mathcal{U}).$$

Assume $\mu^*(F) < \infty$. Then $\sum_{n=k}^{\infty} \mu^*(F \cap B_n) \xrightarrow{k \to \infty} 0$.

**Pf:** By Step 3, $\forall k \in \mathbb{N}$,

$$\sum^{k} \mu^*(F \cap B_{2n}) = \mu^*\left(\coprod^{k} F \cap B_{2n}\right) \leq \mu^*(F) \quad \text{and}$$

$$\sum^{k} \mu^*(F \cap B_{2n+1}) = \mu^*\left(\coprod^{k} F \cap B_{2n+1}\right) \leq \mu^*(F)$$

Taking $k \to \infty$, we have $\sum \mu^*(F \cap B_n) \leq 2\mu^*(F) < \infty$.
Hence the tail must $\to 0$.

**Step 5:** We now calculate $\forall n \in \mathbb{N}$:

$$\mu^*(F \cap \mathcal{U}) + \mu^*(F \setminus \mathcal{U}) \leq \underbrace{\mu^*(F \cap A_n)} + \mu^*(F \cap (\mathcal{U} \setminus A_n)) + \underbrace{\mu^*(F \setminus \mathcal{U})}$$

Can combine these two since
$\rho(F \cap A_n, F \setminus \mathcal{U}) \geq \rho(A_n, X \setminus \mathcal{U}) \geq \frac{1}{n}$.

$$= \mu^*(F \cap [A_n \cup F \setminus \mathcal{U}]) + \mu^*(F \cap (\underbrace{\mathcal{U} \setminus A_n}_{\coprod_{n+1}^{\infty} B_k}))$$

$$\leq \mu^*(F) + \underbrace{\sum_{n+1}^{\infty} \mu^*(F \cap B_k)}_{\to 0 \text{ as } n \to \infty \text{ by Step 4.}}$$

**Def:** Suppose $(X, \rho)$ a metric space, $p \geq 0$, and $\delta > 0$.
For $E \subset X$, define

$$\eta^*_{p,\varepsilon}(E) := \inf \left\{ \sum_1^{\infty} [\operatorname{diam} B_n]^p \;\middle|\; \begin{array}{l} (B_n) \text{ seq. of open balls,} \\ \operatorname{diam}(B_n) \leq \varepsilon \ \forall n, \text{ and} \\ E \subset \bigcup B_n \end{array} \right\}$$

$\operatorname{diam}(S) := \sup\{\rho(x,y) \mid x, y \in S\}$

Convention: $\inf \emptyset := \infty$.

Observe $\eta^*_{p,\varepsilon}$ is the outer measure induced by

$$\delta : \{\emptyset\} \cup \left\{ \begin{array}{c} \text{open balls w/} \\ \operatorname{diam} \leq \varepsilon \end{array} \right\} \longrightarrow [0, \infty]$$

$$B \longmapsto [\operatorname{diam} B]^p$$

$$\emptyset \longmapsto 0$$

Observe that if $\varepsilon < \varepsilon'$, then $\eta^*_{p,\varepsilon}(E) \geq \eta^*_{p,\varepsilon'}(E)$ as as we are taking an infimum over a <u>smaller</u> set. [Every $\varepsilon$-cover is an $\varepsilon'$-cover]. Hence

$$\eta^*_p(E) := \lim_{\varepsilon \to 0} \eta^*_{p,\varepsilon}(E)$$

is well-defined.

<u>Lemma:</u> If $(\mu_i^*)_{i \in I}$ is a family of outer measures on $X$, then

$$\mu^*(E) := \sup_{i \in I} \mu_i^*(E) \text{ is an outer measure.}$$

<u>Pf:</u> Exercise.

<u>Prop:</u> $\eta^*_p$ is a metric outer measure

<u>Pf:</u> Since $\eta^*_p = \sup_{\varepsilon > 0} \eta^*_{p,\varepsilon}$, $\eta^*_p$ is an outer measure by the lemma.

Suppose $\rho(E,F) > \varepsilon > 0$, and choose an $\varepsilon$-covering $(B_n)$ of $E \amalg F$. Then $\forall n$, $B_n$ intersects <u>at most one</u> of $E, F$. So we may partition $(B_n)$ into $(B_n^E)$ and $(B_n^F)$ s.t.

- $E \subset \bigcup B_n^E$ and $B_n^E \cap F = \emptyset$
- $F \subset \bigcup B_n^F$ and $B_n^F \cap E = \emptyset$

Thus
$$\eta^*_{p,\varepsilon}(E) + \eta^*_{p,\varepsilon}(F) \leq \sum \operatorname{diam}(B_n^E)^p + \operatorname{diam}(B_n^F)^p$$
$$= \sum \operatorname{diam}(B_n)^p$$

for any $\varepsilon$-covering. Hence $\forall \varepsilon < \rho(E,F)$,

$$\eta^*_{p,\varepsilon}(E) + \eta^*_{p,\varepsilon}(F) \leq \eta^*_{p,\varepsilon}(E \amalg F)$$

Taking $\varepsilon \to 0$, we get

$$\eta^*_p(E \amalg F) \underset{\eta^*_p \text{ outer measure}}{\leq} \eta^*_p(E) + \eta^*_p(F) \underset{\text{limit } \varepsilon \to 0}{\leq} \eta^*_p(E \amalg F).$$

**Def:** Since the Borel σ-alg $B_X \subset \mathcal{M}^*$ for $\eta^*_p$, get a Borel measure $\eta_p := \eta^*_p|_{B_X}$ called <u>p-dimensional Hausdorff measure</u>.

**Properties:**

① $\forall$ isometry $f: X \to X$, $\eta_p(E) = \eta_p(f(E))$ $\forall E \in B_X$.

$\underbrace{\qquad}_{\text{cts s.t. } \rho(f(x),f(y)) = \rho(x,y)\ \forall x,y \in X}$

Pf: $\forall \varepsilon > 0$, $\eta^*_{p,\varepsilon}(E) = \eta^*_{p,\varepsilon}(f(E))$ as $E \subset \bigcup B_n \iff f(E) \subset \bigcup f(B_n)$.

② $\eta_1$ on $\mathbb{R}$ is $\lambda|_{B_{\mathbb{R}}}$.

Pf: Follows by uniqueness of $\lambda|_{B_{\mathbb{R}}}$ from <u>HW4</u>.

③ If $\eta_p(E) < \infty$, then $\eta_q(E) = 0$ $\forall$ $q > p$.

Pf: Let $\varepsilon > 0$. $\exists (B_n)$ s.t. $E \subset \bigcup B_n$, $\text{diam}(B_n) \le \varepsilon$, and $\sum \text{diam}(B_n)^p \le \eta_p(E) + 1$. But if $q > p$,

$$\sum \text{diam}(B_n)^q = \sum \underbrace{\text{diam}(B_n)^{q-p}}_{\le \varepsilon^{q-p}} \text{diam}(B_n)^p$$

$$\le \varepsilon^{q-p} \sum \text{diam}(B_n)^p \le \varepsilon^{q-p}[\eta_p(E) + 1].$$

Thus $\forall \varepsilon > 0$, $\eta^*_{q,\varepsilon}(E) \le \varepsilon^{q-p}[\eta_p(E) + 1]$.

Letting $\varepsilon \to 0$, $\eta_q(E) = \eta^*_q(E) \le \lim_{\varepsilon \to 0} \varepsilon^{q-p}[\eta_p(E) + 1] = 0$.

④ If $\eta_p(E) > 0$, then $\eta_q(E) = \infty$ $\forall q < p$.

Pf: Contrapositive of ③.

**Def:** If $E \subset B_X$, its <u>Hausdorff dimension</u> is

$$\inf\{p \ge 0 \mid \eta_p(E) = 0\} = \sup\{p \ge 0 \mid \eta_p(E) = \infty\}.$$