**Product topology:** Given topological spaces $(X, \tau)$ and $(Y, \theta)$, the product topology $\tau \times \theta$ on $X \times Y$ is generated by $\{U \times V \mid U \in \tau \text{ and } V \in \theta\}$. [take unions]

**Exercise:** Prove $\tau \times \theta$ is the weakest topology on $X \times Y$ s.t. the projection maps $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are cts.

**Product Spaces:** Given measurable spaces $(X, \mathcal{M})$, $(Y, \mathcal{N})$, the product $\sigma$-alg $\mathcal{M} \times \mathcal{N}$ is generated by $\{E \times F \mid E \in \mathcal{M}, F \in \mathcal{N}\}$.

**Exercise:** Show $\mathcal{M} \times \mathcal{N}$ is smallest $\sigma$-alg s.t. the canonical projection maps are measurable. Deduce $\mathcal{M} \times \mathcal{N}$ is generated by:

- $\{\pi_X^{-1}(E), \pi_Y^{-1}(F) \mid E \in \mathcal{M}, F \in \mathcal{N}\}$ or
- $\pi^{-1}(\mathcal{E}_X) \cup \pi^{-1}(\mathcal{E}_Y)$ where $\mathcal{M} = \mathcal{M}(\mathcal{E}_X)$ and $\mathcal{N} = \mathcal{M}(\mathcal{E}_Y)$.

$\mathcal{E}_X$ generates $\mathcal{M}$
$\mathcal{E}_Y$ generates $\mathcal{N}$

**Prop:** Suppose $(X, \rho_X)$ and $(Y, \rho_Y)$ are metric spaces.

① $B_X \times B_Y$ is generated by $\tau_X \times Y \cup X \times \tau_Y$

② $B_X \times B_Y \subset B_{X \times Y}$

③ If $X, Y$ are separable [$\exists$ countable dense subsets], $B_X \times B_Y = B_{X \times Y}$.

**Pf:** ① follows immediately from the exercise.

② Since $\tau_X \times Y$, $X \times \tau_Y \subset \tau_X \times \tau_Y$, $B_X \times B_Y \subseteq B_{X \times Y}$.

③ Suppose $C \subseteq X$ and $D \subseteq Y$ are countable dense subsets. Let $\mathcal{E}_X, \mathcal{E}_Y$ be the collection of open balls centered at $C, D$ respectively w/ rational radius. Then every open set in $X, Y$ is a countable union of elts of $\mathcal{E}_X, \mathcal{E}_Y$ respectively. Also, $C \times D$ is a countable dense subset of $X \times Y$, and thus $\tau_X \times \tau_Y$ is generated by $\mathcal{E}_X \times \mathcal{E}_Y \subset B_X \times B_Y$. Hence $B_{X \times Y} \subseteq B_X \times B_Y$.

## Exercise:

① Find an example of (non-separable) metric spaces $X, Y$ s.t. $\mathcal{B}_X \times \mathcal{B}_Y \subsetneq \mathcal{B}_{X\times Y}$.

② If one of $X$ or $Y$ is separable, is $\mathcal{B}_X \times \mathcal{B}_Y = \mathcal{B}_{X\times Y}$? Give a proof or a counterexample.

**Recall:** For sets $X, Y$, the product $X\times Y$ satisfies the (categorical) universal property which characterizes it up to canonical bijection:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "\exists!"] & \\ X & X\times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

$\forall$ set $Z$ and fcts $f: Z\to X$, $g: Z \to Y$, $\exists!$ fct $f\times g: Z \longrightarrow X\times Y$ s.t. $f = \pi_X\circ(f\times g)$ and $g = \pi_Y\circ(f\times g)$.

**Exercise:** If $X, Y, Z$ are topological [respectively measurable] spaces above, then the following are equivalent:

① $f\times g$ is cts [respectively measurable]

② $f$ and $g$ are cts [respectively measurable]

**Proposition:** The following fcts are continuous and thus measurable.

① $+: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ [can also replace $\mathbb{R}$ w/ $[0,\infty]$ and $\mathbb{C}$]

② $\cdot: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ [can also replace $\mathbb{R}$ w/ $\overline{\mathbb{R}}$ and $\mathbb{C}$]

**Pf:** Exercise!

**Cor:** If $f: (X,\mathcal{M}) \to \mathbb{R}$ and $g: (Y,\mathcal{N}) \to \mathbb{R}$ are measurable, so are $f+g$ and $fg$. [Can also use other codomains]

**Pf:** $f+g$, $fg$ are composites:

$$\begin{tikzcd} & X \arrow[r, "f"] & \mathbb{R} & \\ X\times Y \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"'] \arrow[rr, dashed, "\text{``}f\times g\text{''}"] & & \mathbb{R}\times\mathbb{R} \arrow[u] \arrow[d] \arrow[r, "+"] \arrow[dr, "\cdot"'] & \mathbb{R} \\ & Y \arrow[r, "g"'] & \mathbb{R} & \mathbb{R} \end{tikzcd}$$

This gives another pf that $f:(X,\mathcal{M})\to\mathbb{C}$ is cts/measurable $\iff$ $\mathrm{Re}(f)$ and $\mathrm{Im}(f) : (X,\mathcal{M})\to\mathbb{R}$ are cts/measurable.

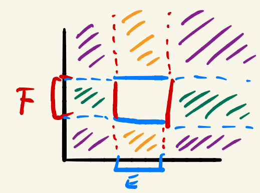
Pf: Observe $f: X\to\mathbb{C}$ is the ! map s.t. the following diagram commutes:

$$\begin{tikzcd}
 & \mathbb{R} \arrow[r, "\mathrm{id}"] & \mathbb{R} \\
X \arrow[ur, "\mathrm{Re}(f)"] \arrow[rr, dashed, "f"] \arrow[dr, "\mathrm{Im}(f)"'] & & \mathbb{C} \arrow[u, "\mathrm{Re}"'] \arrow[d, "\mathrm{Im}"] \\
 & \mathbb{R} \arrow[r, "\text{mult by } i"'] & i\mathbb{R}
\end{tikzcd}$$

<u>Product measures</u>: Fix measure spaces $(X,\mathcal{M},\mu)$, $(Y,\mathcal{N},\nu)$.

<u>Def</u>: A <u>measurable rectangle</u> is a set $E\times F\subseteq X\times Y$ s.t. $E\in\mathcal{M}$ and $F\in\mathcal{N}$. Observe:

- $(E_1\times F_1)\cap(E_2\times F_2) = (E_1\cap E_2)\times(F_1\cap F_2)$
- $(E\times F)^c = (E\times F^c)\sqcup(E^c\times F)\sqcup(E^c\times F^c)$




$\Rightarrow$ $\mathcal{A} := \{$finite disjoint unions of measurable rectangles$\}\subset\mathcal{M}\times\mathcal{N}$ is an <u>algebra</u> which generates $\mathcal{M}\times\mathcal{N}$.

<u>Prop</u>: For $G=\coprod_{k=1}^{n} E_k\times F_k\in\mathcal{A}$, define

$$(\mu\times\nu)_0(G) := \sum_{k=1}^{n}\mu(E_k)\nu(F_k)$$

w/ convention that $0\cdot\infty=0$.

Then $(\mu\times\nu)_0$ is a <u>premeasure</u> on $\mathcal{A}\subset\mathcal{M}\times\mathcal{N}$.

Pf: It suffices to show that if $E\times F=\coprod E_j\times F_j$, then $\mu(E)\nu(F)=\sum\mu(E_j)\nu(F_j)$.

Trick: $\forall x\in E, y\in F$, $\exists!\, j$ s.t. $(x,y)\in E_j\times F_j$. Thus $E=\coprod_{j \text{ s.t. } y\in F_j} E_j$.

This is a <u>disjoint</u> union since we cannot have both $x\in E_j\cap E_k$ and $y\in F_j\cap F_k$, else $(x,y)\in(E_j\times F_j)\cap(E_k\times F_k)=\emptyset$.

Now for $y\in F$,

$$\mu(E) = \sum_{j \text{ s.t. } y\in F_j}\mu(E_j) = \sum\mu(E_j)\chi_{F_j}(y).$$

Then $\mu(E)\chi_F(y) = \sum \mu(E_j)\chi_{F_j}(y)$, and integrating yields

$$\int_Y \mu(E)\chi_F\,d\nu = \int_Y \sum \mu(E_j)\chi_{F_j}\,d\nu \underset{MCT}{=} \sum \int_Y \mu(E_j)\chi_{F_j}\,d\nu$$

$$= \sum \mu(E_j)\int_Y \chi_{F_j}\,d\nu = \sum_j \mu(E_j)\nu(F_j).$$

We now use the outer measure construction to get an outer measure $(\mu\times\nu)^*$ on $\mathcal{P}(X\times Y)$, which restricts to a measure $\mu\times\nu$ on the $(\mu\times\nu)^*$-measurable sets, which is a σ-alg containing $\mathcal{M}\times\mathcal{N}$ (which was gen. by $\mathcal{A}$).

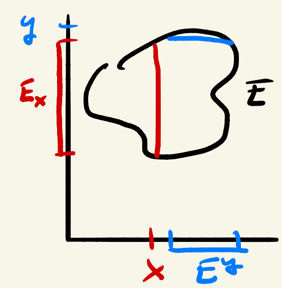
<u>Exercise</u>: Consider the canonical projs $\pi_X, \pi_Y$ on $X\times Y$.

1. If $X, Y$ topological spaces, prove $\pi_X, \pi_Y$ are open maps.
2. If $(X,\mathcal{M}), (Y,\mathcal{N})$ measurable, do $\pi_X, \pi_Y$ map sets in $\mathcal{M}\times\mathcal{N}$ to sets in $\mathcal{M}, \mathcal{N}$ respectively? Give a proof or counterexample.

<u>Cross Sections</u>: Fix meas. spaces $(X,\mathcal{M},\mu)$, $(Y,\mathcal{N},\nu)$.

For $E\subset X\times Y$, define

- <u>x-section</u>: $E_x := \{y\in Y \mid (x,y)\in E\}$ $\leftarrow \pi_Y[E\cap(\{x\}\times Y)]$
- <u>y-section</u>: $E^y := \{x\in X \mid (x,y)\in E\}$ $\leftarrow \pi_X[E\cap(X\times\{y\})]$




<u>Exercise</u>: Suppose $(E_n)\subset \mathcal{M}\times\mathcal{N}$. Then

1. $[\bigcup E_i]_x = \bigcup (E_i)_x$
2. $[\bigcap E_i]_x = \bigcap (E_i)_x$
3. $(E_i\setminus E_j)_x = (E_i)_x \setminus (E_j)_x$
4. $\chi_{E_i}(x,y) = \chi_{(E_i)_x}(y)$

Similar statements hold for y-sections.

**Proposition:** Let $E \in \mathcal{M}\times\mathcal{N}$. $\forall x \in X$, $E_x \in \mathcal{N}$ and $\forall y \in Y$, $E^y \in \mathcal{M}$. Equivalently, $\pi_X(\cdot \cap (\{x\}\times Y))$ and $\pi_Y(\cdot \cap (X \times \{y\}))$ map measurable sets to measurable sets.

**Pf:** We claim $\mathcal{S} := \{E \subset X\times Y \mid E_x \in \mathcal{N}\}$ is a σ-alg containing the measurable rectangles in $\mathcal{M}\times\mathcal{N}$, which generates $\mathcal{M}\times\mathcal{N}$.

0. $\emptyset \in \mathcal{N} \Rightarrow \emptyset \in \mathcal{S}$.
1. If $(E_n) \in \mathcal{S}$ so $(E_n)_x \in \mathcal{N}$ then, observe $(\bigcup E_n)_x = \bigcup (E_n)_x \in \mathcal{N}$. Hence $\bigcup E_n \in \mathcal{S}$.
2. If $E \in \mathcal{S}$ so $E_x \in \mathcal{N}$, observe $(E^c)_x = (E_x)^c \in \mathcal{N} \Rightarrow E^c \in \mathcal{S}$.

- Now if $E\times F$ is a measurable rectangle, $(E\times F)_x = \begin{cases} F & x\in E \\ \emptyset & x \notin E \end{cases} \in \mathcal{N}$.
- Similarly, $\mathcal{T} := \{E \subset X\times Y \mid E^y \in \mathcal{M}\}$ is a σ-alg containing $\mathcal{M}\times\mathcal{N}$.

**Exercise:** Use the proposition to prove $\mathcal{L}\times\mathcal{L}$ is not equal to $\mathcal{L}^2 := (\lambda\times\lambda)^*$-measurable sets in $\mathbb{R}^2$.

**Def:** For $f: X\times Y \longrightarrow \mathbb{R}, \overline{\mathbb{R}}$, or $\mathbb{C}$, define the

- x-section: $f_x : Y \longrightarrow \mathbb{R}, \overline{\mathbb{R}}$, or $\mathbb{C}$ by $f_x(y) := f(x,y)$
- y-section: $f^y : X \longrightarrow \mathbb{R}, \overline{\mathbb{R}}$, or $\mathbb{C}$ by $f^y(x) := f(x,y)$

**Cor:** If $f$ is $\mathcal{M}\times\mathcal{N}$-measurable, then
- $\forall x \in X$, $f_x$ is $\mathcal{N}$-measurable, and
- $\forall y \in Y$, $f^y$ is $\mathcal{M}$-measurable.

**Pf:** We'll prove the first, and the second is similar. Observe $\forall x \in X$ and $G \subset$ codomain measurable,
$(f_x)^{-1}(G) = f^{-1}(G)_x \in \mathcal{N}$.

**Thm:** Suppose $(X,\mathcal{M},\mu)$, $(Y,\mathcal{N},\nu)$ are $\sigma$-finite. Then $\forall E \in \mathcal{M}\times\mathcal{N}$,

① The fcts $x \longmapsto \nu(E_x)$ and $y \longmapsto \mu(E^y)$ are measurable, and

② $(\mu\times\nu)(E) = \int \nu(E_x)\,d\mu(x) = \int \mu(E^y)\,d\nu(y)$.

**Proof:** First, we'll assume $\mu, \nu$ are finite measures.

Let $\Lambda \subseteq \mathcal{M}\times\mathcal{N}$ be the subset for which ①+② hold.

**Step 1:** $\Pi := \{\text{measurable rectangles}\} \subseteq \Lambda$.

Pf: Clear.

**Step 2:** $\Pi$ is a $\pi$-system.

Pf: the intersection of 2 measurable rectangles is a meas. rect.

**Step 3:** $\Lambda$ is a $\lambda$-system. Thus by the $\pi$-$\lambda$ Theorem,

$\mathcal{M}\times\mathcal{N} = \lambda(\Pi) \subseteq \Lambda \subseteq \mathcal{M}\times\mathcal{N}$, so equality holds.

Pf: ⓪ $X\times Y \in \Pi \subset \Lambda$.

① If $E \in \Lambda$ so that ①+② hold, then

$x \longmapsto \nu((E^c)_x) = \nu((E_x)^c) = \nu(Y) - \nu(E_x)$ is measurable,

as is $y \longmapsto \mu((E^c)^y)$. Moreover,

$$\begin{aligned}(\mu\times\nu)(E^c) &= (\mu\times\nu)(X\times Y) - (\mu\times\nu)(E)\\ &= \int \nu(Y)\,d\mu(x) - \int \nu(E_x)\,d\mu(x)\\ &= \int [\nu(Y)-\nu(E_x)]\,d\mu(x)\\ &= \int \nu((E_x)^c)\,d\mu(x)\\ &= \int \nu((E^c)_x)\,d\mu(x)\\ &= \int \mu((E^c)^y)\,d\nu(y) \quad \text{similarly.}\end{aligned}$$

Thus $\Lambda$ is closed under taking complements.

② Suppose $(E_k) \subset \Lambda$ is a seq. of disjoint subsets of $X \times Y$.

Then $\forall k$, $x \longmapsto \nu((E_k)_x)$ is measurable, and so are

$x \longmapsto \sum \nu((E_k)_x) = \nu(\amalg (E_k)_x) = \nu((\amalg E_k)_x)$ and

$y \longmapsto \mu((\amalg E_k)^y)$. We calculate

$$(\mu \times \nu)(\amalg E_k) = \sum (\mu \times \nu)(E_k) = \sum \int \nu((E_k)_x)\, d\mu(x)$$

$$\underset{(MCT)}{=} \int \sum \nu((E_k)_x)\, d\mu(x) = \int \nu((\amalg E_k)_x)\, d\mu(x)$$

$$= \int \mu((\amalg E_k)^y)\, d\nu(y) \text{ similarly.}$$

Step 4: when $\mu, \nu$ $\sigma$-finite, write $X \times Y$ as increasing union $X \times Y = \bigcup X_n \times Y_n$ w/ $X_n \times Y_n$ measurable rectangles s.t. $\mu(X_n), \nu(Y_n) < \infty$ $\forall n$. For $E \in \mathcal{M} \times \mathcal{N}$, write $E_n = E \cap (X_n \times Y_n)$.

$$(\mu \times \nu)(E) = \lim (\mu \times \nu)(E_n) = \lim \int \nu((E_n)_x)\, d\mu(x) \underset{MCT}{=} \int \lim \underline{\nu((E_n)_x)}\, d\mu(x)$$

$$= \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y) \text{ similarly.}$$

$(E_n)_x \nearrow E_x$. cts from below.

Thm (Tonelli): Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ $\sigma$-finite. For $f \in L^+(X \times Y, \mathcal{M} \times \mathcal{N})$,

① $x \longmapsto \int_Y f_x\, d\nu$ is $\mathcal{M}$-measurable (in $L^+(X, \mathcal{M})$)

② $y \longmapsto \int_X f^y\, d\mu$ is $\mathcal{N}$-measurable (in $L^+(Y, \mathcal{N})$)

③ $\int_{X \times Y} f\, d(\mu \times \nu) = \int_X \left[ \int_Y f_x\, d\nu \right] d\mu = \int_Y \left[ \int_X f^y\, d\mu \right] d\nu.$

Remark: If $f \in L^+(X \times Y, \mathcal{M} \times \mathcal{N}) \cap L^1(\mu \times \nu)$, then

- $\int_Y f_x\, d\nu < \infty$ $[f_x \in L^1(\nu)]$ a.e. $x \in X$ and
- $\int_X f^y\, d\mu < \infty$ $[f^y \in L^1(\mu)]$ a.e. $y \in Y$.

Pf: If $f = \chi_E$ for some $E \in \mathcal{M} \times \mathcal{N}$, this is the previous theorem. Since $(cf+g)_x = cf_x + g_x$ [exercise], we get the result for simple fcts by linearity. Suppose $(\varphi_n) \subset SF$ s.t. $\varphi_n \nearrow f$ everywhere. Then $(\varphi_n)_x \nearrow f_x$ and $(\varphi_n)^y \nearrow f^y$, so by MCT, $\int_Y (\varphi_n)_x \, d\nu \nearrow \int_Y f_x \, d\nu$ and $\int_X (\varphi_n)^y \, d\mu \nearrow \int_X f^y \, d\mu$

which implies ① + ② [countable sups of meas. fcts are meas.].

Again by MCT,

$$\int_X \left[ \int_Y f_x \, d\nu \right] d\mu = \int_X \lim \left[ \int_Y (\varphi_n)_x \, d\nu \right] d\mu \underset{MCT}{=} \lim \int_X \left[ \int_Y (\varphi_n)_x \, d\nu \right] d\mu$$

$$\underset{\text{last thm}}{=} \lim \int_{X \times Y} \varphi_n \, d(\mu \times \nu) = \int_{X \times Y} f \, d(\mu \times \nu) \underset{\text{similarly}}{=} \int_Y \left[ \int_X f^y \, d\mu \right] d\nu$$

Cor (Fubini): If $f \in L^1(\mu \times \nu)$, then

→ Exercise: Show Fubini also holds replacing $\mathcal{M} \times \mathcal{N}, \mu \times \nu$ w/ completion $\overline{\mathcal{M} \times \mathcal{N}}, \overline{\mu \times \nu}$.

① $f_x \in L^1(\nu)$ a.e. $x \in X$

② $f^y \in L^1(\mu)$ a.e. $y \in Y$

③ $\left[ x \longmapsto \int_Y f_x \, d\nu \right] \in L^1(\mu)$

④ $\left[ y \longmapsto \int_X f^y \, d\mu \right] \in L^1(\nu)$

⑤ $$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[ \int_Y f_x \, d\nu \right] d\mu = \int_Y \left[ \int_X f^y \, d\mu \right] d\nu.$$

**Pf:** $f = \mathrm{Re}(f)_+ - \mathrm{Re}(f)_- + i\,\mathrm{Im}(f)_+ - i\,\mathrm{Im}(f)_-$ where $\mathrm{Re}(f)_\pm$ and $\mathrm{Im}(f)_\pm \in L^+ \cap L^1$. Hence Tonelli's Thm applies to these 4 fcts, as does the Remark after. The result follows.

Exercise: Show $\sigma$-finiteness is necessary for both $X, Y$.

Application: Convolution on $\mathbb{R}$

Exercise: Suppose $f, g \in \mathcal{L}^1(\mathbb{R}, \lambda)$.

① $[y \longmapsto f(x-y)g(y)] \in \mathcal{L}^1(\mathbb{R})$ a.e. $x \in \mathbb{R}$

② the convolution $f * g$ of $f, g$ given by

$$(f*g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\,d\lambda(y)$$

is in $\mathcal{L}^1(\mathbb{R}, \lambda)$.

③ $\mathcal{L}^1(\mathbb{R},\lambda)$ is a commutative $\mathbb{C}$-algebra under $\cdot, +, *$.

④ $\int |f*g| \leq \int |f| \int |g|$ [$\|\cdot\|_1$ is submultiplicative]

$\mathcal{L}^1(\mathbb{R},\lambda)$ is a Banach algebra.

## n-dim'l Lebesgue Integral:

Def: $(\mathbb{R}^n, \mathcal{L}^n, \lambda^n)$ is the completion of $(\mathbb{R}^n, \underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}, \underbrace{\lambda\times\cdots\times\lambda})$ { n factors }

Properties:

① $\lambda^n$ is $\sigma$-finite

② $\lambda^n$ is regular (inner + outer regular)

③ $\forall E \in \mathcal{L}^n$, $\forall \varepsilon > 0$, $\exists R_1, \dots, R_k$ disjoint rectangles whose sides (proj's) are intervals s.t. $\lambda^n(E \,\Delta\, \bigcup_{j=1}^{k} R_j) < \varepsilon$

$\Delta$ is symmetric difference

4. $SF \cap \mathcal{L}^1(\lambda^n)$ is dense in $\mathcal{L}^1(\lambda^n)$

5. $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^1(\lambda^n)$.

6. Suppose $E \in \mathcal{L}^n$.
   - $\forall r \in \mathbb{R}^n$, $r+E \in \mathcal{L}^n$ and $\lambda^n(r+E) = \lambda^n(E)$.
   - $\forall T \in GL(n, \mathbb{R})$, $T(E) \in \mathcal{L}^n$ and $\lambda^n(E) = |\det T|\, \lambda^n(E)$.

7. $\forall$ $\mathcal{L}^n$-measurable $f: \mathbb{R}^n \to \mathbb{C}$, the following fcts are also $\mathcal{L}^n$-measurable:
   - $x \longmapsto f(x+r)$ for $r \in \mathbb{R}^n$, and
   - $x \longmapsto f(Tx)$ for $T \in GL(n, \mathbb{R})$.

   If moreover $f \in L^+$ or $\mathcal{L}^1$, then

$$\int f(x+r)\, d\lambda^n = \int f(x)\, d\lambda^n \quad \text{and}$$

$$\int f(Tx)\, d\lambda^n = |\det T| \int f(x)\, d\lambda^n.$$

## Differentiation:

Note: Following notes of Sarason, we'll treat $L^1(\lambda^n)$ case and then explain how to extend to $L^1_{loc}(\lambda^n) := \{ f : \mathbb{R}^n \to \mathbb{C} \text{ int'bg on bdd sets} \}$.

**Def:** A cube in $\mathbb{R}^n$ is a set $Q$ of the form $Q = \prod_{j=1}^n I_j$ where each $I_j$ is a closed interval of same length, which we denote by $\ell(Q)$.

- For $x \in \mathbb{R}^n$, let $\mathcal{C}(x) := \{ \text{cubes which contain } x \}$.
- For $Q$ a cube and $r > 0$, $rQ$ is the cube w/ same center as $Q$ but w/ $\ell(rQ) = r\ell(Q)$.

**Goal:** Prove the Lebesgue Differentiation Thm: $\forall f \in L^1_{loc}(\lambda^n)$,

$$\lim_{\substack{l(Q)\to 0\\ x\in Q}} \frac{1}{\lambda^n(Q)} \int_Q f \, d\lambda^n = f(x) \quad \text{a.e.}$$

As a corollary, we get when $n=1$:

Fundamental Thm of Calculus: Suppose $f \in \mathcal{L}^1(\lambda)$. Define

$$F(x) := \int_{(-\infty,x)} f \, d\lambda. \quad \text{Then } F'(x) = f(x) \text{ a.e.}$$

Pf: Observe $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{\substack{h\to 0\\ x\in Q:=[x-\frac{h}{2},x+\frac{h}{2}]}} \frac{1}{\lambda(Q)} \int_Q f \, d\lambda.$

$= f(x)$ a.e. (LDT).

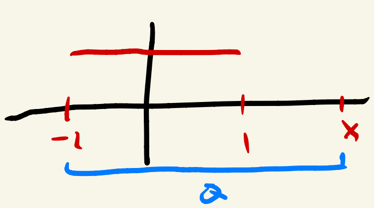
**Def:** For $f \in L^1_{loc} = L^1_{loc}(\lambda^n) = \{f : \mathbb{R}^n \to \mathbb{C} \text{ integ. on bdd sets}\}$, define $Mf : \mathbb{R}^n \to [0,\infty]$ [Hardy-Littlewood maximal fct] by

$$Mf := \sup\left\{ \frac{1}{\lambda^n(Q)} \int_Q |f| \, d\lambda^n \;\middle|\; Q \in \mathcal{C}(x) \right\}.$$

Properties:

- $M(rf) = |r| \cdot Mf \quad \forall r \in \mathbb{R}$.
- $M(f+g) \leq Mf + Mg$
- $Mf > 0$ everywhere unless $f = 0$ a.e.
- $Mf$ is lower semi cts [$\Leftrightarrow \{Mf > a\}$ is open $\forall a \in \mathbb{R}$]
  $\Rightarrow Mf$ is measurable.

Example: $f = \chi_{[-1,1]} : \mathbb{R} \to \mathbb{C}$, $Mf(x) = \begin{cases} 1 & x \in [-1,1] \\ \frac{2}{1+|x|} & x \notin [-1,1] \end{cases} \notin L^1.$

eg: 


$\frac{1}{\lambda(Q)} \int_Q \chi_{[-1,1]} \, d\lambda = \frac{1}{1+x} \cdot 2$

<u>Hardy-Littlewood Maximal Thm</u>: $\exists\, c>0$ only depending on $n$ s.t. $\forall f \in L^1(\lambda^n)$ and $a>0$, $\lambda^n(\{Mf > a\}) \le c \dfrac{\|f\|_1}{a}$.

<u>Remark</u>: This is a generalization of <u>Tchebychev's Inequality</u>:

$$\forall a>0, \int_{\{|f|>a\}} |f|\, d\mu \geq a\mu(\{|f|>a\}) \Rightarrow \mu(\{|f|>a\}) \le \frac{\|f\|_1}{a}$$

we'll use a variation of:

<u>Vitali Covering Lemma</u>: Let $\mathcal{B}$ be a collection of open balls in $\mathbb{R}^n$ and let $\mathcal{U} := \bigcup_{B \in \mathcal{B}} B$. If $c < \lambda^n(\mathcal{U})$, $\exists$ disjoint $B_1, \dots, B_k \in \mathcal{B}$ s.t. $\sum \lambda^n(B_j) > 3^{-n} c$.

<u>Pf</u>: Since $\lambda^n$ is regular, $\exists$ cpt $K \subset \mathcal{U}$ s.t. $c < \lambda^n(K)$ and finitely many of the balls in $\mathcal{B}$ cover $K$, say $A_1, \dots, A_m$. Inductively, define $B_1 =$ largest of the $A_i$ (largest radius) and $B_j =$ largest of $A_i$ <u>disjoint</u> from $B_1, \dots, B_{j-1}$. Since there are finitely many $A_i$, this process terminates giving $B_1, \dots, B_k$.

<u>Trick</u>: If $A_i$ is <u>not</u> one of $B_1, \dots, B_k$, $\exists$ <u>smallest</u> $1 \le j \le k$ s.t. $A_i \cap B_j \neq \emptyset$. Then $\text{rad}(A_i) \le \text{rad}(B_j)$, so $A_i \subset 3B_j$ where $3B_j$ has the same center as $j$ but $3\times$ radius.

Then $K \subset \bigcup_1^k 3B_j$, so $c < \lambda^n(K) \le \sum_1^k \lambda^n(3B_j) = 3^n \sum \lambda^n(B_j)$.

<u>Exercise</u>: Let $E \subset \mathbb{R}^n$ and $\mathcal{C}$ be a collection of cubes covering $E$ s.t. $\sup\{\ell(Q) \mid Q \in \mathcal{C}\} < \infty$. Then $\exists$ a seq. $(Q_n) \subset \mathcal{C}$ of disjoint cubes s.t. $\sum \lambda^n(Q) \geq 5^{-n} (\lambda^n)^*(E)$ (outer measure)

Pf of HLMT: Suppose $f \in L^1(\lambda^n)$ and $a > 0$. Let $E = \{Mf > a\}$.

Set $\mathcal{C} = \{\text{cubes } Q \mid \underbrace{\frac{1}{\lambda^n(Q)} \int_Q |f| \, d\lambda^n > a}\}$. Then $\mathcal{C}$ covers $E$.

$\Rightarrow \ell(Q)^{-n} \|f\|_1 > a \Rightarrow \ell(Q) < \left[\frac{\|f\|_1}{a}\right]^{1/n}$

By the exercise, $\exists$ seq. $(Q_i) \subset \mathcal{C}$ of disjoint cubes s.t.

$\sum \lambda^n(Q_i) \geq 5^{-n} \lambda^n(E)$. Then

$$\lambda^n(E) \leq 5^n \sum \underbrace{\lambda^n(Q_i)}_{\leq \frac{1}{a}\int_{Q_i} |f| \, d\lambda^n} < 5^n \sum \frac{1}{a} \int_{Q_i} |f| \, d\lambda^n \leq \frac{5^n \|f\|_1}{a}.$$

Lebesgue's Differentiation Thm: If $f \in L^1_{loc}(\lambda^n)$, then

$$\lim_{\substack{\ell(Q) \to 0 \\ Q \in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q f \, d\lambda^n = f(x) \quad \text{a.e.} \qquad (LD)$$

Step 1: The result for $f \in L^1 \Rightarrow$ the result for $f \in L^1_{loc}$.

Pf: Suppose $f \in L^1_{loc}$. It suffices to show that for $R > 0$, (LD) holds a.e. for $x \in Q_R(0) := \prod^n [-R, R]$. For $x \in Q_R(0)$, and $Q \in \mathcal{C}(x)$ w/ $\ell(Q) \leq 1$, the value of $\frac{1}{\lambda^n(Q)} \int_Q |f| \, d\lambda^n$ only depends on $f(y)$ for $y \in Q_{R+1}(0)$. So we can replace $f$ with $f \chi_{Q_{R+1}(0)} \in L^1$.

Step 2: The result for $f \in C_c(\mathbb{R}^n) \Rightarrow$ the result for $f \in L^1$.

Pf: For $Q \in \mathcal{C}(0)$ and $f \in L^1$, define $I_Q f(x) := \frac{1}{\lambda^n(Q)} \int_{Q+x} f \, d\lambda^n$.

Observe $I_Q$ is linear, and $|I_Q f| \leq Mf$ everywhere. Now fix $f \in L^1$ and $\varepsilon > 0$.

Let $E := \left\{ x \in \mathbb{R}^n \;\middle|\; \limsup_{\substack{\ell(Q) \to 0 \\ Q \in \mathcal{C}(0)}} |I_Q f(x) - f(x)| > \varepsilon \right\}$. We'll show $(\lambda^n)^*(E) = 0$.

$\Rightarrow E \in \mathcal{L}^n$, $\lambda^n(E) = 0$.

Fix $\delta>0$. Since $C_c(\mathbb{R}^n)\subset L^1$ is dense [exercise], $\exists$ cts $g\in C_c(\mathbb{R}^n)$ s.t. $\|f-g\|_1<\delta$. Then:

$$
\begin{aligned}
|I_Q f - f| &= |I_Q(f-g) + [I_Q g - g] + [g-f]| \\
&\le |I_Q(f-g)| + |I_Q g - g| + |g-f| \\
&\le M(f-g) + \underbrace{|I_Q g - g|}_{\to 0 \text{ as } \substack{\ell(Q)\to 0\\ Q\in\mathcal{C}_0} \text{ by assumption}} + |g-f|
\end{aligned}
$$

Hence $E\subset\{M(f-g)>\frac{\varepsilon}{2}\}\cup\{|g-f|>\frac{\varepsilon}{2}\}$. Now by HLMT and Tchebychev's Inequality,

$$
\begin{aligned}
(\lambda^n)^*(E) &\le \lambda(\{M(f-g)>\tfrac{\varepsilon}{2}\}) + \lambda(\{|g-f|>\tfrac{\varepsilon}{2}\}) \\
&\le \frac{C\|f-g\|_1}{\varepsilon/2} + \frac{\|g-f\|_1}{\varepsilon/2} = \frac{2(C+1)}{\varepsilon}\|f-g\|_1 \\
&< \frac{2(C+1)}{\varepsilon}\delta.
\end{aligned}
$$

But $\delta>0$ was arbitrary, so $(\lambda^n)^*(E)=0$.

Step 3: The result holds for $g\in C_c(\mathbb{R}^n)$.

Pf: Observe $g$ is uniformly cts. Let $\varepsilon>0$. Pick $\delta>0$ s.t. $x,y\in Q$ w/ $\ell(Q)<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$. Then for such $Q$,

$$
\left|\frac{1}{\lambda^n(Q)}\int_Q f(y)\,d\lambda^n(y) - f(x)\right| \le \frac{1}{\lambda^n(Q)}\int_Q |f(x)-f(y)|\,d\lambda^n(y) < \varepsilon.
$$

Since $\varepsilon$ was arbitrary, the result follows.

Def: Suppose $E\in\mathcal{L}^n$. A pt $x\in E$ is called a Lebesgue point of density of $E$ if $\displaystyle\lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{\lambda^n(Q\cap E)}{\lambda^n(Q)} = 1$.

**Cor:** For $E \in \mathcal{L}^n$, almost all points of $E$ are L.P.o.D's.

**Pf:** Apply Lebesgue's Diff. Thm to $\chi_E$.

**Def:** For $f \in L^1(\lambda^n)$, $x \in \mathbb{R}^n$ is a Lebesgue pt of $f$ if

$$\lim_{\substack{\ell(Q) \to 0 \\ Q \in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f - f(x)| \, d\lambda^n = 0.$$

**Cor:** For $f \in L^1_{loc}$, almost all pts of $\mathbb{R}^n$ are Lebesgue pts of $f$.

**Pf:** As in proof of Lebesgue Diff Thm, we may assume $f \in L^1$. Let $D \subset \mathbb{C}$ be a countable dense subset ($\mathbb{Q} + i\mathbb{Q}$ will suffice). For $d \in D$, set

$$E_d := \left\{ x \in \mathbb{R}^n \,\middle|\, \lim_{\substack{\ell(Q) \to 0 \\ Q \in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f - d| - |f(x) - d| \, d\lambda^n = 0 \right\}.$$

By LDT, $E_d^c$ is $\lambda^n$-null. Set $E := \bigcap_{d \in D} E_d$, so $E^c$ is still $\lambda^n$-null. We claim every $x \in E$ is a Lebesgue pt of $f$. Indeed, if $x \in E$, then $\forall d \in D$,

$$|f - f(x)| \le |f - d| + |f(x) - d| = \big[ |f - d| - |f(x) - d| \big] + 2|f(x) - d|.$$

$$\limsup_{\substack{\ell(Q) \to 0 \\ Q \in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f - f(x)| \, d\lambda^n \le \underbrace{2|f(x) - d|}_{d \text{ arbitrary!}} + \underbrace{\limsup_{\substack{\ell(Q) \to 0 \\ Q \in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int |f - d| - |f(x) - d| \, d\lambda^n}_{=0}$$

Since $D \subset \mathbb{C}$ dense, can approximate $f(x)$ by $d \in D$ up to any $\varepsilon > 0$. We conclude $x$ is a Lebesgue pt of $f$.