$K := [a,b] \setminus U$, which is compact, and observe that f is continuous at all points of K (not $f|_K$!). For each $x \in K$, pick $\delta_x > 0$ such that $y \in [a,b]$ (not K!) and $|x-y| < \delta_x$ implies $|f(x) - f(y)| < \varepsilon'$. Then $\{B_{\delta_x/2}(x)\}_{x\in K}$ is an open cover of K, so there are $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n B_{\delta_{x_i}/2}(x_i)$. Set $\delta := \min\{\delta_{x_i}/2 | i = 1, \ldots, n\}$.

Claim. *If $x \in K$ and $y \in [a,b]$ and $|x-y| < \delta/2$, then $|f(x) - f(y)| < 2\varepsilon'$.*

Proof. Without loss of generality, $x \in B_{\delta_1/2}(x_1)$. Then $y \in B_{\delta_1}(x_1)$, and thus

$$|f(x) - f(y)| \le |f(x) - f(x_1)| + |f(x_1) - f(y)| < 2\varepsilon'. \qquad \square$$

Let P be any partition of $[a,b]$ whose intervals have length at most δ. Let P' consist of the intervals that intersect K and let P'' be the intervals that do not intersect K. By the claim, if $J \in P'$, then $M_J - m_j \le 2\varepsilon'$. Thus

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{J\in P}(M_J - m_J)\lambda(J) \\
&= \sum_{J\in P'}(M_J - m_J)\lambda(J) + \sum_{J\in P''}(M_J - m_J)\lambda(J) \\
&\le \sum_{J\in P'} 2\varepsilon'\lambda(J) + \sum_{J\in P''}(M-m)\lambda(J) \\
&\le 2\varepsilon'(b-a) + (M-m)\lambda(U) && \left(\bigcup_{J\in P''} J \subseteq U\right) \\
&< \varepsilon'(2(b-a) + (M-m))
\end{aligned}$$

where $M = \sup_{x\in[a,b]} f(x)$ and $m := \inf_{x\in[a,b]} f(x)$. Taking $\varepsilon' = \varepsilon/(2(b-a) + (M-m))$ works. $\square$

3.8. **Product measures.**

Definition 3.8.1. Given measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a *measurable rectangle* is a set of the form $E \times F \subset X \times Y$ where $E \in \mathcal{M}$ and $F \in \mathcal{N}$. The *product σ-algebra* $\mathcal{M} \times \mathcal{N} \subset P(X \times Y)$ is the σ-algebra generated by the measurable rectangles.

Exercise 3.8.2. Prove that $\mathcal{M} \times \mathcal{N}$ is the smallest σ-algebra such that the canonical projection maps $\pi_X : X \times Y \to Y$ and $\pi_Y : X \times Y \to X$ are measurable. Deduce that $\mathcal{M} \times \mathcal{N}$ is generated by $\pi_X^{-1}(\mathcal{E}_X) \cup \pi_Y^{-1}(\mathcal{E}_Y)$ for any generating sets $\mathcal{E}_X$ of $\mathcal{M}$ and $\mathcal{E}_Y$ of $\mathcal{N}$.

Warning 3.8.3. Recall that given topological spaces X, Y, the canonical projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are open maps. When $(X, \mathcal{M}), (Y, \mathcal{N})$ are measurable, however, π_X, π_Y need not map measurable sets to measurable sets. (Unfortunately, actually constructing a set in $\mathcal{M} \times \mathcal{N}$ whose projection to X is not measurable is quite difficult.)

Exercise 3.8.4. Show that the subset of $P(X \times Y)$ consisting of finite disjoint unions of measurable rectangles is an algebra which generates $\mathcal{M} \times \mathcal{N}$.
Hint: For $E, E_1, E_2 \in \mathcal{M}$ and $F, F_1, F_2 \in \mathcal{N}$,

- $(E_1 \times F_1) \cap (E_2 \times F_2) = (E_1 \cap E_2) \times (F_1 \cap F_2)$, *and*
- $(E \times F)^c = (E \times F^c) \amalg (E^c \times F) \amalg (E^c \times F^c)$.

Proposition 3.8.5. *Suppose (X, d_X) and (Y, d_Y) are metric spaces.*

(1) *$\mathcal{B}_X \times \mathcal{B}_Y$ is generated by $(\mathcal{T}_X \times Y) \cup (X \times \mathcal{T}_Y)$.*
(2) *$\mathcal{B}_X \times \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$.*
(3) *If X, Y are separable, then $\mathcal{B}_X \times \mathcal{B}_Y = \mathcal{B}_{X\times Y}$.*

Proof.
(1) This is an immediate consequence of Exercise 3.8.2.
(2) Since $\mathcal{T}_X \times Y, X \times \mathcal{T}_Y \subset \mathcal{T}_X \times \mathcal{T}_Y$, we have $\mathcal{B}_X \times \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$.
(3) Suppose $C \subset X$ and $D \subset Y$ are countable dense subsets. Let $\mathcal{E}_X, \mathcal{E}_Y$ be the collections of open balls centered at C, D respectively with rational radii. Note that $C \times D$ is a countable dense subset of $X \times Y$, and thus $\mathcal{T}_X \times \mathcal{T}_Y$ is generated by $\mathcal{E}_X \times \mathcal{E}_Y \subset \mathcal{B}_X \times \mathcal{B}_Y$. Hence $\mathcal{B}_{X\times Y} \subset \mathcal{B}_X \times \mathcal{B}_Y$. □

Exercise 3.8.6.

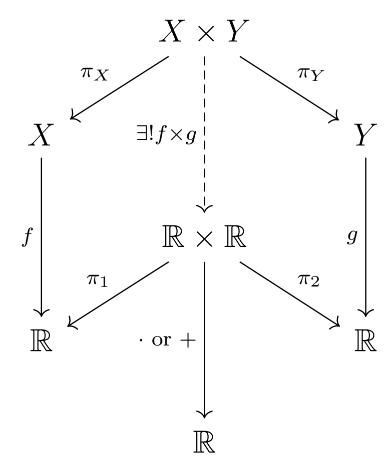
(1) Find an example of (non-separable) metric spaces X, Y such that $\mathcal{B}_X \times \mathcal{B}_Y \subsetneq \mathcal{B}_{X\times Y}$.
(2) If one of X or Y is separable, is $\mathcal{B}_X \times \mathcal{B}_Y = \mathcal{B}_{X\times Y}$? Find a proof or a counterexample.

Exercise 3.8.7. Suppose $(X, \mathcal{M}), (Y, \mathcal{N}), (Z, \mathcal{P})$ are measurable spaces and $f : Z \to X$ and $g : Z \to Y$. Show that $f \times g : Z \to X \times Y$ (the unique map from the universal property of the product) is measurable if and only if f and g are measurable. Deduce that the category of measurable spaces and measurable functions has finite categorical products.

Exercise 3.8.8. Prove that $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and thus (Borel) measurable.

Corollary 3.8.9. *If $f : (X, \mathcal{M}) \to \mathbb{R}$ and $g : (Y, \mathcal{N}) \to \mathbb{R}$ are measurable, then so are $f + g$ and fg. (This also holds for other codomains such as $\mathbb{C}$ and $\overline{\mathbb{R}}$ if the sum is well-defined.)*

Proof. Observe that fg and $f + g$ are composites:



The composite of these measurable functions is $\mathcal{M} \times \mathcal{N}$-measurable. □

Exercise 3.8.10. Adapt the proof of Corollary 3.8.9 to give another proof that $f : (X, \mathcal{M}) \to \mathbb{C}$ is measurable if and only if $\mathrm{Re}(f), \mathrm{Im}(f)$ are measurable.

For the rest of this section, suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces, and let $\mathcal{A}$ be the algebra of finite disjoint unions of measurable rectangles from Exercise 3.8.4.

Proposition 3.8.11. *For $G = \coprod_{k=1}^n E_k \times F_k \in \mathcal{A}$, define*

$$(\mu \times \nu)_0(G) := \sum_{k=1}^n \mu(E_k)\nu(F_k)$$

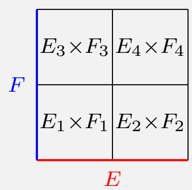
with the convention that $0 \cdot \infty = 0$. Then $(\mu \times \nu)_0$ is a premeasure on $\mathcal{A}$.

Proof. It suffices to show that if $E \in \mathcal{M}$ and $F \in \mathcal{N}$ such that $E \times F = \coprod E_n \times F_n$ for some (non-disjoint!) sequences $(E_n) \subset \mathcal{M}$ and $(F_n) \subset \mathcal{N}$, then $\mu(E)\nu(E) = \sum \mu(E_n)\nu(E_n)$.

Trick. For all $x \in E$ and $y \in F$, there is a unique k such that $(x, y) \in E_k \times F_k$. Hence, for any fixed $y \in F$, $(x, y) \in E \times F$ for all $x \in E$, and thus

$$E = \coprod_{k \text{ s.t. } y \in F_k} E_k.$$

This is a disjoint union, since if $x \in E_j \cap E_k$ and $y \in F_j \cap F_k$, then $(x, y) \in (E_j \times F_j) \cap (E_k \times F_k)$, so $j = k$. Here is a cartoon of this trick:



$E = E_1 \amalg E_2 = E_3 \amalg E_4$

$E_1 = E_3$ and $E_2 = E_4$

$F = F_1 \amalg F_3 = F_2 \amalg F_4$

$F_1 = F_2$ and $F_3 = F_4$

Hence for $y \in F$,

$$\mu(E) = \sum_{k \text{ s.t. } y \in F_k} \mu(E_k) = \sum \mu(E_k)\chi_{F_k}(y),$$

and thus $\mu(E)\chi_F(y) = \sum \mu(E_k)\chi_{F_k}(y)$. Integrating over y yields

$$\mu(E)\nu(F) = \int_Y \mu(E)\chi_F(y)\, d\nu(y) = \int_Y \sum \mu(E_k)\chi_{F_k}(y)\, d\nu(y)$$
$$\underset{\text{(MCT)}}{=} \sum \int_Y \mu(E_k)\chi_{F_k}(y)\, d\nu(y) = \sum \mu(E_k)\nu(F_k).$$

□

Now use Carathéodory's outer measure construction, we get an outer measure $(\mu \times \nu)^*$ on $P(X \times Y)$, which restricts to a measure $\mu \times \nu$ on the $(\mu \times \nu)^*$-measurable sets, which is a σ-algebra containing $\mathcal{M} \times \mathcal{N}$ (as sets in $\mathcal{A}$ are $(\mu \times \nu)^*$-measurable, and $\mathcal{A}$ generates $\mathcal{M} \times \mathcal{N}$).

Exercise 3.8.12. Suppose X, Y are topological spaces and μ, ν are σ-finite Borel measures on X, Y respectively.

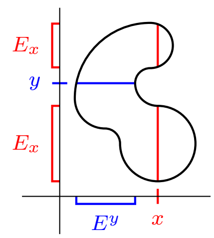
(1) Prove that $\mu \times \nu$ is σ-finite.
(2) Show that if μ, ν are both outer regular, then so is $\mu \times \nu$.
(3) Show that (2) fails when the σ-finite condition is dropped.
Hint: Consider a Dirac mass δ at x_0 such that $\delta(\{x_0\}) = \infty$.

3.9. **The Fubini and Tonelli Theorems.** For this section, fix measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$.

Definition 3.9.1. For $E \subset X \times Y$, we define

- (x-section) $E_x := \{y \in Y | (x, y) \in E\} = \pi_Y(E \cap (\{x\} \times Y))$
- (y-section) $E^y := \{x \in X | (x, y) \in E\} = \pi_X(E \cap (X \times \{y\}))$

Here is a cartoon of x- and y-sections:



Exercise 3.9.2. Suppose $(E_n) \subset P(X \times Y)$. Prove the following assertions.

1. $(\bigcup E_n)_x = \bigcup (E_n)_x$
2. $(\bigcap E_n)_x = \bigcap (E_n)_x$
3. $(E_n \setminus E_k)_x = (E_n)_x \setminus (E_k)_x$
4. $\chi_{E_n}(x, y) = \chi_{(E_n)_x}(y)$.

Deduce similar statements also hold for y-sections.

Proposition 3.9.3. *Let $E \in \mathcal{M} \times \mathcal{N}$. For all $x \in X$, $E_x \in \mathcal{N}$ and for all $y \in Y$, $E^y \in \mathcal{M}$.*

Proof. We prove the first statement, and the second is similar.

> **Trick.** We'll show that the following set is a σ-algebra on $X \times Y$:
> $$\mathcal{S} := \{E \subset X \times Y | E_x \in \mathcal{N}\}.$$

This implies the result, since $\mathcal{S}$ contains the measurable rectangles in $\mathcal{M} \times \mathcal{N}$, which generates $\mathcal{M} \times \mathcal{N}$. Thus $\mathcal{M} \times \mathcal{N} \subset \mathcal{S}$.

(0) Observe $\emptyset \in \mathcal{N}$ implies $\emptyset \in \mathcal{S}$.

(1) If $(E_n) \subset \mathcal{S}$, then $(E_n)_x \in \mathcal{N}$ for all $n \in \mathbb{N}$. By Exercise 3.9.2, $(\bigcup E_n)_x = \bigcup (E_n)_x \in \mathcal{N}$. Thus $\bigcup E_n \in \mathcal{S}$.

(2) If $E \in \mathcal{S}$, then $E_x \in \mathcal{N}$. Observe $(E^c)_x = (E_x)^c \in \mathcal{N}$, and thus $E^c \in \mathcal{S}$. □

Exercise 3.9.4. Use Proposition 3.9.3 to show that $\mathcal{L} \times \mathcal{L}$ is not equal to $\mathcal{L}^2$, where $\mathcal{L}$ is the Lebesgue σ-algebra and $\mathcal{L}^2$ denotes the σ-algebra of $(\lambda \times \lambda)^*$-measurable sets in $\mathbb{R}^2$.

Definition 3.9.5. For $f : X \times Y \to \mathbb{R}, \overline{\mathbb{R}}$, or $\mathbb{C}$, we define

- (x-section) $f_x : Y \to \mathbb{R}, \overline{\mathbb{R}}$, or $\mathbb{C}$ by $f_x(y) := f(x, y)$, and
- (y-section) $f^y : X \to \mathbb{R}, \overline{\mathbb{R}}$, or $\mathbb{C}$ by $f^y(x) := f(x, y)$.

Corollary 3.9.6. *If $f : X \times Y \to \mathbb{R}, \overline{\mathbb{R}}$, or $\mathbb{C}$ is $\mathcal{M} \times \mathcal{N}$-measurable, then*

- *for all $x \in X$, f_x is $\mathcal{N}$-measurable, and*
- *for all $y \in Y$, f^y is $\mathcal{M}$-measurable.*

Proof. We'll prove the first statement, and the second is similar. Observe that for all $x \in X$ and measurable G contained in the codomain, $f_x^{-1}(G) = f^{-1}(G)_x \in \mathcal{N}$. □

Exercise 3.9.7. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is such that each x-section f_x is Borel measurable and each y-section f^y is continuous. Show f is Borel measurable.

Theorem 3.9.8 (Tonelli for characteristic functions)**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. Then for all $E \in \mathcal{M} \times \mathcal{N}$,*

(1) The functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable, and
(2) $(\mu \times \nu)(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$.

Proof. First, we'll assume μ, ν are finite measures. Let $\Lambda \subset \mathcal{M} \times \mathcal{N}$ be the subset for which (1) and (2) above hold. Observe that $\Pi := \{\text{measurable rectangles in } \mathcal{M} \times \mathcal{N}\}$ is contained in Λ.

Step 1: Π is a π-system.

Proof. The intersection of 2 measurable rectangles is a measurable rectangle. □

Step 2: Λ is a λ-system. Thus by the $\pi - \lambda$ Theorem,

$$\mathcal{M} \times \mathcal{N} = \Lambda(\Pi) \subset \Lambda \subset \mathcal{M} \times \mathcal{N},$$

and thus equality holds.

Proof.
(0) First, note $X \times Y \in \Pi \subset \Lambda$.
(1) If $E \in \Lambda$ so that (1) and (2) hold for E, then as we assumed ν is finite,

$$x \longmapsto \nu((E^c)_x) = \nu((E_x)^c) = \nu(Y) - \nu(E_x)$$

is measurable (as a constant function minus a measurable function), as is $y \mapsto \mu((E^c)_y)$, so (1) holds for E^c. Moreover, $\mu \times \nu$ is finite, so

$$\begin{aligned}
(\mu \times \nu)(E^c) &= (\mu \times \nu)(X \times Y) - (\mu \times \nu)(E) \\
&= \int_X \nu(Y)\, d\mu(x) - \int \nu(E_x)\, d\mu(x) \\
&= \int_X (\nu(Y) - \nu(E_x))\, d\mu(x) \\
&= \int_X \nu((E_x)^c)\, d\mu(x) \\
&= \int_X \nu((E^c)_x)\, d\mu(x) && \text{proving part of (2) for } E^c \\
&= \int_Y \mu((E^c)^y)\, d\nu(y) && \text{similarly.}
\end{aligned}$$

Thus Λ is closed under taking complements.
(2) Suppose $(E_n) \subset \Lambda$ is a sequence of disjoint subsets. Observe for all $x \in X$, $((E_n)_x) \subset \mathcal{N}$ is disjoint. Then for all n, $x \mapsto \nu((E_n)_x)$ is measurable, and thus so is

$$x \longmapsto \sum \nu((E_n)_x) = \nu\left(\coprod (E_n)_x\right) = \nu\left(\left(\coprod E_n\right)_x\right).$$

Similarly, $y \mapsto \mu\left((\coprod E_n)^y\right)$ is measurable, proving (1) for $\coprod E_n$. We calculate

$$
\begin{aligned}
(\mu \times \nu)\left(\coprod E_n\right) &= \sum (\mu \times \nu)(E_n) \\
&= \sum \int_X \nu((E_n)_x)\, d\mu(x) \\
&= \int_X \sum \nu((E_n)_x)\, d\mu(x) && \text{(by the MCT 3.3.9)} \\
&= \int_X \nu\left(\coprod (E_n)_x\right) d\mu(x) \\
&= \int_X \nu\left(\left(\coprod E_n\right)_x\right) d\mu(x) && \text{proving part of (2) for } \coprod E_n \\
&= \int_Y \mu\left(\left(\coprod E_n\right)^y\right) d\nu(y) && \text{similarly.}
\end{aligned}
$$

Thus Λ is closed under taking countable disjoint unions. □

Step 3: When μ, ν are σ-finite, write $X \times Y$ as an *increasing* union $X \times Y = \bigcup X_n \times Y_n$ with $X_n \times Y_n$ measurable rectangles such that $\mu(X_n), \nu(Y_n) < \infty$ for all $n \in \mathbb{N}$. For $E \in \mathcal{M} \times \mathcal{N}$, write $E_n := E \cap (X_n \times Y_n)$, and observe $E_n \nearrow E$, so $(E_n)_x \nearrow E_x$. Thus the function

$$x \longmapsto \nu(E_x) = \lim \nu((E_n)_x)$$

is measurable (as a pointwise limit of measurable functions), as is $y \longmapsto \mu(E^y)$. We then calculate

$$
\begin{aligned}
(\mu \times \nu)(E) &= \lim (\mu \times \nu)(E_n) \\
&= \lim \int_X \nu((E_n)_x)\, d\mu(x) \\
&= \int_X \lim \nu((E_n)_x)\, d\mu(x) && \text{(by the MCT 3.3.9)} \\
&= \int_X \nu(E_x)\, d\mu(x) \\
&= \int_Y \mu(E^y)\, d\nu(y) && \text{similarly.}
\end{aligned}
$$

□

Theorem 3.9.9 (Tonelli)**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. For $f \in L^+(X \times Y, \mathcal{M} \times \mathcal{N})$,*

(1) $x \mapsto \int_Y f_x\, d\nu$ is $\mathcal{M}$-measurable (an element of $L^+(X, \mathcal{M})$),

(2) $y \mapsto \int_X f^y\, d\mu$ is $\mathcal{N}$-measurable (an element of $L^+(Y, \mathcal{N})$), and

(3) $\int_{X \times Y} f\, d(\mu \times \nu) = \int_X \left(\int_Y f_x\, d\nu\right) d\mu = \int_Y \left(\int_X f^y\, d\mu\right) d\nu$.

Proof. If $f = \chi_E$ for some $E \in \mathcal{M} \times \mathcal{N}$, this is exactly the previous theorem. Since $(cf+g)_x = c(f_x)+g_x$ (this is an exercise), we get the result for non-negative simple functions by linearity.

Suppose now $f \in L^+$ is arbitrary and $(\psi_n) \subset \mathsf{SF}^+$ such that $\psi_n \nearrow f$ everywhere. Then $(\psi_n)_x \nearrow f_x$ and $(\psi_n)^y \nearrow f^y$, so by the MCT 3.3.9,

$$\int_Y (\psi_n)_x \, d\nu \nearrow \int_Y f_x \, d\nu \qquad \text{and} \qquad \int_X (\psi_n)^y \, d\mu \nearrow \int_X f^y \, d\mu,$$

which implies (1) and (2) (countable supremums of measurable functions are measurable). Again by the MCT 3.3.9,

$$\begin{aligned}
\int_X \left(\int_Y f_x \, d\nu \right) d\mu &= \int_X \left(\lim \int_Y (\psi_n)_x \, d\nu \right) d\mu \\
&= \lim \int_X \left(\int_Y (\psi_n)_x \, d\nu \right) d\mu \\
&= \lim \int_{X \times Y} \psi_n \, d(\mu \times \nu) && \text{by previous theorem} \\
&= \int_{X \times Y} f \, d(\mu \times \nu) \\
&= \int_Y \left(\int_X f^y \, d\mu \right) d\nu && \text{similarly.}
\end{aligned}$$

□

Exercise 3.9.10 (Counterexample: Folland §2.5, #46)**.** Let $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, $\mu = \lambda$ Lebesgue measure, and ν counting measure. Let $\Delta = \{(x, x) | x \in [0, 1]\}$ be the diagonal. Prove that $\int\int \chi_\Delta \, d\mu \, d\nu$, $\int\int \chi_\Delta \, d\nu \, d\mu$, and $\int \chi_\Delta \, d(\mu \times \nu)$ are all distinct.

Exercise 3.9.11. Suppose $f : \mathbb{R} \to [0, \infty)$ is Borel measurable.

(1) Show that $E := \{(x, y) \in \mathbb{R}^2 | 0 \le y \le f(x)\}$ is Borel measurable.
(2) Show that $\int f(x) \, d\lambda(x) = (\lambda \times \lambda)(E)$.

Remark 3.9.12. Under the hypotheses of Tonelli's Theorem 3.9.9, if in addition $f \in L^+(X \times Y, \mathcal{M} \times \mathcal{N}) \cap L^1(\mu \times \nu)$, then

- $\displaystyle\int_Y f_x \, d\nu < \infty$ ($f_x \in L^1(\nu)$) a.e. $x \in X$, and
- $\displaystyle\int_X f^y \, d\mu < \infty$ ($f^y \in L^1(\mu)$) a.e. $y \in Y$.

Corollary 3.9.13 (Fubini)**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $f \in L^1(\mu \times \nu)$, then*

(1) $f_x \in L^1(\nu)$ *a.e.* $x \in X$ *and* $f^y \in L^1(\mu)$ *a.e.* $y \in Y$,
(2) $\left(x \mapsto \displaystyle\int_Y f_x \, d\nu \right) \in L^1(\mu)$ *and* $\left(y \mapsto \displaystyle\int_X f^y \, d\mu \right) \in L^1(\nu)$, *and*
(3) $\displaystyle\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f_x \, d\nu \right) d\mu = \int_Y \left(\int_X f^y \, d\mu \right) d\nu.$

Proof. Recall that

$$f = \operatorname{Re}(f)_+ - \operatorname{Re}(f)_- + i \operatorname{Im}(f)_+ - i \operatorname{Im}(f)_-,$$

where $\operatorname{Re}(f)_\pm, \operatorname{Im}(f)_\pm \in L^+(X \times Y, \mathcal{M} \times \mathcal{N}) \cap L^1(\mu \times \nu)$. Hence Tonelli's Theorem 3.9.9 applies to the 4 functions, as does Remark 3.9.12. The result follows. □

Exercise 3.9.14 (Counterexample: Folland §2.5, #48)**.** Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = P(\mathbb{N})$, and $\mu = \nu$ counting measure. Define

$$f(m,n) := \begin{cases} 1 & \text{if } m = n \\ -1 & \text{if } m = n+1 \\ 0 & \text{else.} \end{cases}$$

Prove that $\int |f| d(\mu \times \nu) = \infty$, and $\int \int f \, d\mu \, d\nu$ and $\int \int f \, d\nu \, d\mu$ both exist and are unequal.

Exercise 3.9.15. Let $f, g \in L^1([0,1], \lambda)$ where λ is Lebesgue measure. For $0 \leq x \leq 1$, define

$$F(x) := \int_{[x,1]} f \, d\lambda \qquad \text{and} \qquad G(x) := \int_{[x,1]} g \, d\lambda.$$

(1) Prove that F and G are continuous on $[0,1]$.
(2) Compute

$$\int_{\underbrace{[0,1]^2 = \square}_{\textit{Hint!}}} f(x)g(y) \, d(\lambda \times \lambda)$$

to prove the *integration by parts* formula:

$$\int_{[0,1]} Fg \, d\lambda = F(0)G(0) - \int_{[0,1]} Gf \, d\lambda.$$

Exercise 3.9.16. Prove the Fubini Theorem (Corollary 3.9.13) also holds replacing $(\mathcal{M} \times \mathcal{N}, \mu \times \nu)$ with its completion $(\overline{\mathcal{M} \times \mathcal{N}}, \overline{\mu \times \nu})$

Exercise 3.9.17. Show that the conclusions of the Fubini and Tonelli Theorems hold when $(X, \mathcal{M}, \mu)$ is an arbitrary measure space (not necessarily σ-finite) and Y is a countable set, $\mathcal{N} = P(Y)$, and ν is counting measure.

Exercise 3.9.18. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces *which are not assumed to be σ-finite.* Let $f \in L^1(\mu, \mathbb{R})$ and $g \in L^1(\nu, \mathbb{R})$, and define $h(x,y) := f(x)g(y)$.

(1) Prove that h is $\mathcal{M} \times \mathcal{N}$-measurable.
(2) Prove that $h \in L^1(\mu \times \nu)$.
(3) Prove that $\displaystyle\int_{X \times Y} h \, d(\mu \times \nu) = \int_X f \, d\mu \int_Y g \, d\nu$.

Remark: Since $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are not assumed to be σ-finite, you cannot directly apply the Fubini or Tonelli Theorems!

As an application, we give the following exercise on convolution multiplication on $\mathcal{L}^1(\mathbb{R}, \lambda)$.

Exercise 3.9.19. Suppose $f, g \in \mathcal{L}^1(\mathbb{R}, \lambda)$.

(1) Show that $y \mapsto f(x-y)g(y)$ is measurable for all $x \in \mathbb{R}$ and in $\mathcal{L}^1(\mathbb{R}, \lambda)$ for a.e. $x \in \mathbb{R}$.
(2) Define the *convolution* of f and g by

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y) \, d\lambda(y).$$

Show that $f * g \in \mathcal{L}^1(\mathbb{R}, \lambda)$.
(3) Show that $\mathcal{L}^1(\mathbb{R}, \lambda)$ is a commutative $\mathbb{C}$-algebra under $\cdot, +, *$.

(4) Show that $\int_{\mathbb{R}} |f * g| \leq \int_{\mathbb{R}} |f| \int_{\mathbb{R}} |g|$, i.e., $\|\cdot\|_1$ is submultiplicative.

Since we already know that $\mathcal{L}^1(\mathbb{R}, \lambda)$ is complete, this shows that the $\mathbb{C}$-algebra $\mathcal{L}^1(\mathbb{R}, \lambda)$ is a *Banach algebra.*

3.10. **Then n-dimensional Lebesgue integral.** Recall that $\mathcal{L}$ is the Lebesgue σ-algebra on $\mathbb{R}$ and λ is Lebesgue measure.

Definition 3.10.1. We define $(\mathbb{R}^n, \mathcal{L}^n, \lambda^n)$ as the completion of $(\mathbb{R}^n, \underbrace{\mathcal{L} \times \cdots \times \mathcal{L}}_{n \text{ factors}}, \underbrace{\lambda \times \cdots \times \lambda}_{n \text{ factors}})$.

Facts 3.10.2. Here are some properties of Lebesgue measure. Verification is left as an exercise.

(1) λ^n is σ-finite.
(2) λ^n is regular.
(3) For all $E \in \mathcal{L}^n$, for all $\varepsilon > 0$, there are disjoint rectangles $R_1, \ldots, R_n$ whose sides (projections) are intervals such that $\lambda^n(E \bigtriangleup \coprod^n R_k) < \varepsilon$, where $\bigtriangleup$ denotes symmetric difference.
(4) $\mathsf{ISF} = \mathsf{SF} \cap \mathcal{L}^1(\lambda^n)$ is dense in $\mathcal{L}^1(\lambda^n)$.
(5) $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^1(\lambda^n)$.
(6) Suppose $E \in \mathcal{L}^n$.
   - For all $r \in \mathbb{R}^n$, $r + E \in \mathcal{L}^n$, and $\lambda^n(r + E) = \lambda^n(E)$.
   - For all $T \in GL(n, \mathbb{R})$, $TE \in \mathcal{L}^n$ and $\lambda^n(TE) = |\det(T)| \cdot \lambda^n(E)$.
(7) For all $\mathcal{L}^n$-measurable $f : \mathbb{R}^n \to \mathbb{C}$, the following functions are also $\mathcal{L}^n$-measurable:
$$x \longmapsto f(x + r) \text{ for } r \in \mathbb{R}^n, \text{ and}$$
$$x \longmapsto f(Tx) \text{ for } T \in GL(n, \mathbb{R}).$$
   If moreover $f \in L^+$ or $\mathcal{L}^1(\lambda^n)$, then
$$\int f(x + r)\, d\lambda^n(x) = \int f(x)\, d\lambda^n(x) \qquad \text{and}$$
$$\int f(x)\, d\lambda^n(x) = |\det(T)| \cdot \int f(Tx)\, d\lambda^n(x).$$

Exercise 3.10.3. Suppose μ is a translation-invariant measure on $\mathcal{B}_{\mathbb{R}^n}$ such that $\mu([0, 1]^n) = 1$. Show that $\mu = \lambda^n|_{\mathcal{B}_{\mathbb{R}^n}}$.

Exercise 3.10.4. Prove some assertions from Facts 3.10.2.

Exercise 3.10.5. Suppose $T \in GL_n(\mathbb{C})$ and $f \in L^+$ or $\mathcal{L}^1(\lambda^n)$.

(1) Prove that $f \circ T \in L^+$ or $\mathcal{L}^1(\lambda^n)$ respectively.
(2) Show that
$$\int f(x)\, d\lambda^n(x) = |\det(T)| \cdot \int f(Tx)\, d\lambda^n(x).$$

4.1. **Signed measures.** For this section, let $(X, \mathcal{M})$ be a measurable space.

Definition 4.1.1. A function $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ is called a *signed measure* if

- ν takes on *at most one* of the values $\pm\infty$,
- (vacuum) $\nu(\emptyset) = 0$, and
- (σ-additivity) for every disjoint sequence $(E_n) \subset \mathcal{M}$, $\nu\left(\coprod E_n\right) = \sum \nu(E_n)$.

We call ν *finite* if ν does not take on the values $\pm\infty$.

Remark 4.1.2. If ν is a signed measure and $(E_n) \subset \mathcal{M}$ are disjoint, then σ-additivity of ν *implies* that the sum $\sum \nu(E_n)$ must converge *absolutely* if $\left|\nu\left(\coprod E_n\right)\right| < \infty$. Indeed, reindexing the sets (E_n) does not change $\coprod E_n$, and thus it must not change the sum $\sum \nu(E_n)$.

Exercise 4.1.3.

(1) If μ_1, μ_2 are measures on $(X, \mathcal{M})$ with at least one of μ_1, μ_2 finite, then $\nu := \mu_1 - \mu_2$ is a signed measure.
(2) Suppose μ is a measure on $(X, \mathcal{M})$. If $f : X \to \overline{\mathbb{R}}$ is measurable and *extended μ-integrable*, i.e., at least one of $\int f_\pm < \infty$, then $\nu(E) := \int_E f\, d\mu$ is a signed measure.

It is now our goal to prove these are really the *only* ways to construct signed measures!

Definition 4.1.4. Suppose ν is a signed measure on $(X, \mathcal{M})$. We call $E \in \mathcal{M}$:

- *positive* if for all measurable $F \subseteq E$, $\mu(F) \geq 0$,
- *negative* if for all measurable $F \subseteq E$, $\mu(F) \leq 0$, and
- *null* if for all measurable $F \subseteq E$, $\mu(F) = 0$.

Observe that $N \in \mathcal{M}$ is null if and only if N is both positive and negative.

Facts 4.1.5. For ν a signed measure on $(X, \mathcal{M})$, we have the following facts about positive measurable sets. Similar statements hold for negative and null measurable sets.

(1) E positive implies $\nu(E) \geq 0$.
(2) E positive and $F \subseteq E$ measurable implies F is positive.
(3) $(E_n) \subset \mathcal{M}$ positive implies $\bigcup E_n$ positive.

Proof. Disjointify the E_n so that $\bigcup E_n = \coprod F_n$ where $F_1 := E_1$ and $F_n := E_n \setminus \bigcup^{n-1} E_k$ is positive for all $n \in \mathbb{N}$. If $G \subset \bigcup E_n = \coprod F_n$, then

$$\nu(G) = \nu\left(G \cap \coprod F_n\right) = \sum \nu(G \cap F_n) \geq 0. \qquad \square$$

(4) If $0 < \nu(E) < \infty$, there is a positive $F \subseteq E$ such that $\nu(F) > 0$.

Proof. If E is positive, we win. Otherwise, let $n_1 \in \mathbb{N}$ be minimal such that that there is a measurable $E_1 \subset E$ and $\nu(E_1) < -\frac{1}{n_1}$. Observe that $\nu(E \setminus E_1) > 0$, so if $E \setminus E_1$ is positive, we win. Otherwise, let $n_2 \in \mathbb{N}$ minimal such that there is a measurable $E_2 \subset E \setminus E_1$ with $\nu(E_2) < -\frac{1}{n_2}$. We can inductively iterate this procedure. Either $E \setminus \coprod^n E_k$ is positive for some n, or we have constructed a disjoint sequence (E_k) with $\nu(E_k) < -\frac{1}{n_k}$ for all k. Set $F := E \setminus \coprod E_k$. Since $\nu(E) < \infty$ and $E = F \amalg \coprod E_k$, by countable additivity, $\sum |\nu(E_k)| < \infty$, so $\sum_k -\frac{1}{n_k}$ converges. Hence $n_k \to \infty$ as $k \to \infty$. Since $\nu(E) > 0$ and $\nu(E_k) < 0$ for all k, $\nu(F) > 0$. Suppose $G \subset F$ is measurable. Then $\nu(G) \geq -\frac{1}{n_k - 1}$ for all k with $n_k > 1$, and thus $\nu(G) \geq 0$. So F is positive. □

Theorem 4.1.6 (Hahn Decomposition)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. There is a positive set $P \in \mathcal{M}$ such that P^c is negative. Moreover, if $Q \in \mathcal{M}$ is another positive set such that Q^c is negative, then $P \triangle Q$ and $P^c \triangle Q^c$ are null.*

A positive $P \in \mathcal{M}$ such that P^c is negative is called a Hahn decomposition *of X with respect to ν.*

Proof.
Existence: We may assume $\infty \notin \operatorname{im}(\nu) \subset \overline{\mathbb{R}}$ (otherwise, replace ν with $-\nu$). Define

$$r := \sup \{\nu(E) | E \text{ is positive}\}.$$

Then there is a sequence (E_n) of positive sets such that $\nu(E_n) \to r$. Take $P := \bigcup E_n$, which is positive. Since a signed measure restricted to a positive set is a positive measure, $\nu(P) = \lim \nu(E_n) = r$ by continuity from below ($\mu3$). We claim that P^c is negative. If $F \subset P^c$ such that $\nu(F) > 0$, by Facts 4.1.5(4), there is a positive $G \subset F$ such that $\nu(G) > 0$. Then $P \amalg G$ is positive with $\nu(P \amalg G) = \nu(P) + \nu(G) > r$, a contradiction.
Uniqueness: Suppose $P, Q \subset X$ are positive such that P^c, Q^c are negative. Then

$$P \triangle Q = (P \setminus Q) \cup (Q \setminus P) = \underbrace{(P \cap Q^c)}_{\text{pos. and neg.}} \cup \underbrace{(Q \cap P^c)}_{\text{pos. and neg.}}$$

is ν-null. Similarly, $P^c \triangle Q^c$ is ν-null. □

Definition 4.1.7. We say positive measures μ_1, μ_2 on $(X, \mathcal{M})$ are *mutually singular*, denoted $\mu_1 \perp \mu_2$, if there exist disjoint $E, F \in \mathcal{M}$ such that $X = E \amalg F$ and $\mu_1(F) = 0 = \mu_2(E)$.

Theorem 4.1.8 (Jordan decomposition)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. There exist unique mutually singular measures $\nu_\pm$ on $(X, \mathcal{M})$ such that $\nu = \nu_+ - \nu_-$, which we call the* Jordan decomposition *of ν.*

Proof.
Existence: Given a Hahn decomposition $X = P \amalg P^c$, $\nu_+(E) := \nu(E \cap P)$ and $\nu_-(E) := -\nu(E \cap P^c)$ are positive measures on $\mathcal{M}$, such that $\nu_+(P^c) = 0 = \nu_-(P)$ and $\nu = \nu_+ - \nu_-$. (Observe $\nu_\pm$ are *independent* of the Hahn decomposition.)
Uniqueness: Suppose that $\nu = \mu_+ - \mu_- = \nu_+ - \nu_-$ where $\mu_\pm$ and $\nu_\pm$ are all positive measures with $\mu_+ \perp \mu_-$ and $\nu_+ \perp \nu_-$. Then by definition of mutual singularity, there exist two Hahn

decompositions for ν: $X = P \amalg P^c$ such that $\mu_+(P^c) = 0 = \mu_-(P)$ and $X = Q \amalg Q^c$ such that $\nu_+(Q^c) = 0 = \nu_-(Q)$. Thus $P \triangle Q$ and $P^c \triangle Q^c$ are ν-null, and for all $E \in \mathcal{M}$,

$$\begin{aligned}\mu_+(E) = \mu_+(E \cap P) = \nu(E \cap P) &= \nu(E \cap P \cap Q) + \nu(E \cap \underbrace{P \cap Q^c}_{\nu\text{-null}})\\ &= \nu(E \cap P \cap Q) = \nu(E \cap P \cap Q) + \nu(E \cap \underbrace{P^c \cap Q}_{\nu\text{-null}}) = \nu(E \cap Q)\\ &= \nu_+(E \cap Q) = \nu_+(E).\end{aligned}$$

Hence $\mu_+ = \nu_+$, and similarly, $\mu_- = \nu_-$. $\square$

Definition 4.1.9. For a signed measure ν on $(X, \mathcal{M})$, define $L^1(\nu) := L^1(\nu_+) \cap L^1(\nu_-)$. For $f \in L^1(\nu)$, define

$$\int f \, d\nu := \int f \, d\nu_+ - \int f \, d\nu_-.$$

Clearly $L^1(\nu)$ is a $\mathbb{C}$-vector space and $\int$ is a linear functional. We define $\mathcal{L}^1(\nu)$ to be the quotient of $L^1(\nu)$ by the equivalence relation $f = g$ ν_+-a.e. and ν_--a.e.

Exercise 4.1.10. Suppose ν is a signed measure on $(X, \mathcal{M})$. Prove that $E \in \mathcal{M}$ is ν-null if and only if E is ν_+-null and ν_--null. Deduce that $f = g$ ν_+-a.e. and ν_--a.e if and only if $f = g$ up to a ν-null set.

Definition 4.1.11. For a signed measure ν on $(X, \mathcal{M})$, define the *total variation* of $\nu = \nu_+ - \nu_-$ by $|\nu| := \nu_+ + \nu_-$, which is a positive measure. Observe that

$$|\nu(E)| = |\nu_+(E) - \nu_-(E)| \leq \nu_+(E) + \nu_-(E) = |\nu|(E) \qquad \forall\, E \in \mathcal{M}.$$

Hence ν is finite if and only if $|\nu|$ is finite.

Exercise 4.1.12. Suppose ν is a signed measure on $(X, \mathcal{M})$, let $\nu = \nu_+ - \nu_-$ be its Jordan decomposition, and let $|\nu|$ be its total variation.

(1) Prove that for $E \in \mathcal{M}$, $\nu_+(E) = \sup \{\nu(F) | F \in \mathcal{M} \text{ with } F \subset E\}$.
(2) Prove that for $E \in \mathcal{M}$, $\nu_-(E) = -\inf \{\nu(F) | F \in \mathcal{M} \text{ with } F \subset E\}$.
(3) Prove that for $E \in \mathcal{M}$,

$$|\nu|(E) = \sup \left\{ \sum_{i=1}^n |\nu(E_i)| \,\middle|\, E_1, \ldots, E_n \in \mathcal{M} \text{ disjoint with } E = \coprod_{i=1}^n E_i \right\}.$$

Exercise 4.1.13. Suppose $(X, \mathcal{M})$ is a measurable space, ν is a signed measure on $(X, \mathcal{M})$, and λ, μ are positive measures on $(X, \mathcal{M})$ such that $\nu = \lambda - \mu$. Show that $\nu_+ \leq \lambda$ and $\nu_- \leq \mu$ where $\nu = \nu_+ - \nu_-$ is the Jordan decomposition of ν.

Lemma 4.1.14. *Suppose μ_1, μ_2 are measures on X with at least one of μ_1, μ_2 finite, and set $\nu = \mu_1 - \mu_2$. Then $|\nu|(X) \leq \mu_1(X) + \mu_2(X)$.*

Proof. Let $\nu = \nu_+ - \nu_-$ be the Jordan decomposition of ν, and let $X = P \amalg P^c$ be a Hahn decomposition such that $\nu_+(P^c) = 0 = \nu_-(P)$. Then

$$\begin{aligned}&0 \leq \nu_+(X) = \nu(X \cap P) = \nu(P) = \mu_1(P) - \mu_2(P) \leq \mu_1(P) \leq \mu_1(X)\\ &0 \leq \nu_-(X) = -\nu(X \cap P^c) = -\nu(P^c) = \mu_2(P^c) - \mu_1(P^c) \leq \mu_2(P^c) \leq \mu_2(X)\end{aligned}$$

Hence $|\nu|(X) = \nu_+(X) + \nu_-(X) \leq \mu_1(X) + \mu_2(X)$. $\square$

Exercise 4.1.15 (Folland §3.1, #3)**.** Suppose μ is a positive measure on $(X,\mathcal{M})$ and ν is a signed measure on $(X,\mathcal{M})$. Prove that the following are equivalent.

(1) $\nu \perp \mu$
(2) $|\nu| \perp \mu$
(3) $\nu_+ \perp \mu$ and $\nu_- \perp \mu$.

Exercise 4.1.16 (Folland §3.1, #3)**.** Let ν be a signed measure on $(X,\mathcal{M})$. Prove the following assertions:

(1) $\mathcal{L}^1(\nu) = \mathcal{L}^1(|\nu|)$.
(2) If $f \in \mathcal{L}^1(\nu)$, $\left|\int f\,d\nu\right| \leq \int |f|d|\nu|$.
(3) If $E \in \mathcal{M}$, $|\nu|(E) = \sup\left\{\left|\int_E f\,d\nu\right| \middle| -1 \leq f \leq 1\right\}$.

Exercise 4.1.17. Suppose μ,ν are finite signed measures on the measurable space $(X,\mathcal{M})$.

(1) Prove that the signed measure $\mu \wedge \nu := \frac{1}{2}(\mu + \nu - |\mu - \nu|)$ satisfies $(\mu\wedge\nu)(E) \leq \min\{\mu(E),\nu(E)\}$ for all $E \in \mathcal{M}$.
(2) Suppose in addition that μ,ν are positive. Prove that $\mu \perp \nu$ if and only if $\mu \wedge \nu = 0$.

Exercise 4.1.18 (Folland §3.1, #6)**.** Suppose

$$\nu(E) := \int_E f\,d\mu \qquad\qquad E \in \mathcal{M}$$

where μ is a positive measure on $(X,\mathcal{M})$ and and f is an extended μ-integrable function. Describe the Hahn decompositions of ν and the positive, negative, and total variations of ν in terms of f and μ.

Exercise 4.1.19. In this exercise, we will show that

$$M := M(X,\mathcal{M},\mathbb{R}) := \{\text{finite signed measures on } (X,\mathcal{M})\}$$

is a Banach space with $\|\nu\| := |\nu|(X)$.

(1) Prove $\|\nu\| := |\nu|(X)$ is a norm on M.
(2) Show that $(\nu_n) \subset M$ Cauchy implies $(\nu_n(E)) \subset \mathbb{R}$ is uniformly Cauchy for all $E \in \mathcal{M}$. That is, show that for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and $E \in \mathcal{M}$, $|\nu_m(E) - \nu_n(E)| < \varepsilon$.
(3) Use part (2) to define a candidate limit signed measure μ on $\mathcal{M}$. Prove that ν is σ-additive.
Hint: first prove ν is finitely additive.
(4) Prove that $\sum \nu(E_n)$ converges absolutely when $(E_n) \subset \mathcal{M}$ is disjoint, and thus ν is a finite signed measure.
(5) Show that $\nu_n \to \nu$ in M.

4.2. **Absolute continuity and the Lebesgue-Radon-Nikodym Theorem.** For this section, we fix a measurable space $(X,\mathcal{M})$.

Definition 4.2.1. Let ν be a signed measure and μ a positive measure on $(X,\mathcal{M})$. We say ν is *absolutely continuous with respect to* μ, denoted $\nu \ll \mu$, if $\mu(E) = 0$ implies $\nu(E) = 0$.

Example 4.2.2. Let $f \in L^1(\mu,\mathbb{R})$ and set $\nu(E) := \int_E f\,d\mu$. (This is sometimes denoted by $d\nu := f d\mu$.) Then $\nu \ll \mu$.

Exercise 4.2.3 (Folland §3.2, #8)**.** Suppose μ is a positive measure on $(X, \mathcal{M})$ and ν is a signed measure on $(X, \mathcal{M})$. Prove that the following are equivalent.

(1) $\nu \ll \mu$
(2) $|\nu| \ll \mu$
(3) $\nu_+ \ll \mu$ and $\nu_- \ll \mu$.

Exercise 4.2.4. Suppose $(X, \mathcal{M})$ is a measurable space and ν is a signed measure and λ, μ are positive measures on $(X, \mathcal{M})$ such that $\nu = \lambda - \mu$. Show that $\nu_+ \leq \lambda$ and $\nu_- \leq \mu$ where $\nu = \nu_+ - \nu_-$ is the Jordan decomposition of ν.

Exercise 4.2.5 (Adapted from Folland §3.2, #9)**.** Suppose $\{\nu_j\}$ is a sequence of positive measures on $(X, \mathcal{M})$ and μ is a positive measure on $(X, \mathcal{M})$. Prove the following assertions.

(1) If $\{\nu_j\}$ is a sequence of positive measures on $(X, \mathcal{M})$ with $\nu_j \perp \mu$ for all j, then $\sum_{j=1}^\infty \nu_j \perp \mu$.
(2) If ν_1, ν_2 are positive measures on $(X, \mathcal{M})$ with at least one of ν_1, ν_2 is finite and $\nu_j \perp \mu$ for $j = 1, 2$, then $(\nu_1 - \nu_2) \perp \mu$.
(3) If $\{\nu_j\}$ is a sequence of positive measures on $(X, \mathcal{M})$ with $\nu_j \ll \mu$ for all j, then $\sum_{j=1}^\infty \nu_j \ll \mu$.
(4) If ν_1, ν_2 are positive measures on $(X, \mathcal{M})$ with at least one of ν_1, ν_2 is finite and $\nu_j \ll \mu$ for $j = 1, 2$, then $(\nu_1 - \nu_2) \ll \mu$.

Proposition 4.2.6. *Suppose ν is a finite signed measure and μ is a positive measure on $(X, \mathcal{M})$. The following are equivalent:*

(1) $\nu \ll \mu$, and
(2) For all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $E \in \mathcal{M}$, $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Proof. Since $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ and $|\nu(E)| \leq |\nu|(E)$, we may assume ν is positive. The result now follows from a previous exercise. For completeness, we'll provide the proof below.

First, it is clear that (2) implies (1). Suppose (2) fails. Then there is an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there is an $E_n \in \mathcal{M}$ with $\mu(E_n) < 2^{-n}$, but $\nu(E_n) \geq \varepsilon$. Set $F := \bigcap_{k=1}^\infty \bigcup_{k=n}^\infty E_n$. Since

$$\mu\left(\bigcup_{k=n}^\infty E_n\right) < \sum_{n=k}^\infty 2^{-k} = 2^{1-k} \qquad \forall\, k \in \mathbb{N},$$

$\mu(F) = 0$. But since ν is finite, $\nu(F) = \lim_k \left(\bigcup_{k=n}^\infty E_n\right) \geq \varepsilon$. Hence (1) fails. $\square$

Example 4.2.7. On $(\mathbb{N}, P(\mathbb{N}))$, define $\mu(E) := \sum_{n \in E} 2^{-n}$ and $\nu(E) := \sum_{n \in E} 2^n$. Then $\nu \ll \mu$ and $\mu \ll \nu$, but (2) above fails as ν is not finite.

Lemma 4.2.8. *Suppose μ, ν are finite measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$ or there is an $\varepsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\nu \geq \varepsilon\mu$ on E, i.e., E is positive for $\nu - \varepsilon\mu$.*

Proof. Let $X = P_n \amalg P_n^c$ be a Hahn decomposition for $\nu - n^{-1}\mu$ for all $n \in \mathbb{N}$. Set $P := \bigcup P_n$ so $P^c = \bigcap P_n^c$. Then P^c is negative for all $\nu - n^{-1}\mu$. Observe

$$0 \leq \nu(P^c) \leq \frac{1}{n} \underbrace{\mu(P^c)}_{<\infty} \qquad \forall\, n \in \mathbb{N},$$

so $\nu(P^c) = 0$. If $\mu(P) = 0$, then $\nu \perp \mu$. If $\mu(P) > 0$, then $\mu(P_n) > 0$ for some n, and P_n is positive for $\nu - n^{-1}\mu$. □

Theorem 4.2.9 (Lebesgue-Radon-Nikodym)**.** *Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There are unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ called the* Lebesgue decomposition of ν *such that*

$$\lambda \perp \mu, \qquad \rho \ll \mu, \qquad \text{and} \qquad \nu = \lambda + \rho.$$

Moreover, there exists a unique extended μ-integrable function f called the Radon-Nikodym derivative of ρ with respect to μ *such that $d\rho = f d\mu$. If ν is positive or finite, then so are λ and ρ respectively, and $f \in L^+$ or $L^1(\mu)$ respectively.*

Proof.
Case 1: Suppose μ, ν are finite positive measures.
Uniqueness: Suppose λ, λ' are finite signed measures such that $\lambda, \lambda' \perp \mu$ and $f, f' \in \mathcal{L}^1$ such that $d\nu = d\lambda + f d\mu = d\lambda' + f' d\mu$. Then as signed measures, $d(\lambda - \lambda') = (f' - f)d\mu$. But $(\lambda - \lambda') \perp \mu$ and $(f' - f)d\mu \ll d\mu$, so as signed measures by Exercise 4.2.5, $d(\lambda - \lambda') = 0 = (f' - f)d\mu$. We conclude that $\lambda = \lambda'$ and $f = f'$ in $\mathcal{L}^1$.
Existence: Set

$$A := \left\{ f \in L^1(X, \mu, [0, \infty]) \middle| \int_E f \, d\mu \le \nu(E) \text{ for all } E \in \mathcal{M} \right\}.$$

Observe that $0 \in A$.

Claim. *$f, g \in A$ implies $f \vee g \in A$.*

Proof. For all $E \in \mathcal{M}$,

$$\int_E f \vee g \, d\mu = \int_{E \cap \{g < f\}} f \, d\mu + \int_{E \setminus \{g < f\}} g \, d\mu \le \nu(E \cap \{g < f\}) + \nu(E \setminus \{g < f\}) = \nu(E).$$

□

Set $M := \sup \left\{ \int f \, d\mu \middle| f \in A \right\}$, and note that $M \le \nu(X) < \infty$. Choose $(f_n) \subset A$ such that $\int f_n \, d\mu \nearrow M$. Set $g_n := \max\{f_1, \ldots, f_n\} \in A$ and $f := \sup g_n$. Then by the Squeeze Theorem,

$$\int f_n \, d\mu \le \int g_n \, d\mu \nearrow M.$$

Since $g_n \nearrow f$ pointwise,

$$\int_E f \, d\mu \underset{\text{(MCT)}}{=} \lim_n \int_E g_n \, d\mu \le \nu(E) \qquad \forall E \in \mathcal{M}.$$

So $f \in A$ and $\int f \, d\mu = M$.

Claim. $\lambda(E) := \nu(E) - \int_E f\,d\mu \geq 0$ *is mutually singular with respect to* μ*. So setting* $d\rho := f d\mu$*, we have* $\lambda \perp \mu$*,* $\rho \ll \mu$*,* $\nu = \lambda + \rho$*, and* $d\rho = f d\mu$*.*

Proof. Suppose λ is not mutually singular with respect to μ. Then by Lemma 4.2.8, there is a $E \in \mathcal{M}$ and $\varepsilon > 0$ such that $\mu(E) > 0$ and $\lambda \geq \varepsilon\mu$ on E. But then for all $F \in \mathcal{M}$,

$$\begin{aligned}\int_F f + \varepsilon\chi_E\,d\mu &= \int_F f\,d\mu + \varepsilon\mu(E \cap F)\\ &\leq \int_F f\,d\mu + \lambda(E \cap F)\\ &= \int_F f\,d\mu + \nu(E \cap F) - \int_{E\cap F} f\,d\mu\\ &= \int_{F\setminus E} f\,d\mu + \nu(E \cap F)\\ &\leq \nu(F \setminus E) + \nu(E \cap F)\\ &= \nu(F).\end{aligned}$$

Hence $f + \varepsilon\chi_E \in A$, but $\int f + \varepsilon\chi_E\,d\mu = M + \varepsilon\mu(E) > M$, a contradiction. □

Case 2: Suppose μ, ν are σ-finite positive measures.

Existence: Write $X = \coprod X_n$ such that $\mu(X_n) < \infty$ and $\nu(X_n) < \infty$ for all n. Set $\mu_n(E) := \mu(E \cap X_n)$ and $\nu_n(E) := \nu(E \cap X_n)$ for all n. By Case 1, there exist positive measures $\lambda_n \perp \mu_n$ and $f_n \in \mathcal{L}^1_+(X_n, \mu_n)$ such that $d\nu_n = d\lambda_n + f_n d\mu_n$. Since $\mu_n(X_n^c) = \nu_n(X_n^c) = 0$, we have

$$\lambda_n(X_n^c) = \nu_n(X_n^c) - \int_{X_n^c} f_n\,d\mu_n = 0.$$

Hence we may assume $f_n|_{X_n^c} = 0$. Set $\lambda := \sum \lambda_n$ and $f := \sum f_n \in L^+$. Then $\lambda \perp \mu$ by Exercise 4.2.5, λ and $f d\mu$ are σ-finite, and $d\nu = d\lambda + f d\mu$.

Uniqueness: If λ' is another positive measure such that $\lambda' \perp \mu$ and $f' \in L^+$ such that $d\nu = d\lambda' + f' d\mu$. Setting $\lambda'_n(E) := \lambda'(E \cap X_n)$ for $E \in \mathcal{M}$ and $f'_n := f'\chi_{X_n}$, by Uniqueness from Case 1, we have $\lambda'_n = \lambda_n$ and $f'_n = f_n$ in $\mathcal{L}^1(\mu_n)$. Then

$$\begin{aligned}\lambda' &= \sum \lambda'_n = \sum \lambda_n = \lambda && \text{on } X\text{, and}\\ f' &= \sum f'_n = \sum f_n = f && \text{in } \mathcal{L}^1(\mu).\end{aligned}$$

Case 3: Suppose μ is σ-finite positive and ν is σ-finite signed. In this case, we use the Jordan Decomposition Theorem 4.1.8 to get $\nu = \nu_+ - \nu_-$ with $\nu_+ \perp \nu_-$. We apply Case 2 to $\nu_\pm$ separately and subtract the results. This shows existence and uniqueness. □

Remark 4.2.10. If μ is σ-finite positive and ν is σ-finite signed with $\nu \ll \mu$, there is a unique extended μ-integrable function $\frac{d\nu}{d\mu}$ called the *Radon-Nikodym derivative of* ν *with respect to* μ such that $d\nu = \frac{d\nu}{d\mu}d\mu$.

Exercise 4.2.11. Suppose ν is a σ-finite signed measure.

(1) Show that $\left|\frac{d\nu}{d|\nu|}\right| = 1$, $|\nu|$-a.e.
(2) Suppose further that $\nu \ll \mu$ for some σ-finite positive measure μ on $(X, \mathcal{M})$. Show that for all $f \in \mathcal{L}^1(\nu)$, $f\frac{d\nu}{d\mu} \in \mathcal{L}^1(\mu)$ and $\int f\,d\nu = \int f\frac{d\nu}{d\mu}\,d\mu$.
(3) Suppose even further that $\mu \ll \lambda$ for some σ-finite positive measure λ. Show $\nu \ll \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$.

Definition 4.2.12. A signed measure ν on a topological space $(X, \mathcal{T})$ is called *regular* if $|\nu|$ is regular.

Exercise 4.2.13. Suppose ν is a finite signed Borel measure on the LCH space X. Determine which of the conditions below are equivalent.
(1) ν is regular.
(2) $\nu_\pm$ is regular.
(3) For every $E \in \mathcal{B}_X$ and $\varepsilon > 0$, there is an open $U \subset X$ with $E \subset U$ such that $|\nu(U) - \nu(E)| < \varepsilon$.

Which of the above conditions are equivalent if
- X is σ-compact?
- ν is not finite?

4.3. **Complex measures.** For this section, fix a measurable space $(X, \mathcal{M})$.

Definition 4.3.1. A function $\nu : \mathcal{M} \to \mathbb{C}$ is called a *complex measure* if
- (vacuum) $\nu(\emptyset) = 0$, and
- (σ-additivity) For every disjoint sequence $(E_n) \subset \mathcal{M}$, $\nu\left(\coprod E_n\right) = \sum \nu(E_n)$.

Observe that if ν is a complex measure on $(X, \mathcal{M})$, then $\operatorname{Re}(\nu)$ and $\operatorname{Im}(\nu)$ are finite signed measures on $(X, \mathcal{M})$.

Remark 4.3.2. As in Remark 4.1.2, if ν is a complex measure and $(E_n) \subset \mathcal{M}$ are disjoint, then σ-additivity of ν *implies* that the sum $\sum \nu(E_n)$ converges *absolutely*.

Exercise 4.3.3. Prove the following assertions.
(1) If $\mu_0, \mu_1, \mu_2, \mu_3$ are finite measures on $(X, \mathcal{M})$, then $\sum_{k=0}^3 i^k \mu_k$ is a complex measure.
(2) For μ a measure on $(X, \mathcal{M})$ and $f \in L^1(\mu)$, $\nu(E) := \int_E f\,d\mu$ is a complex measure on $(X, \mathcal{M})$.

By the Jordan Decomposition Theorem 4.1.8, we get the following corollary:

Corollary 4.3.4. *If ν is a complex measure on $(X, \mathcal{M})$, there exist unique pairs of mutually singular finite measures $\operatorname{Re}(\nu)_\pm$ and $\operatorname{Im}(\nu)_\pm$ such that*

$$\nu = \underbrace{\operatorname{Re}(\nu)_+}_{=:\nu_0} - \underbrace{\operatorname{Re}(\nu)_-}_{=:\nu_2} + i(\underbrace{\operatorname{Im}(\nu)_+}_{=:\nu_1} - \underbrace{\operatorname{Im}(\nu)_-}_{=:\nu_3}) =: \sum_{k=0}^{3} i^k \nu_k.$$

Definition 4.3.5. For a complex measure ν on $(X, \mathcal{M})$, we define $L^1(\nu) := \bigcap_{k=0}^3 L^1(\nu_k)$. We define $\mathcal{L}^1(\nu)$ to be the quotient under the equivalence relation $f = g$ ν_k-a.e. for $k = 0, 1, 2, 3$. For $f \in L^1(\nu_k)$, we define

$$\int f\,d\nu := \sum_{k=0}^{3} i^k \int f\,d\nu_k.$$

Warning 4.3.6. The total variation of a complex measure $\nu = \sum_{k=0}^{3} i^k \nu_k$ is *not* $\sum_{k=0}^{3} \nu_k$. We must use the complex Radon-Nikodym Theorem 4.3.9 below.

Definition 4.3.7. Suppose ν is a complex measure and μ is a positive measure on $(X, \mathcal{M})$. We say:

- $\nu \perp \mu$ if $\operatorname{Re}(\nu) \perp \mu$ and $\operatorname{Im}(\nu) \perp \mu$, and
- $\nu \ll \mu$ if $\operatorname{Re}(\nu) \ll \mu$ and $\operatorname{Im}(\nu) \ll \mu$.

Exercise 4.3.8. Suppose ν is a complex measure and μ is a positive measure on $(X, \mathcal{M})$. Show that $\nu \ll \mu$ if and only if for all $E \in \mathcal{M}$, $\mu(E) = 0$ implies $|\nu(E)| = 0$.

Theorem 4.3.9 (Complex Lebesgue-Radon-Nikodym)**.** *If ν is a complex measure on $(X, \mathcal{M})$ and μ is a σ-finite positive measure on $(X, \mathcal{M})$, there exists a unique complex measure λ on $(X, \mathcal{M})$ and a unique $f \in \mathcal{L}^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$.*

Proof. Apply the Lebesgue-Radon-Nikodym Theorem 4.2.9 to $\operatorname{Re}(\nu)$ and $\operatorname{Im}(\nu)$ separately and then recombine. ☐

Exercise 4.3.10. Repeat Exercise 4.2.11 for ν a complex measure on $(X, \mathcal{M})$.

Lemma 4.3.11. *Suppose ν is a complex measure on $(X, \mathcal{M})$. There is a unique positive measure $|\nu|$ on $(X, \mathcal{M})$ satisfying the following property:*

- *For all positive measures μ on $(X, \mathcal{M})$ and $f \in \mathcal{L}^1(\mu)$ such that $d\nu = f d\mu$, $d|\nu| = |f| d\mu$.*

We call $|\nu|$ the total variation *of ν.*

Proof. First consider $\mu := |\operatorname{Re}(\nu)| + |\operatorname{Im}(\nu)|$. Since $|\operatorname{Re}(\nu)| \ll \mu$ and $|\operatorname{Im}(\nu)| \ll \mu$, we have $\operatorname{Re}(\nu) \ll \mu$ and $\operatorname{Im}(\nu) \ll \mu$, and thus $\nu \ll \mu$. By the complex Lebesgue-Radon-Nikodym Theorem 4.3.9, there is an $f \in \mathcal{L}^1(\mu)$ such that $d\nu = f d\mu$. Define $d|\nu| := |f| d\mu$. Observe this uniquely determines $|\nu|$ if it satisfies the uniqueness property in the bullet point above. So suppose that $d\nu = g d\rho$ for another positive measure ρ on $(X, \mathcal{M})$ and $g \in \mathcal{L}^1(\rho)$. Consider $\mu + \rho$ on $(X, \mathcal{M})$ and observe that $\nu \ll \mu$, $\mu \ll \mu + \rho$, and $\rho \ll \mu + \rho$. Hence

$$d\mu = \frac{d\mu}{d(\mu+\rho)} d(\mu+\rho) \qquad \text{and} \qquad d\rho = \frac{d\rho}{d(\mu+\rho)} d(\mu+\rho).$$

Since

$$f \frac{d\mu}{d(\mu+\rho)} d(\mu+\rho) = f d\mu = d\nu = g d\rho = g \frac{d\rho}{d(\mu+\rho)} d(\mu+\rho),$$

by Exercise 4.3.10 we have

$$f \frac{d\mu}{d(\mu+\rho)} = \frac{d\nu}{d(\mu+\rho)} = g \frac{d\rho}{d(\mu+\rho)} \qquad (\mu+\rho)\text{-a.e.}$$

This implies

$$|f| \frac{d\mu}{d(\mu+\rho)} = \left| f \frac{d\mu}{d(\mu+\rho)} \right| = \left| g \frac{d\rho}{d(\mu+\rho)} \right| = |g| \frac{d\rho}{d(\mu+\rho)} \qquad (\mu+\rho)\text{-a.e.}$$

Again by Exercise 4.3.10, $|f| d\mu = d|\nu| = |g| d\rho$, and thus ν satisfies the uniqueness condition in the bullet point. ☐

Facts 4.3.12. Suppose ν is a complex measure on $(X, \mathcal{M})$.

(1) $\nu \ll |\nu|$, as

$$|\nu(E)| = \left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu = |\nu|(E) \qquad \forall, E \in \mathcal{M}.$$

(2) If ν is a finite signed measure ($\operatorname{Im}(\nu) = 0$), then $d\nu = (\chi_P - \chi_{P^c})d|\nu|$, and so $d|\nu|' = (\chi_P + \chi_{P^c})d|\nu| = d|\nu|$ for any Hahn decomposition $X = P \amalg P^c$ for ν. This means this new definition $|\nu|'$ for a complex measure agrees with the old definition $|\nu|$ for a finite signed measure.

(3) Observe that if $d\nu = f d\mu$, then

$$\begin{aligned} d\operatorname{Re}(\nu) &= \operatorname{Re}(f)d\mu \\ d\operatorname{Im}(\nu) &= \operatorname{Im}(f)d\mu \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} d|\operatorname{Re}(\nu)| &= |\operatorname{Re}(f)|d\mu \\ d|\operatorname{Im}(\nu)| &= |\operatorname{Im}(f)|d\mu \end{aligned}$$

Since $|f|^2 = |\operatorname{Re}(f)|^2 + |\operatorname{Im}(f)|^2$, we have

$$\frac{d|\nu|}{d\mu} = |f| = \left(|\operatorname{Re}(f)|^2 + |\operatorname{Im}(f)|^2\right)^{1/2} = \left(\left(\frac{d|\operatorname{Re}(\nu)|}{d\mu}\right)^2 + \left(\frac{d|\operatorname{Im}(\nu)|}{d\mu}\right)^2 \right)^{1/2}.$$

Exercise 4.3.13. Suppose ν is a complex measure on $(X, \mathcal{M})$. Prove that $|\operatorname{Re}(\nu)| \leq |\nu|$, $|\operatorname{Im}(\nu)| \leq |\nu|$, and $|\nu| \leq |\operatorname{Re}(\nu)| + |\operatorname{Im}(\nu)|$ as $[0, \infty)$-valued functions on $\mathcal{M}$.

Exercise 4.3.14. Suppose ν is a complex measure on $(X, \mathcal{M})$.

(1) Prove that $L^1(\nu) = L^1(|\nu|)$.

(2) Show that for $f \in L^1(\nu)$,

$$\left| \int f d\nu \right| \leq \int |f| d|\nu|.$$

Exercise 4.3.15. In this exercise, we will show that

$$M := M(X, \mathcal{M}, \mathbb{C}) := \{\text{complex measures on } (X, \mathcal{M})\}$$

is a Banach space with $\|\nu\| := |\nu|(X)$.

(1) Prove that $\max\{\|\operatorname{Re}(\nu)\|, \|\operatorname{Im}(\nu)\|\} \leq \|\nu\| \leq 2\max\{\|\operatorname{Re}(\nu)\|, \|\operatorname{Im}(\nu)\|\}$.

(2) Show that if $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ are normed vector spaces, then $\|(v, w)\|_\infty := \max\{\|v\|, \|w\|\}$ is a norm on $V \oplus W$. Moreover, show that if $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are complete, then so is $(V \oplus W, \|\cdot\|_\infty)$.

(3) Show that $M(X, \mathcal{M}, \mathbb{C}) = M(X, \mathcal{M}, \mathbb{R}) \oplus iM(X, \mathcal{M}, \mathbb{R})$, where $M(X, \mathcal{M}, \mathbb{R})$ was defined in Exercise 4.1.19.

(4) Show that $\|\cdot\|$ on $M(X, \mathcal{M}, \mathbb{C})$ is equivalent to $\|\cdot\|_\infty$ on $M(X, \mathcal{M}, \mathbb{R}) \oplus iM(X, \mathcal{M}, \mathbb{R})$. Deduce that $M(X, \mathcal{M}, \mathbb{C})$ is complete.

Definition 4.3.16. A complex Borel measure ν on a topological space $(X, \mathcal{T})$ is called *regular* if $|\nu|$ is regular.

Exercise 4.3.17. Repeat Exercise 4.2.13 for a complex Borel measure ν, where (2) is replaced by

(2') $\operatorname{Re}(\nu)$ and $\operatorname{Im}(\nu)$ are regular signed measures.

4.4. **Lebesgue differentiation.** Here, I will be following notes from a graduate course I took in Fall 2005 at UC Berkeley from Sarason. We will treat differentiation of $f \in L^1(\lambda^n)$, and we'll then explain how to extend these results to

$$L^1_{\text{loc}} := L^1_{\text{loc}}(\lambda^n) := \{f : \mathbb{R}^n \to \mathbb{C} | f \text{ is integrable on bounded measurable sets}\}.$$

Definition 4.4.1. A *cube* in $\mathbb{R}^n$ is a set $Q \subset \mathbb{R}^n$ of the form $Q = \prod_{k=1}^n I_k$ where each I_k is a closed interval of the same length, which we denote by $\ell(Q)$.

- For $x \in \mathbb{R}^n$, define $\mathcal{C}(x) := \{\text{cubes } Q | x \in Q \text{ and } 0 < \ell(Q) < \infty\}$.
- For Q a cube and $r > 0$, rQ is the cube with the same center as Q, but with $\ell(rQ) = r\ell(Q)$.

Our goal is to prove the following theorem.

Theorem 4.4.2 (Lebesgue Differentiation)**.** *For all* $f \in L^1_{\text{loc}}$,

$$\lim_{\substack{\ell(Q)\to 0\\ x\in Q}} \frac{1}{\lambda^n(Q)} \int_Q f\, d\lambda^n = f(x) \qquad a.e. \tag{LDT}$$

As a direct corollary, we get (for $n = 1$):

Theorem 4.4.3 (Fundamental Theorem of Calculus)**.** *Suppose* $f \in \mathcal{L}^1(\lambda)$. *Define* $F(x) := \int_{(-\infty,x)} f\, d\lambda$. *Then* $F'(x) = f(x)$ *a.e.*

Proof. Observe

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{\substack{h\to 0\\ x\in Q_h := [x,x+h]}} \frac{1}{\lambda(Q_h)} \int_{Q_h} f\, d\lambda \underset{\text{(LDT)}}{=} f(x) \qquad a.e. \qquad \square$$

Definition 4.4.4 (Hardy-Littlewood Maximal Function)**.** For $f \in L^1_{\text{loc}}$, define $Mf := \mathbb{R}^n \to [0, \infty]$ by

$$(Mf)(x) := \sup \left\{ \frac{1}{\lambda^n(Q)} \int_Q |f|\, d\lambda^n \middle| Q \in \mathcal{C}(x) \right\}.$$

The function $M : L^1_{\text{loc}} \to \{f : \mathbb{R}^n \to [0, \infty]\}$ is called the *Hardy-Littlewood maximal function.*

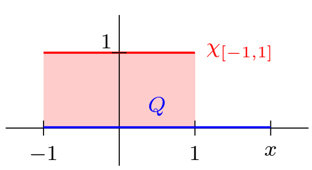
Facts 4.4.5. The Hardy-Littlewood maximal function satisfies the following properties:

1. $M(rf) = |r| \cdot Mf$ for all $r \in \mathbb{R}$.
2. $M(f + g) \le Mf + Mg$ for all $f, g \in L^1_{\text{loc}}$.
3. $Mf > 0$ everywhere unless $f = 0$ a.e.
4. Mf is lower semicontinuous ($\{Mf > r\}$ is open for all $r \in \mathbb{R}$), and thus measurable.

Example 4.4.6. For $\chi_{[-1,1]} : \mathbb{R} \to \mathbb{C}$,

$$M\chi_{[-1,1]}(x) = \begin{cases} 1 & x \in [-1, 1] \\ \dfrac{2}{1 + |x|} & x \notin [-1, 1] \end{cases}$$

and thus $M\chi_{[-1,1]} \notin L^1$. Here is a cartoon:



$$\frac{1}{\lambda(Q)} \int_Q \chi_{[-1,1]}\, d\lambda = \frac{2}{1 + x}.$$

Exercise 4.4.7 (Sarason)**.** Prove that for the f defined below, $f \in L^1(\lambda)$, but $Mf \notin L^1_{\text{loc}}$:

$$f(x) := \begin{cases} \dfrac{1}{|x|(\ln|x|)^2} & \text{if } |x| \le \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases}$$

Theorem 4.4.8 (Hardy-Littlewood Maximal, a.k.a. HLMT)**.** *There is a $c > 0$, only depending on n, such that for all $f \in L^1(\lambda^n)$ and $a > 0$,*

$$\lambda^n(\{Mf > a\}) \le c \cdot \frac{\|f\|_1}{a}.$$

Remark 4.4.9. The HLMT 4.4.8 is a generalization of Chebyshev's Inequality for a measure space $(X, \mathcal{M}, \mu)$: for all $a \ge 0$, $\int_{\{a \le |f|\}} |f| \, d\mu \ge a\mu(\{a \le |f|\})$. Hence for all $f \in L^1(\mu)$ and $a \ge 0$,

$$\mu(\{a \le |f|\}) \le \frac{\|f\|_1}{a}. \tag{4.4.10}$$

To prove the HLMT 4.4.8, we'll use a variation of the Vitali Covering Lemma. We'll prove the more general Vitali Covering Lemma, and I'll leave the exact variation that we'll use to prove the HLMT as an exercise.

Lemma 4.4.11 (Vitali Covering)**.** *Let $\mathcal{B}$ be some collection of open balls in $\mathbb{R}^n$, and let $U = \bigcup_{B \in \mathcal{B}} B$. If $c < \lambda^n(U)$, then there exist disjoint $B_1, \ldots, B_k \in \mathcal{B}$ such that $\sum_{j=1}^k \lambda^n(B_j) > 3^{-n}c$.*

Proof. Since λ^n is regular, there is a compact $K \subset U$ such that $c < \lambda^n(K)$. Then there exist finitely many balls in $\mathcal{B}$ which cover K, say $A_1, \ldots, A_m$. Define B_1 to be the largest (in terms of radius) of the A_i, and inductively for $j \ge 2$, define B_j to be the larges of the the A_i *disjoint* from $B_1, \ldots, B_{j-1}$. Since there are finitely many A_i, this process terminates, giving $B_1, \ldots, B_k$.

> **Trick.** If A_i is *not* one of $B_1, \ldots, B_k$, there is a smallest $1 \le j \le k$ such that $A_i \cap B_j \neq \emptyset$. Then $\text{rad}(A_i) \le \text{rad}(B_j)$, so $A_i \subset 3B_j$, where $3B_j$ has the same center as B_j, but three times the radius.

Then $K \subset \bigcup^k 3B_j$, so

$$c < \lambda^n(K) \le \sum^k \lambda^n(3B_j) = 3^n \sum^k \lambda^n(B_j). \qquad \square$$

Exercise 4.4.12 (Sarason, variation of Vitali Covering Lemma 4.4.11)**.** Suppose $E \subset \mathbb{R}^n$ (not assumed to be Borel measurable) and let $\mathcal{C}$ be a family of cubes covering E such that

$$\sup\{\ell(Q) | Q \in \mathcal{C}\} < \infty.$$

Show there exists a sequence $(Q_k) \subset \mathcal{C}$ of disjoint cubes such that

$$\sum_{k=1}^{\infty} \lambda^n(Q_k) \ge 5^{-n}(\lambda^n)^*(E).$$

Hint: Inductively choose Q_k such that $2\ell(Q_k)$ is larger than the sup of the lengths of all cubes which do not intersect $Q_1, \ldots, Q_{k-1}$, with $Q_0 = \emptyset$ by convention.

Proof of HLMT 4.4.8. Suppose $f \in L^1(\lambda^n)$ and $a > 0$. Let $E = \{a < Mf\}$ and

$$\mathcal{C} = \left\{ \text{cubes } Q \middle| a < \frac{1}{\lambda^n(Q)} \int_Q |f| \, d\lambda^n \right\}.$$

By definition, the cubes in $\mathcal{C}$ cover E. Observe that $a < \ell(Q)^{-n}\|f\|_1$ implies $\ell(Q) < \left(\frac{\|f\|_1}{a}\right)^{1/n}$. By Exercise 4.4.12, there is a sequence $(Q_i) \subset \mathcal{C}$ of disjoint cubes such that $\sum \lambda^n(Q_i) \geq 5^{-n}\lambda^n(E)$. Then

$$\lambda^n(E) \leq 5^n \sum \lambda^n(Q_i) \leq 5^n \sum \frac{1}{a} \int_{Q_i} |f| \, d\lambda^n \leq 5^n \cdot \frac{\|f\|_1}{a}. \qquad \square$$

Proof of the Lebesgue Differentiation Theorem 4.4.2.
Step 1: (LDT) for all $f \in L^1$ implies (LDT) for all $f \in L^1_{\text{loc}}$.

Proof. Suppose $f \in L^1_{\text{loc}}$. It suffices to show that for all $R > 0$, (LDT) holds a.e. $x \in Q_R(0) := \prod^n [-R, R]$. For $x \in Q_R(0)$ and $Q \in \mathcal{C}(x)$ with $\ell(Q) \leq 1$, the value of $\frac{1}{\ell(Q)^n} \int_Q f \, d\lambda^n$ only depends on $f(y)$ for $y \in Q_{R+1}(0)$. So we can replace f with $f\chi_{Q_{R+1}(0)} \in L^1$. $\square$

Step 2: (LDT) for all $f \in C_c(\mathbb{R}^n)$ implies (LDT) for all $f \in L^1$.

Proof. For $Q \in \mathcal{C}(0)$ and $f \in L^1$, define $(I_Q f)(x) := \frac{1}{\lambda^n(Q)} \int_{Q+x} f \, d\lambda^n$. Observe I_Q is linear, and $|I_Q f| \leq Mf$ everywhere. Now fix $f \in L^1$ and $\varepsilon > 0$. Let

$$E_\varepsilon := \left\{ x \in \mathbb{R}^n \middle| \limsup_{\substack{\ell(Q) \to 0 \\ Q \in \mathcal{C}(0)}} |I_Q f(x) - f(x)| > \varepsilon \right\}.$$

We'll show $(\lambda^n)^*(E_\varepsilon) = 0$, which implies $E_\varepsilon \in \mathcal{L}^n$ and $\lambda^n(E_\varepsilon) = 0$. If $\varepsilon' < \varepsilon$, then $E_\varepsilon \subset E_{\varepsilon'}$. Hence $\bigcup E_{1/n}$ has measure zero, which implies the result.
In order to show $(\lambda^n)^*(E_\varepsilon) = 0$, let $\delta > 0$. Since $C_c(\mathbb{R}^n) \subset L^1$ is dense, there is a continuous $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_1 < \delta$. Then

$$\begin{aligned} |I_Q f - f| &= |I_Q(f-g) + (I_Q g - g) + (g - f)| \\ &\leq |I_Q(f-g)| + |(I_Q g - g)| + |(g-f)| \\ &\leq M(f-g) + \underbrace{|(I_Q g - g)|}_{\to 0} + |g - f| \end{aligned}$$

By assumption, as $\ell(Q) \to 0$ for $Q \in \mathcal{C}(0)$, $|(I_Q g - g)| \to 0$. Hence

$$E_\varepsilon \subset \left\{ \frac{\varepsilon}{2} < M(f-g) \right\} \cup \left\{ \frac{\varepsilon}{2} < |f - g| \right\}.$$

By the HLMT 4.4.8 and Chebyshev's Inequality (4.4.10),

$$\begin{aligned}(\lambda^n)^*(E_\varepsilon) &\leq \lambda^n\left(\left\{\frac{\varepsilon}{2} < M(f-g)\right\}\right) + \lambda^n\left(\left\{\frac{\varepsilon}{2} < |f-g|\right\}\right)\\ &\leq \frac{c\|f-g\|_1}{\varepsilon/2} + \frac{\|f-g\|_1}{\varepsilon/2}\\ &= \frac{2(c+1)}{\varepsilon}\cdot\|f-g\|_1\\ &< \frac{2(c+1)}{\varepsilon}\cdot\delta.\end{aligned}$$

But $\delta > 0$ was arbitrary, so $(\lambda^n)^*(E_\varepsilon) = 0$. □

Step 3: (LDT) holds for all $g \in C_c(\mathbb{R}^n)$.

Proof. Observe that g is uniformly continuous. Let $\varepsilon > 0$, and pick $\delta > 0$ such that $x, y \in Q$ with $\ell(Q) < \delta$ implies $|g(x) - g(y)| < \varepsilon$. Then for all such Q,

$$\left|g(x) - \frac{1}{\lambda^n(Q)}\int_Q g(y)\,d\lambda^n(y)\right| \leq \frac{1}{\lambda^n(Q)}\int_Q |g(x) - g(y)|\,d\lambda^n(y) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

Combining Steps 1-3 yields the result. □

Definition 4.4.13. Suppose $E \in \mathcal{L}^n$. A point $x \in E$ is called a *Lebesgue point of density of* E if

$$\lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{\lambda^n(Q\cap E)}{\lambda^n(Q)} = 1.$$

Corollary 4.4.14. *For $E \in \mathcal{L}^n$, almost all points of E are Lebesgue points of density.*

Proof. Apply the Lebesgue Differentiation Theorem 4.4.2 to χ_E. □

Exercise 4.4.15 (Steinhaus Theorem, version 2)**.** Suppose that $A, B \subset \mathbb{R}$ are sets with positive Lebesgue measure. Prove that there is an interval I with $\lambda(I) > 0$ such that

$$I \subseteq A + B = \{a + b | a \in A \text{ and } b \in B\}.$$

Definition 4.4.16. For $f \in L^1(\lambda^n)$, $x \in \mathbb{R}^n$ is called a *Lebesgue point of* f if

$$\lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{1}{\lambda^n(Q)}\int_Q |f - f(x)|\,d\lambda^n = 0.$$

Corollary 4.4.17. *For $f \in L^1_{\text{loc}}$, almost all points of $\mathbb{R}^n$ are Lebesgue points of f.*

Proof. As in the proof of the Lebesgue Differentiation Theorem 4.4.2, we may assume $f \in L^1$. Let $D \subset \mathbb{C}$ be a countable dense subset ($\mathbb{Q} + i\mathbb{Q}$ will suffice). For $d \in D$, set

$$E_d := \left\{x \in \mathbb{R}^n \,\middle|\, \lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{1}{\lambda^n(Q)}\int_Q |f-d| - |f(x)-d|\,d\lambda^n = 0\right\}.$$

By the Lebesgue Differentiation Theorem 4.4.2, E_d^c is λ^n-null, which implies $E_d \in \mathcal{L}^n$. Set $E := \bigcap_{d\in D} E_d \in \mathcal{L}^n$, and observe $E^c = \bigcup_{d\in D} E_d^c$ is still λ^n-null. We claim that every $x \in E$ is a Lebesgue point of f. Indeed, if $x \in E$, then for all $d \in D$,

$$|f - f(x)| \leq |f - d| + |f(x) - d| = (|f - d| - |f(x) - d|) + 2|f(x) - d|.$$

This implies for all $d \in D$,

$$\begin{aligned}\limsup_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f - f(x)|\, d\lambda^n &\leq 2|f(x) - d| + \underbrace{\limsup_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f - d| - |f(x) - d|\, d\lambda^n}_{=0} \\ &= 2|f(x) - d|.\end{aligned}$$

But since D is dense in $\mathbb{C}$, we can approximate $f(x)$ by $d \in D$ up to any $\varepsilon > 0$. We conclude that x is a Lebesgue point of f. □

4.5. **Functions of bounded variation.** Recall that the Lebesgue-Stieltjes measures on $\mathbb{R}$ were constructed from non-decreasing right continuous functions $F : \mathbb{R} \to \mathbb{R}$. They enjoyed the properties of being a complete measure which is equal to the completion of the restriction to $\mathcal{B}_{\mathbb{R}}$, which is a regular Borel measure.

We can adapt this construction to get a complex measure from a function $F : \mathbb{R} \to \mathbb{C}$ with *bounded variation*.

Definition 4.5.1. For a function $F : \mathbb{R} \to \mathbb{C}$, define its *total variation* $T_F : \mathbb{R} \to [0, \infty]$ by

$$T_F(x) := \sup\left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| \,\middle|\, n \in \mathbb{N} \text{ and } -\infty < x_0 < x_1 < \cdots < x_n = x \right\}.$$

Observe that T_F is a non-decreasing function. We say F has *bounded variation* if T_F is bounded, which is equivalent to $\lim_{x\to\infty} T_F(x) < \infty$. We define

$$\mathsf{BV} := \{F : \mathbb{R} \to \mathbb{C} | F \text{ has bounded variation}\}.$$

Exercise 4.5.2. Prove that for all $a, b \in \mathbb{R}$ with $a < b$ and $F : \mathbb{R} \to \mathbb{C}$,

$$T_F(b) = T_F(a) + \sup\left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| \,\middle|\, n \in \mathbb{N} \text{ and } a = x_0 < x_1 < \cdots < x_n = b \right\}.$$

The sup on the right hand side is called the *total variation of F on* $[a, b]$. We say F has bounded variation on $[a, b]$ if this number is bounded.

Exercise 4.5.3. Show that if F is differentiable and F' is bounded, then $F \in \mathsf{BV}[a, b]$ for all $a < b$ in $\mathbb{R}$.

Facts 4.5.4. Here are some facts about functions with bounded variation.

(BV1) If $F : \mathbb{R} \to \mathbb{R}$ is increasing, then $F \in \mathsf{BV}$ if and only if F is bounded.

Proof. For any $-\infty < x_0 < x_1 < \cdots < x_n = x$,

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = F(x) - F(x_0).$$

Hence T_F is bounded if and only if F is bounded. $\square$

(BV2) $F \in \mathsf{BV}$ if and only if $T_F \in \mathsf{BV}$.

Proof. If $F \in \mathsf{BV}$, then $T_F : \mathbb{R} \to [0, \infty]$ is increasing and bounded, and thus in BV by (BV1). Conversely, if $T_F \in \mathsf{BV}$, then T_F is bounded by (BV1), and thus $F \in \mathsf{BV}$. $\square$

(BV3) BV is a complex vector space which is closed under complex conjugation.

Proof. The triangle inequality implies $T_{F+G} \le T_F + T_G$, homogeneity ($|wz| = |w| \cdot |z|$) implies $T_{zF} \le |z| \cdot T_F$, and $|\overline{z}| = |z|$ for $z \in \mathbb{C}$ implies $T_{\overline{F}} = T_F$. $\square$

(BV4) $F \in \mathsf{BV}$ if and only if $\mathrm{Re}(F), \mathrm{Im}(F) \in \mathsf{BV}$.

Proof. Just observe that $\mathrm{Re}(F) = \frac{1}{2}(F + \overline{F})$ and $\mathrm{Im}(F) = \frac{1}{2i}(F - \overline{F})$, so the result follows from (BV3). $\square$

(BV5) If $F : \mathbb{R} \to \mathbb{R}$ and $F \in \mathsf{BV}$, then $T_F \pm F$ are increasing (and in BV).

Proof. Suppose $a < b$ in $\mathbb{R}$. Let $\varepsilon > 0$, and choose $x_0 < x_1 < \cdots < x_n = a$ such that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \ge T_F(a) - \varepsilon.$$

Then since $F(b) = (F(b) - F(a)) + F(a)$,

$$\begin{aligned}
T_F(b) \pm F(b) &\ge \underbrace{\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(b) - F(a)|}_{\le T_F(b)} \pm F(b) \\
&= \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \underbrace{|F(b) - F(a)| \pm (F(b) - F(a))}_{\ge 0} \pm F(a) \\
&\ge T_F(a) - \varepsilon \pm F(a)
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $T_F \pm F$ is increasing. (The parenthetical follows from (BV3).) $\square$

(BV6) If $F : \mathbb{R} \to \mathbb{C}$, then $F \in \mathsf{BV}$ if and only if $F = \sum_{k=0}^{3} i^k F_k$ where $F_k : \mathbb{R} \to \mathbb{R}$ is bounded and increasing for $k = 0, 1, 2, 3$.

Proof. By (BV4), $F \in \mathsf{BV}$ if and only if $\operatorname{Re}(F), \operatorname{Im}(F) \in \mathsf{BV}$, so we may assume $F : \mathbb{R} \to \mathbb{R}$. If $F \in \mathsf{BV}$, just observe

$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F).$$

The converse follows from (BV1) and (BV3). □

(BV7) If $F \in \mathsf{BV}$, then $F(x+) := \lim_{y \searrow x} F(y)$ and $F(x-) := \lim_{y \nearrow x} F(y)$ exist for all $x \in \mathbb{R}$, as do $F(\pm\infty) := \lim_{y \to \pm\infty} F(y)$.

Proof. This follows from (BV6). □

Remark 4.5.5. For an $\mathbb{R}$-valued $F \in \mathsf{BV}$, we call

$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$$

the Jordan decomposition of F. We call $\frac{1}{2}(T_F \pm F)$ the positive/negative variations of F respectively.

Definition 4.5.6. The space of *normalized* functions of bounded variation is

$$\mathsf{NBV} := \{F \in \mathsf{BV} | F \text{ is right continuous and } F(-\infty) = 0\}.$$

Observe that NBV is a complex vector subspace of BV closed under complex conjugation.

Exercise 4.5.7. Suppose $f \in L^1(\lambda)$ where λ is Lebesgue measure on $\mathbb{R}$. Consider the function $F : \mathbb{R} \to \mathbb{C}$ by $F(x) = \int_{-\infty}^{x} f(t)\, dt$.

(1) Prove directly from the definitions that $F \in \mathsf{NBV}$.
(2) Describe T_F to the best of your ability. Justify your answer.

Lemma 4.5.8. *Suppose $F : \mathbb{R} \to \mathbb{C}$.*

(1) If $F \in \mathsf{BV}$, then $T_F(-\infty) = 0$.
(2) If moreover F is right-continuous, then so is T_F.

Hence $F \in \mathsf{NBV}$ implies $T_F \in \mathsf{NBV}$.

Proof.

(1) Let $\varepsilon > 0$. For $x \in \mathbb{R}$, choose $x_0 < x_1 < \cdots < x_n = x$ such that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \geq T_F(x) - \varepsilon.$$

By Exercise 4.5.2

$$T_F(x) - T_F(x_0) \geq T_F(x) - \varepsilon,$$

and thus $T_F(y) \leq \varepsilon$ for all $y \leq x_0$. Since $\varepsilon > 0$ was arbitrary, $T_F(-\infty) = 0$.

(2) Now suppose F is right continuous. Fix $x \in \mathbb{R}$, and define

$$\alpha := \lim_{y \searrow x} T_F(y) - T_F(x).$$

To show $\alpha = 0$, fix $\varepsilon > 0$, and let $\delta > 0$ such that $0 < h < \delta$ implies both $|F(x+h)-F(x)| < \varepsilon$ and

$$T_F(x+h) - T_F(x) - \alpha = T_F(x+h) - \lim_{y \searrow x} T_F(y) < \varepsilon. \tag{4.5.9}$$

Now fixing $0 < h < \delta$, by Exercise 4.5.2, there are $x = x_0 < x_1 < \cdots < x_n = x+h$ such that

$$\frac{3}{4}\alpha \leq \frac{3}{4}(T_F(x+h) - T_F(x)) \leq \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|$$

which by right continuity of F and the choice of δ implies

$$\frac{3}{4}\alpha - \varepsilon \leq \frac{3}{4}(T_F(x+h) - T_F(x)) - |F(x_1) - F(x_0)| \leq \sum_{j=2}^{n} |F(x_j) - F(x_{j-1})|. \tag{4.5.10}$$

Again using Exercise 4.5.2,

$$\frac{3}{4}\alpha \leq \frac{3}{4}(T_F(x_1) - T_F(x)) \leq \sum_{i=1}^{k} |F(t_i) - F(t_{i-1})|. \tag{4.5.11}$$

Combining these inequalities, we have

$$\begin{aligned}
\alpha + \varepsilon &> T_F(x+h) - T_F(x) && \text{by (4.5.9)}\\
&\geq \sum_{i=1}^{k} |F(t_i) - F(t_{i-1})| + \sum_{j=2}^{n} |F(x_j) - F(x_{j-1})| && \text{by Exercise 4.5.2}\\
&\geq \frac{3}{4}\alpha + \frac{3}{4}\alpha - \varepsilon && \text{by (4.5.10) and (4.5.11)}\\
&= \frac{3}{2}\alpha - \varepsilon.
\end{aligned}$$

This implies $\alpha \leq 4\varepsilon$, but since $\varepsilon > 0$ was arbitrary, $\alpha = 0$. □

Theorem 4.5.12.

(1) *If ν is a complex Borel measure on $\mathbb{R}$, then $F_\nu(x) := \nu((-\infty, x])$ defines a function in* NBV.

(2) *If $F \in$ NBV, there is a unique complex Borel measure ν_F such that $F(x) = \nu_F((-\infty, x])$.*

Proof. For a complex Borel measure ν, we have $\nu = \sum_{k=0}^{3} i^k \nu_k$ where each ν_k is a finite positive measure. If we set $F_k := \nu_k((-\infty, x])$, then F_k is increasing and right continuous, $F_k(-\infty) = 0$, and $F_k(\infty) = \nu_k(\mathbb{R}) < \infty$. Thus each $F_k \in$ NBV, and thus $F_\nu := \sum_{k=0}^{3} i^k F_k$ is in NBV.

Conversely, by (BV6) and Lemma 4.5.8, any $F \in$ NBV can be written as $F = \sum_{k=0}^{3} i^k F_k$ where each $F_k : \mathbb{R} \to [0, \infty)$ is increasing and in NBV. By the Lebesgue-Stieltjes construction, for each F_k, there is a finite regular Borel measure ν_k on $\mathbb{R}$ with $\nu_k((-\infty, x]) = F_k(x)$. Setting $\nu := \sum_{k=0}^{3} i^k \nu_k$ gives a complex Borel measure such that $F(x) = \nu((-\infty, x])$. Uniqueness follows by being determined on h-intervals together with the $\pi - \lambda$ Theorem. □

Exercise 4.5.13. Suppose $F \in$ NBV, and let ν_F be the corresponding complex Borel measure from Theorem 4.5.12.

(1) Prove that ν_F is regular.

(2) Prove that $|\nu_F| = \nu_{T_F}$.

One could proceed as follows.

(a) *Define* $G(x) := |\nu_F|((-\infty, x])$. *Show that* $|\nu_F| = \nu_{T_F}$ *if and only if* $G = T_F$.

(b) *Show* $T_F \leq G$.

(c) *Show that* $|\nu_F(E)| \leq \nu_{T_F}(E)$ *whenever* E *is an interval.*

(d) *Show that* $|\nu_F| \leq \nu_{T_F}$.

Exercise 4.5.14. Show that if $F \in \mathsf{NBV}$, then $(\nu_F)_\pm = \nu_{\frac{1}{2}(T_F \pm F)}$, i.e., the positive/negative variations of F exactly correspond to the positive/negative parts of the Jordan decomposition of ν_F.
Hint: Use Exercise 4.5.13.

4.6. **Bounded variation, differentiation, and absolute continuity.** We now want to connect functions of bounded variation and ordinary differentiation on $\mathbb{R}$.

Definition 4.6.1. Recall that $F : \mathbb{R} \to \mathbb{C}$ is called *absolutely continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any finite set of disjoint open intervals $(a_1, b_1), \dots, (a_n, b_n)$,

$$\sum_{i=1}^n (b_i - a_i) < \delta \qquad \Longrightarrow \qquad \sum_{i=1}^n |F(b_i) - F(a_i)| < \varepsilon.$$

Exercise 4.6.2. Suppose $F \in \mathsf{NBV}$. Show F is absolutely continuous if and only if T_F is absolutely continuous.
Hint: Use Exercise 4.5.2.

Proposition 4.6.3. *If* $F \in \mathsf{NBV}$, *then* F *is absolutely continuous if and only if* $\nu_F \ll \lambda$.

Proof.

Claim. *We may assume* F *is* $[0, \infty)$*-valued and increasing. Thus* $\nu_F = \mu_F$ *is an honest Lebesgue-Stieltjes measure.*

Proof. By Exercises 4.3.13 and 4.5.13(2), $\nu_F \ll \lambda$ if and only if $|\nu_F| = \nu_{T_F} \ll \lambda$. By Exercise 4.6.2, F is absolutely continuous if and only if T_F is absolutely continuous. Hence we may replace F with $T_F \in \mathsf{NBV}$ which is $[0, \infty)$-valued and increasing. ☐

That $\mu_F \ll \lambda$ for a Lebesgue-Stieltjes measure is equivalent to absolute continuity of a bounded, right-continuous $F : \mathbb{R} \to [0, \infty)$ with $F(-\infty) = 0$ now follows Exercise 2.5.20. We provide a proof here for completeness and convenience using Proposition 4.2.6 which states:

- $\mu_F \ll \lambda$ if and only if for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $E \in \mathcal{M}$, $\mu_F(E) < \varepsilon$ whenever $\lambda(E) < \delta$.

First, suppose $\mu_F \ll \lambda$. For any finite set of disjoint h-intervals $((a_i, b_i])_{i=1}^n$, we have

$$\sum_{i=1} (b_i - a_i) = \lambda\left(\coprod (a_i, b_i]\right) < \delta \qquad \Longrightarrow \qquad \mu_F\left(\coprod (a_i, b_i]\right) = \sum_{i=1}^n \mu_F((a_i, b_i]) < \varepsilon.$$

This immediately implies F is absolutely continuous.

Conversely, suppose F is absolutely continuous, and $\varepsilon > 0$. Pick $\delta > 0$ for F as in the definition of absolute continuity for any $0 < \varepsilon' < \varepsilon$. Suppose $E \in \mathcal{L}$ such that $\lambda(E) < \delta$. By outer regularity of λ and μ_F (by Exercise 4.5.13(1)), there is an open set U with $E \subset U$ such that $\lambda(U) < \delta$. Then U is a countable disjoint union of open intervals by Exercise 1.1.24, say $U = \coprod(a_i, b_i)$. For each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(b_i - a_i) \leq \lambda(U) < \delta \qquad \Longrightarrow \qquad \sum_{i=1}^{n}\mu_F((a_i, b_i]) = \sum_{i=1}^{n} F(b_i) - F(a_i) < \varepsilon'.$$

Taking the limit as $n \to \infty$, we have

$$\sum_{i=1}^{\infty}(b_i - a_i) \leq \lambda(U) < \delta \qquad \Longrightarrow \qquad \sum_{i=1}^{\infty}\mu_F((a_i, b_i]) = \sum_{i=1}^{\infty} F(b_i) - F(a_i) \leq \varepsilon' < \varepsilon.$$

Hence $\mu_F \ll \lambda$. □

Exercise 4.6.4. Prove that if $F : [a,b] \to \mathbb{C}$ with $a, b \in \mathbb{R}$ is absolutely continuous, then $F \in \mathsf{BV}[a,b]$.

Exercise 4.6.5 (cf. Folland Thm. 3.22)**.** Denote by λ^n Lebesgue measure on $\mathbb{R}^n$. Suppose ν is a regular signed or complex Borel measure on $\mathbb{R}^n$ which is finite on compact sets (and thus Radon and σ-finite). Let $d\nu = d\rho + f d\lambda^n$ be its Lebesgue-Radon-Nikodym representation from Theorem 4.3.9. Then for λ^n-a.e. $x \in \mathbb{R}^n$,

$$\lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{\nu(Q)}{\lambda^n(Q)} = f(x).$$

Hint: One could proceed as follows.

(1) Show that $d|\nu| = d|\rho| + |f|d\lambda^n$. Deduce that ρ and $f d\lambda^n$ are regular, and $f \in L^1_{\text{loc}}$.
(2) Use the Lebesgue Differentiation Theorem to reduce the problem to showing

$$\lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{|\rho|(Q)}{\lambda^n(Q)} = 0 \qquad\qquad \lambda^n\text{-a.e. } x \in \mathbb{R}^n.$$

Thus we may assume ρ is positive.
(3) Since $\rho \perp \lambda^n$, pick $P \subset \mathbb{R}^n$ Borel measurable such that $\rho(P) = \lambda^n(P^c) = 0$. For $a > 0$, define

$$E_a := \left\{ x \in P \,\middle|\, \lim_{\substack{\ell(Q)\to 0\\ Q\in\mathcal{C}(x)}} \frac{|\rho|(Q)}{\lambda^n(Q)} > a \right\}.$$

Let $\varepsilon > 0$. Since ρ is regular, there is an open $U_\varepsilon \supset P$ such that $\rho(U_\varepsilon) < \varepsilon$. Adapt the proof of the HLMT to show there is a constant $c > 0$, depending only on n, such that for all $a > 0$,

$$\lambda^n(E_a) \leq c \cdot \frac{\rho(U_\varepsilon)}{a} = c \cdot \frac{\varepsilon}{a}$$

(Choose your family of cubes to be contained in U_ε.) Deduce that $\lambda^n(E_a) = 0$.

Lemma 4.6.6. *Suppose that $F : \mathbb{R} \to \mathbb{R}$ is increasing or $F \in \mathsf{BV}$.*

(1) The set of points at which F is discontinuous is countable.

(2) Suppose in addition F is right continuous. Let μ_F be the corresponding (regular, σ-finite) Lebesgue-Stieltjes measure, and let $d\lambda = d\rho + f d\lambda$ be its Lebesgue-Radon-Nikodym representation from Theorem 4.3.9. Then F is differentiable λ-a.e. with $F'(x) = f(x)$ λ-a.e.

(3) Setting $G(x) := \lim_{y \searrow x} F(y)$, F and G are differentiable a.e., with $F' = G'$ a.e.

Proof. Since every $F \in \mathsf{BV}$ is a linear combination of four increasing, bounded functions $\mathbb{R} \to \mathbb{R}$ by (BV6), we may assume $F : \mathbb{R} \to \mathbb{R}$ is an arbitrary increasing function.

(1) Observe that at every discontinuity $x \in \mathbb{R}$, the open interval

$$\left(\lim_{y \nearrow x} F(y), \lim_{y \searrow x} F(y)\right) \neq \emptyset$$

and thus contains a rational point. Since F is increasing, these open intervals at distinct discontinuities will be disjoint, and we can construct an injective mapping from the set of discontinuities to $\mathbb{Q}$.

(2) Suppose in addition that F is right-continuous. Let $D \subset \mathbb{R}$ be the countable set of discontinuities of F, and observe that $\lambda(D) = 0$. By Exercise 4.6.5,

$$\lim_{\substack{\ell(Q) \to 0 \\ Q \in \mathcal{C}(x)}} \frac{\mu_F(Q)}{\lambda(Q)} = f(x) \qquad \lambda\text{-a.e. } x \in \mathbb{R}.$$

Now observe that for $x \notin D$ and $h > 0$, by Exercise 2.5.9,

$$\mu_F([x, x+h]) = \lim_{y \nearrow x} \mu_F((y, x+h]) = \lim_{y \nearrow x} F(x+h) - F(y) = F(x+h) - F(x)$$

If in addition $x - h \notin D$, then we also have

$$\mu_F([x-h, x]) = F(x) - F(x-h).$$

Since D is countable and F is increasing, we may take the following limit for $x \in D^c$ along $h \to 0$ such that $x - |h| \notin D$ to conclude that

$$\begin{aligned}\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} &= \lim_{\substack{h \to 0 \\ x - |h| \notin D}} \frac{\mu_F([\min\{x, x+h\}, \max\{x, x+h\}])}{\lambda([\min\{x, x+h\}, \max\{x, x+h\}])} \\ &= f(x) \qquad \lambda\text{-a.e. } x \in D^c \text{ by Exercise 4.6.5.}\end{aligned}$$

(3), Step 1: G is increasing and right-continuous, and thus G is differentiable a.e. by (2).

If $a < b$ in $\mathbb{R}$, then since F is increasing,

$$G(a) = \lim_{x \searrow a} F(x) = \lim_{\substack{x \searrow a \\ a < x < b}} F(x) \leq F(b) \leq G(b),$$

and thus G is increasing. To show G is right continuous at $x \in \mathbb{R}$, let $\varepsilon > 0$. Since $G(x) = \lim_{y \searrow x} F(y)$, we can pick $\delta' > 0$ such that $0 < h' < \delta'$ implies $F(x+h') - G(x) < \varepsilon$. Then for any $0 \leq h < \delta < h' < \delta'$,

$$G(x+h) - G(x) \leq F(x+h') - G(x) < \varepsilon.$$

(2), Step 2: Setting $H := G - F \geq 0$, H' exists and is zero a.e.

First, note $H(d) > 0$ for all $d \in D$, and

$$\sum_{\substack{d \in D \\ |d|<N}} H(d) = \sum_{\substack{d \in D \\ |d|<N}} G(d) - F(d) \leq G(N) - F(N) < \infty. \tag{4.6.7}$$

Claim. *Setting $\eta := \sum_{d \in D} H(d)\delta_d$ where δ_d is the Dirac point mass at d, η is a regular Borel measure such that $\eta \perp \lambda$.*

Proof. Observe η is finite on compact sets by (4.6.7). We define $h : \mathbb{R} \to \mathbb{R}$ by picking an arbitrary $r_0 \in D^c$, setting $h(r_0) = 0$, and setting

$$h(r) := \begin{cases} \sum_{\substack{d \in D \\ r_0 < d \leq r}} H(d) & \text{if } r > r_0 \\ -\sum_{\substack{d \in D \\ r < d < r_0}} H(d) & \text{if } r < r_0. \end{cases}$$

Observe that h is increasing and right-continuous, and by construction, the Lebesgue-Stieltjes measure $\mu_h = \eta$, which is thus regular. Since η is supported on D and $\lambda(D) = 0$, we have $\eta \perp \lambda$. □

Now for $|h| \neq 0$, again by Exercise 4.6.5,

$$\left|\frac{H(x+h) - H(x)}{h}\right| \leq \frac{H(x+h) + H(x)}{|h|} \leq 2\frac{\eta([x-|h|, x+|h|])}{\lambda([x-|h|, x+|h|])} \xrightarrow{h \to 0} 0 \quad \text{a.e. } x \in \mathbb{R}.$$

We conclude that $H' = 0$ a.e.

This concludes the proof. □

Facts 4.6.8. Suppose $F \in \mathsf{NBV}$, and let $\nu_F = \rho_F + f d\lambda$ where $f \in L^1(\lambda)$ be the Lebesgue-Radon-Nikodym Representation of ν_F from Theorem 4.3.9.

($\mathsf{NBV}'1$) F' exists λ-a.e. with $F' = f \in L^1(\lambda)$.

Proof. By (BV6), $F = \sum_{k=0}^{3} i^k F_k$ where each $F_k : \mathbb{R} \to \mathbb{R}$ is an increasing right-continuous function in NBV. Let $\mu_{F_k} = \rho_{F_k} + f_k d\lambda$ where $f_k \in L^1(\lambda)$ for $k = 0, 1, 2, 3$ be the Lebesgue-Radon-Nikodym representation of the Lebesgue-Stieltjes measure μ_{F_k} from Theorem 4.2.9. By Lemma 4.6.6(2), F_k' exists λ-a.e., and $F_k' = f_k$ λ-a.e. By the proof of the Complex Lebesgue-Radon-Nikodym Theorem 4.3.9, we have $f = \sum_{k=0}^{3} i^k f_k$. Hence

$$F' = \sum_{k=0}^{3} i^k F_k' = \sum_{k=0}^{3} i^k f_k = f \qquad \lambda\text{-a.e.}$$ □

($\mathsf{NBV}'2$) $\nu_F \perp \lambda$ if and only if $F' = 0$ a.e.

Proof. This follows immediately from ($\mathsf{NBV}'1$) and the Lebesgue-Radon-Nikodym Representation of ν_F. □

(NBV′3) $\nu_F \ll \lambda$ if and only if $F(x) = \int_{-\infty}^x F'(t)\,dt$.

Proof. Observe $\nu_F \ll \lambda$ if and only if $\rho_F = 0$ if and only if $d\nu_F = F'd\lambda$ by (NBV′1). This last condition is equivalent to

$$F(x) = \nu_F((-\infty, x]) = \int_{-\infty}^x F'(t)\,dt. \qquad \square$$

Proposition 4.6.9. *The following are equivalent for $F : \mathbb{R} \to \mathbb{C}$.*

(1) $F \in \mathsf{NBV}$ is absolutely continuous.
(2) F is differentiable a.e., $F' \in L^1(\lambda)$, and $F(x) = \int_{-\infty}^x F'(t)\,dt$.
(3) There is an $f \in L^1(\lambda)$ such that $F(x) = \int_{-\infty}^x f(t)\,dt$.

Proof.
(1) ⇒ (2): If $F \in \mathsf{NBV}$ is absolutely continuous, then $\nu_F \ll \lambda$ by Proposition 4.6.3. By (NBV′1), F is differentiable a.e. with $F' \in L^1(\lambda)$, and by (NBV′3), $F(x) = \int_{-\infty}^x F(t)\,dt$.
(2) ⇒ (3): Trivial.
(3) ⇒ (1): Since $f \in L^1(\lambda)$, $d\nu := f d\lambda$ is a complex Borel measure. Thus

$$F(x) = \int_{-\infty}^x f(t)\,dt = \nu((-\infty, x])$$

defines a function in NBV by Theorem 4.5.12(1). Since $\nu \ll \lambda$ by construction, F is absolutely continuous by Proposition 4.6.3. □

We leave the proof of the following corollary to the reader.

Corollary 4.6.10 (Fundamental Theorem of Calculus for Lebesgue Integrals)**.** *Let $a, b \in \mathbb{R}$ with $a < b$, and suppose $F : [a, b] \to \mathbb{C}$. The following are equivalent.*

(1) F is absolutely continuous on $[a, b]$.
(2) F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], \lambda)$, and $F(x) - F(a) = \int_a^x F'(t)\,dt$.
(3) $F(x) - F(a) = \int_a^x f(t)\,dt$ for some $f \in L^1([a, b], \lambda)$.

Exercise 4.6.11 (Folland §3.5, #37)**.** Show that $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous (there is an $M > 0$ such that $|F(x) - F(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$) if and only if F is absolutely continuous and $|F'| \le M$ a.e.