Let X be an LCH space.

Goal: Compute $C_0(X)^*$

Recall: A Borel measure μ on X is called outer regular on E if

$$\mu(E) = \inf\{\mu(U) \mid E \subset U \text{ open}\}$$

and inner regular on E if

$$\mu(E) = \sup\{\mu(K) \mid E \supset K \text{ cpt}\}.$$

If μ is both inner + outer reg. on all Borel sets, call μ regular. Example: Lebesgue–Stieltjes measures on $\mathbb{R}$.

Note: If X is not σ-cpt [$X = \bigcup X_n$ w/ X_n cpt $\forall n$], regularity is too strong.

Def: A Radon measure on X is a Borel measure which is

- finite on all cpt sets $K \subset X$,
- outer regular on all Borel sets, and
- inner regular on open sets.

We'll see later that a Radon measure is inner regular on σ-finite sets.

Consider $C_c(X)$, cts fcts of cpt support. A Radon integral on X is a positive linear functional $\varphi : C_c(X) \to \mathbb{C}$, i.e.,

$$\varphi(f) \geq 0 \quad \forall f \in C_c(X) \text{ s.t. } f \geq 0.$$

Lemma: Radon integrals are bdd on cpt subsets, i.e., $\forall K \subset X$ cpt, $\exists C_K > 0$ s.t. $\forall f \in C_c(X)$ w/ $\operatorname{supp}(f) \subset K$, $|\varphi(f)| \leq C_K \|f\|_\infty$.

Pf: By taking Re + Im parts, we may assume f is $\mathbb{R}$-valued. Let $K \subset X$ be cpt and choose $g \in C_c(X)$ s.t. $g = 1$ on K by LCH Urysohn's Lemma. If $\operatorname{supp}(f) \subset K$, $|f| \leq \|f\|_\infty g$ on X. Then $\|f\|_\infty g - |f| \geq 0$, so $\|f\|_\infty g \pm f \geq 0$. Thus $\|f\|_\infty \varphi(g) \pm \varphi(f) \geq 0 \Rightarrow |\varphi(f)| \leq \underbrace{\varphi(g)}_{C_K} \|f\|_\infty$

Thm (Riesz Representation): $\forall$ Radon integral φ on $\mathfrak{X}$, $\exists!$ Radon measure μ_φ on $\mathfrak{X}$ s.t. $\varphi(f) = \int f \, d\mu_\varphi$ $\forall f \in C_c(\mathfrak{X})$.

Moreover, μ_φ satisfies:

$0 \le f \le 1$ and $\overline{\operatorname{supp}(f)} \subset \mathcal{U}$

a) $\mu_\varphi(\mathcal{U}) = \sup\{\varphi(f) \mid f \in C_c(\mathfrak{X}),\ f \prec \mathcal{U}\}$ $\forall$ open $\mathcal{U}$

b) $\mu_\varphi(K) = \inf\{\varphi(f) \mid f \in C_c(\mathfrak{X}),\ f \ge \chi_K\}$ $\forall$ cpt K.

Pf: Uniqueness: If μ is a Radon measure s.t. $\varphi(f) = \int f \, d\mu$ $\forall f \in C_c(\mathfrak{X})$ and $\mathcal{U} \subset \mathfrak{X}$ open, then $\varphi(f) \le \mu(\mathcal{U})$ $\forall f \prec \mathcal{U}$. If $K \subset \mathcal{U}$ cpt, by LCH Urysohn, $\exists f \in C_c(\mathfrak{X})$ s.t. $f \prec \mathcal{U}$ and $f|_K = 1$. So $\mu(K) \le \int f \, d\mu = \varphi(f)$. Since μ is inner reg on $\mathcal{U}$, a) is satisfied. So μ is determined by φ on open sets, and thus on all Borel sets by outer regularity of μ.

Existence: For $\mathcal{U} \subset \mathfrak{X}$ open, define $\mu(\mathcal{U}) := \sup\{\varphi(f) \mid f \prec \mathcal{U}\}$ and $\mu^*(E) := \sup\{\mu(\mathcal{U}) \mid E \subset \mathcal{U} \text{ open}\}$ $\forall$ Borel E.

Outline of proof:

Step 1: μ^* is an outer measure.

Step 2: Every open set is μ^*-measurable.

$\Rightarrow$ By Carathéodory, $\mathcal{B}_{\mathfrak{X}} \subset \mathcal{M}^*$ and $\mu_\varphi := \mu^*|_{\mathcal{B}_{\mathfrak{X}}}$ is a Borel meas.

By defn, μ_φ is outer regular and satisfies a).

Step 3: μ_φ satisfies b)

$\Rightarrow \mu_\varphi$ finite on cpt sets + inner reg on open sets [If $\mathcal{U} \subset \mathfrak{X}$ open and $\alpha < \mu(\mathcal{U})$, choose $f \in C_c(\mathfrak{X})$ s.t. $f \prec \mathcal{U}$ and $\varphi(f) > \alpha$. Let $K := \overline{\operatorname{supp}(f)}$. $\forall g \in C_c(\mathfrak{X})$ w/ $g \ge \chi_K$, $g - f \ge 0$, so $\varphi(g) \ge \varphi(f) > \alpha$. Since b) holds, $\mu(K) > \alpha$, so μ is inner reg on $\mathcal{U}$.] So μ_φ Radon.

Step 4: $\varphi(f) = \int f \, d\mu_\varphi$ $\forall f \in C_c(\mathfrak{X})$.

Step 1: $\mu^*(E) := \inf\{\mu(\mathcal{U}) \mid E \subset \mathcal{U} \text{ open}\}$ is an outer measure.

Pf: It suffices to prove if $(\mathcal{U}_n)$ a seq. of open sets, then $\mu^*(\bigcup \mathcal{U}_n) \leq \sum \mu^*(\mathcal{U}_n)$. This shows that

$$\mu^*(E) = \inf\{\textstyle\sum \mu(\mathcal{U}_n) \mid \mathcal{U}_n \text{ open and } E \subset \bigcup \mathcal{U}_n\},$$

and we know the RHS defines an outer measure. Now if $f \in C_c(X)$ w/ $f \prec \bigcup \mathcal{U}_n$, let $K := \overline{\operatorname{supp}(f)}$. Since K cpt, $K \subset \bigcup_1^N \mathcal{U}_n$ for some $N \in \mathbb{N}$.

$\boxed{\text{Exercise: } \exists g_1, \ldots, g_N \in C_c(X) \text{ s.t. } g_i \prec \mathcal{U}_i \text{ and } \sum^N g_i = 1 \text{ on } K.}$

Then $f = f \sum_{i=1}^N g_i$ w/ $fg_i \prec \mathcal{U}_i$, so

$$\varphi(f) = \sum_1^N \varphi(fg_i) \leq \sum_1^N \varphi(\chi_{\mathcal{U}_i}) = \sum_1^N \mu(\mathcal{U}_i) \leq \sum \mu(\mathcal{U}_i).$$

Since $f \prec \mathcal{U}$ arbitrary, $\mu(\mathcal{U}) = \sup\{\varphi(f) \mid f \prec \mathcal{U}\} \leq \sum \mu(\mathcal{U}_i)$.

Step 2: Every open set is μ^*-measurable.

Pf: Let $\mathcal{U}$ be open and $E \subset X$ s.t. $\mu^*(E) < \infty$. We must show $\mu^*(E) \geq \mu^*(E \cap \mathcal{U}) + \mu^*(E \setminus \mathcal{U})$.

- If E open, $E \cap \mathcal{U}$ open. Given $\varepsilon > 0$, $\exists f \prec E \cap \mathcal{U}$ s.t. $\varphi(f) > \mu(E \cap \mathcal{U}) - \frac{\varepsilon}{2}$. Since $E \setminus \overline{\operatorname{supp}(f)}$ is open, $\exists g \prec E \setminus \overline{\operatorname{supp}(f)}$ s.t. $\varphi(g) > \mu(E \setminus \overline{\operatorname{supp}(f)}) - \frac{\varepsilon}{2}$. Then $f+g \prec E$, so

$$\begin{aligned}\mu(E) &\geq \varphi(f) + \varphi(g) \\ &> \mu(E \cap \mathcal{U}) + \mu(E \setminus \overline{\operatorname{supp}(f)}) - \varepsilon \\ &\geq \mu^*(E \cap \mathcal{U}) + \mu^*(E \setminus \mathcal{U}) - \varepsilon.\end{aligned}$$

Taking limit as $\varepsilon \to 0$ gives the inequality.

- For general E, $\exists$ open $V \supset E$ s.t. $\mu(V) < \mu^*(E) + \varepsilon$, so

$$\begin{aligned}\mu^*(E) + \varepsilon &> \mu(V) \\ &\geq \mu^*(V \cap \mathcal{U}) + \mu^*(V \setminus \mathcal{U}) \\ &\geq \mu^*(E \cap \mathcal{U}) + \mu^*(E \setminus \mathcal{U}).\end{aligned}$$

Again, let $\varepsilon \to 0$ to get the result.

Step 3: μ_φ satisfies b)

Pf: For $K \subset X$ cpt and $f \geq \chi_K$, set $\mathcal{U}_\varepsilon := \{f > 1-\varepsilon\}$, open.
If $g \prec \mathcal{U}_\varepsilon$, $(1-\varepsilon)^{-1} f - g \geq 0 \Rightarrow \varphi(g) \leq (1-\varepsilon)^{-1}\varphi(f)$. Hence

$$\mu_\varphi(K) \leq \mu_\varphi(\mathcal{U}_\varepsilon) \leq (1-\varepsilon)^{-1}\varphi(f).$$

$\uparrow$ taking $\sup\{\varphi(g) \mid g \prec \mathcal{U}_\varepsilon\}$

Letting $\varepsilon \to 0$, $\mu_\varphi(K) \leq \varphi(f)$.
But $\forall$ open $\mathcal{U} \supset K$, $\exists f \prec \mathcal{U}$ s.t. $f \geq \chi_K$ by LCH Urysohn, and $\varphi(f) \leq \mu_\varphi(\mathcal{U})$. Since μ_φ is outer regular on K,

$$\begin{aligned}\mu_\varphi(K) &= \inf\{\mu_\varphi(\mathcal{U}) \mid K \subset \mathcal{U} \text{ open}\} \\ &= \inf\{\varphi(f) \mid f \geq \chi_K\}.\end{aligned}$$

Step 4: $\varphi(f) = \int f \, d\mu$ $\forall f \in C_c(X)$.

Pf: we may assume $f \in C_c(X, [0,1])$ as this set spans $C_c(X)$.
Fix $N \in \mathbb{N}$, and set $\underbrace{K_j = \{f \geq j/N\}}$ and $K_0 = \overline{\text{supp}(f)}$. Define $\underbrace{f_j}_{1 \leq j \leq N}$ by

$\emptyset = K_{N+1} \subset K_N \subset \cdots \subset K_1 \subset K_0 = \overline{\text{supp}(f)}$.

$$f_j := \left[f - \tfrac{j-1}{N} \vee 0\right] \wedge \tfrac{1}{N} \iff f_j(x) = \begin{cases} 0 & x \notin K_{j-1} \\ f(x) - \frac{j-1}{N} & x \in K_{j-1} \setminus K_j \\ N^{-1} & x \in K_j \end{cases}$$

Observe: $\frac{\chi_{K_j}}{N} \leq f_j \leq \frac{\chi_{K_{j-1}}}{N}$ $\forall j$ and $\sum_1^N f_j = f$.

This means: ① $\frac{1}{N}\mu_\varphi(K_j) \leq \int f_j \, d\mu_\varphi \leq \frac{1}{N}\mu(K_{j-1})$.
$\forall$ open $\mathcal{U} \supset K_{j-1}$, $Nf_j \prec \mathcal{U} \Rightarrow \varphi(f_j) \leq \frac{1}{N}\mu_\varphi(\mathcal{U})$. By b) + outer reg,

$\frac{1}{N}\mu_\varphi(K_j) \leq \varphi(f_j) \leq \frac{1}{N}\mu_\varphi(K_{j-1})$. Now

$$\frac{1}{N}\sum_1^N \mu_\varphi(K_j) \leq \int f \, d\mu_\varphi \leq \frac{1}{N}\sum_0^{N-1}\mu_\varphi(K_j) \quad \text{and}$$

$$\frac{1}{N}\sum_1^N \mu_\varphi(K_j) \leq \varphi(f) \leq \frac{1}{N}\sum_0^{N-1}\mu_\varphi(K_j)$$

$$\Rightarrow \left|\varphi(f) - \int f \, d\mu_\varphi\right| \leq \frac{\mu_\varphi(K_0) - \mu_\varphi(K_N)}{N} \leq \frac{\mu_\varphi(\overline{\text{supp}(f)})}{N}.$$

$< \infty$; N arbitrary $\Big\} \Rightarrow$ result!

Properties of Radon measures: Let μ be a Radon meas. on X. + LCH

① If $E \subset X$ σ-finite, μ is inner regular on E.

$\Rightarrow$ Every σ-finite Radon measure is regular

$\Rightarrow$ If X σ-cpt, every Radon measure is σ-finite $\Rightarrow$ regular.

Pf: If $\mu(E)<\infty$, $\forall \varepsilon>0$, $\exists$ open $U \supset E$ s.t. $\mu(U) \overset{①}{<} \mu(E)+\frac{\varepsilon}{2}$ and a cpt $F \subset U$ s.t. $\mu(F) \overset{②}{>} \mu(U)-\frac{\varepsilon}{2}$. Since $\mu(U \setminus E)<\frac{\varepsilon}{2}$, $\exists$ open $V \supset U \setminus E$ s.t. $\mu(V) \overset{③}{<} \frac{\varepsilon}{2}$. Let $K := F \setminus V \subset E$, cpt and

$$\mu(K) = \mu(F) - \mu(F \cap V) \underset{②}{>} \mu(U) - \frac{\varepsilon}{2} - \mu(V)$$
$$\underset{(E \subset U)}{>} \mu(E) - \frac{\varepsilon}{2} - \mu(V) \underset{③}{>} \mu(E) - \varepsilon.$$

Hence μ is inner reg. on E. If $\mu(E)=\infty$, $E = \bigcup E_j$ w/ $E_j \subset E_{j+1}$ and $\mu(E_j)<\infty$ $\forall j$. So $\forall N \in \mathbb{N}$, $\exists j$ s.t. $\mu(E_j)>N$, and $\exists$ cpt $K \subset E_j \subset E$ s.t. $\mu(K)>N$. Hence μ inner reg on E.

② If μ is a σ-finite Radon measure on X and $E \subset X$ is Borel, then $\forall \varepsilon>0$, $\exists F \subset E \subset U$ w/ F closed, U open, and $\mu(U \setminus F)<\varepsilon$.

Pf: Exercise!

③ Suppose X is LCH s.t. every open set is σ-cpt [e.g. if X is second countable]. Then every Borel measure which is finite on cpt sets is Radon. [X open $\Rightarrow$ X σ-cpt $\Rightarrow$ regular by ①]

Pf: If μ finite on cpt sets, $C_c(X) \subset L^1(\mu)$, so $f \overset{\varphi}{\mapsto} \int f\,d\mu$ is a positive linear ftl on $C_c(X)$. Let ν be the ! Radon measure on X s.t. $\varphi(f) = \int f\,d\nu$. Show $\mu = \nu$. For $U \subset X$ open, write $U = \bigcup K_n$ w/ K_n cpt $\forall n$. Inductively, find $f_n \in C_c(X)$ s.t. $f_n \prec U$, $f_n = 1$ on $\bigcup\limits^{n} K_j$ and on the cpt set $\bigcup\limits^{n-1} \operatorname{supp}(f_j)$. Then $f_n \nearrow \chi_U$ ptwise, so by MCT, $\mu(U) = \lim \int f_n\,d\mu = \lim \varphi(f_n) = \lim \int f_n\,d\nu = \nu(U)$.

If E Borel, $\varepsilon>0$, take $F \subset E \subset U$ as in ② so $\mu(\underbrace{U \setminus F}_{\text{open!}}) = \nu(U \setminus F) < \varepsilon$.

Then $\mu(U) - \varepsilon \le \mu(E) \le \mu(U)$ $\Rightarrow$ μ outer reg $\Rightarrow$ $\mu = \nu$.

Lemma: Suppose X LCH and μ a Radon measure on X. Define $\varphi(f) := \int f \, d\mu$ on $C_c(X)$. TFAE:

① φ extends continuously to $C_0(X)$.

② φ is bdd wrt $\|\cdot\|_\infty$

③ $\mu(X)$ is finite.

Pf: ①$\Leftrightarrow$②: follows since $C_c(X) \subset C_0(X)$ is dense wrt $\|\cdot\|_\infty$.

②$\Leftrightarrow$③: follows by $\mu(X) = \sup\{ \underbrace{\int f \, d\mu}_{=\varphi(f)} \mid f \in C_c(X),\ 0 \le f \le 1 \}$

Cor: Positive f'ls $\varphi \in C_0(X)^*$ are of the form $\varphi = \int \cdot \, d\mu$ where μ is a finite Radon measure.

- We want to describe all of $C_0(X)^*$.

Prop: If $\varphi \in C_0(X, \mathbb{R})^*$, $\exists$ positive $\varphi_\pm \in C_0(X, \mathbb{R})^*$ s.t. $\varphi = \varphi_+ - \varphi_-$.

Pf: For $f \in C_0(X, [0,\infty))$, define $\varphi_+(f) := \sup\{\varphi(g) \mid 0 \le g \le f\}$.

Since $|\varphi(g)| \le \|\varphi\| \cdot \|g\|_\infty \le \|\varphi\| \cdot \|f\|_\infty$ $\forall\, 0 \le g \le f$ and $\varphi(0) = 0$,

$0 \le \varphi_+(f) \le \|\varphi\| \cdot \|f\|_\infty$ $\forall f \in C_0(X, [0,\infty))$.

① $\varphi_+(cf) = c\varphi_+(f)$ $\forall c \ge 0$.

② $\forall f_1, f_2 \in C_0(X, [0,\infty))$, $\varphi_+(f_1) + \varphi_+(f_2) = \varphi_+(f_1 + f_2)$.

Pf: whenever $0 \le g_1 \le f_1$ and $0 \le g_2 \le f_2$, $0 \le g_1 + g_2 \le f_1 + f_2$

$\Rightarrow \varphi_+(f_1 + f_2) \ge \varphi_+(f_1) + \varphi_+(f_2)$.

But if $0 \le g \le f_1 + f_2$, set $g_1 := g \wedge f_1$ and $g_2 = g - g_1$. Then $0 \le g_1 \le f_1$ and $0 \le g_2 \le f_2$

$\Rightarrow \varphi(g) = \varphi(g_1) + \varphi(g_2) \le \varphi_+(f_1) + \varphi_+(f_2)$ $\Rightarrow \varphi_+(f_1 + f_2) \le \varphi_+(f_1) + \varphi_+(f_2)$

For $f \in C_0(X, \mathbb{R})$, set $\varphi_+(f) := \varphi_+(f_+) - \varphi_+(f_-)$ as $f_\pm \in C_0(X, \mathbb{R})$.

If $f = g - h$ where $g, h \ge 0$, $g + f_- = h + f_+ \Rightarrow \varphi_+(g) + \varphi_+(f_-) = \varphi_+(h) + \varphi_+(f_-)$

So $\varphi_+(f) = \varphi_+(g) - \varphi_+(h)$. Hence φ_+ is linear on $C_0(X, \mathbb{R})$. Also,

$|\varphi_+(f)| \le \max\{\varphi_+(f_+), \varphi_+(f_-)\} \le \|\varphi\| \cdot \max\{\|f_+\|_\infty, \|f_-\|_\infty\} = \|\varphi\| \cdot \|f\|_\infty$.

Thus $\|\varphi_+\| \le \|\varphi\|$. Setting $\varphi_- := \varphi - \varphi_+$, $\varphi_- \in C_0(X, \mathbb{R}^*)$ is also positive.

Cor: If $\varphi \in C_0(X,\mathbb{R})^*$, $\exists$ finite Radon [$\iff$ finite regular Borel] measures μ_1, μ_2 on X s.t. $\forall f \in C_0(X,\mathbb{R})$, $\varphi(f) = \underbrace{\int f\,d\mu_1 - \int f\,d\mu_2}_{\text{" } = \int f\,d(\mu_1-\mu_2) \text{ "}}$

- Describing complex fcls is trickier at this pt.

Exercise: For $\varphi \in C_0(X)^*$, $\exists$ finite Radon measures $\mu_0, \mu_1, \mu_2, \mu_3$ on X s.t. $\forall f \in C_0(X)$, $\varphi(f) = \sum_{k=0}^{3} i^k \int_X f\,d\mu_k$ " $= \int f\,d(\sum_{k=0}^{3} i^k \mu_k)$ "

- We want to make sense of $\underbrace{\text{"}\mu_1 - \mu_2\text{"}}_{\text{signed measures}}$ and $\underbrace{\text{"}\sum_{k=0}^{3} i^k \mu_k\text{"}}_{\text{complex measures}}$.

Signed measures

Def: Let $(X, \mathcal{M})$ be a measurable space. A fct $\mu: \mathcal{M} \to \overline{\mathbb{R}}$ is called a signed measure if

- μ takes on at most one of the values $\pm\infty$,
  - ↳ if μ takes neither value, call μ finite.
- $\mu(\emptyset) = 0$, and
- $\forall$ disjoint seq. $(E_n) \subset \mathcal{M}$, $\mu(\amalg E_n) = \sum \mu(E_n)$, where the sum converges absolutely if $|\mu(\amalg E_n)| < \infty$.

Examples: ① If μ_1, μ_2 are measures on $(X, \mathcal{M})$ w/ at least one of μ_1, μ_2 finite, then $\mu_1 - \mu_2$ is a signed measure.

② Suppose μ is a measure on $(X, \mathcal{M})$. If $f: X \to \overline{\mathbb{R}}$ measurable s.t. at least one of $\int f_\pm\,d\mu$ is finite, then

Call f extended μ-integrable

$\nu(E) := \int_E f\,d\mu$ is a signed measure.

- We want to show these are really the only examples!

Def: Suppose μ is a signed measure on $(X, \mathcal{M})$. Call $E \in \mathcal{M}$

- positive
- negative
- null

} if $\forall$ measurable $F \subseteq E$, $\mu(F) \begin{cases} \geq 0 \\ \leq 0 \\ = 0. \end{cases}$

Observe $N \in \mathcal{M}$ is null $\iff$ N is both positive + negative.

Facts:

① E positive $\Rightarrow$ $\mu(E) \geq 0$. [similarly for negative, null]

② E positive and $F \subseteq E$ $\Rightarrow$ F positive. [negative, null]

③ (E_n) positive $\Rightarrow$ $\bigcup E_n$ positive.

Pf: Disjointify so $\bigcup E_n = \coprod F_n$ where F_n positive $\forall n$. If $G \subset \bigcup E_n = \coprod F_n$,
$\mu(G) = \mu(G \cap \coprod F_n) = \sum \mu(G \cap F_n) \geq 0$.

④ If $0 < \mu(E) < \infty$, $\exists$ positive $F \subset E$ s.t. $\mu(F) > 0$.

Pf: If E positive, done. Else, let $n_1 \in \mathbb{N}$ be minimal s.t. $\exists E_1 \subset E$ and $\mu(E_1) < -\frac{1}{n_1}$. If $E \setminus E_1$ positive, done. Else, let $n_2 \in \mathbb{N}$ be minimal s.t. $\exists E_2 \subset E \setminus E_1$ s.t. $\mu(E_2) < -\frac{1}{n_2}$. Inductively continue. Either $E \setminus \coprod^{k} E_j$ is positive for some k, or (E_i) is disjoint w/ $\mu(E_i) < -\frac{1}{n_i}$. Let $F := E \setminus \coprod E_i$. Since $\mu(E) < \infty$, $\sum |\mu(E_i)| < \infty$, so $\sum -\frac{1}{n_i}$ converges, so $n_i \to \infty$ as $i \to \infty$. Since $\mu(E) > 0$ and $\mu(E_i) < 0$ $\forall i$, $\mu(F) > 0$. Suppose $G \subset F$ m'ble. Then $\mu(G) \geq \frac{-1}{n_i - 1}$ $\forall i$ $\Rightarrow$ $\mu(G) \geq 0$. So F is positive.

Thm (Hahn Decomposition): Let μ be a signed measure on $(X, \mathcal{M})$. $\exists$ positive set P s.t. P^c is negative. Moreover, If $Q \subset X$ is another positive set s.t. Q^c is negative, $P \Delta Q$ and $P^c \Delta Q^c$ are μ-null sets.

Def: We call a pos. P s.t. P^c is negative a Hahn decomposition of X.

Existence: WLOG, we may assume $\infty \notin \mathrm{im}(\mu) \subset \overline{\mathbb{R}}$ [else consider $-\mu$].
Define $\lambda := \sup\{\mu(E) \mid E \text{ is positive}\}$. Then $\exists (E_n)$ s.t. $\mu(E_n) \to \lambda$. Take $P := \bigcup E_n$. Then P is positive and $\mu(P) \leq \lambda$. Also $E_n \subset P$ and $P \setminus E_n \subset P$, so $\mu(P \setminus E_n) \geq 0$ and

$$\mu(P) = \mu(E_n) + \mu(P \setminus E_n) \geq \mu(E_n) \to \lambda \text{ as } n \to \infty.$$

Thus $\mu(P) = \lambda$. We claim P^c is negative. If $F \subset P^c$ s.t. $\mu(F) > 0$, by 4 on previous page, $\exists$ positive $G \subset F$ s.t. $\mu(G) > 0$. Then $P \amalg G$ is positive w/ $\mu(P \amalg G) = \mu(P) + \mu(G) > \lambda$, a contradiction.

Uniqueness: If $P, Q \subseteq X$ are positive s.t. P^c, Q^c negative, then

$$P \Delta Q = [P \setminus Q] \cup [Q \setminus P] = [\underset{\text{pos}}{P} \cap \underset{\text{neg}}{Q^c}] \cup [\underset{\text{pos}}{Q} \cap \underset{\text{neg}}{P^c}] \text{ is } \mu\text{-null},$$

and similarly for $P^c \Delta Q^c$.

Def: For a signed measure μ on X, let $X = P \amalg P^c$ be a Hahn decomposition. Then $\mu_+(E) := \mu(E \cap P)$ and $\mu_-(E) := -\mu(E \cap P^c)$ are measures on X, and $\mu = \mu_+ - \mu_-$. These $\mu_\pm$ are independent of Hahn decomposition. Moreover, $\mu_+(P^c) = 0 = \mu_-(P)$.

Def: We say measures μ_1, μ_2 on $\mathcal{M}$ are mutually singular, denoted $\mu_1 \perp \mu_2$, if $\exists$ disjoint $E, F \in \mathcal{M}$ s.t. $X = E \amalg F$ and $\mu_1(F) = 0 = \mu_2(E)$.

Thm (Jordan Decomposition): Let μ be a signed measure on $(X, \mathcal{M})$. $\exists!$ mutually singular measures μ_+, μ_- s.t. $\mu = \mu_+ - \mu_-$. Called the Jordan decomposition of μ.

Pf: Existence follows from Hahn decomposition. Suppose now $\mu = \mu_+ - \mu_- = \nu_+ - \nu_-$ where $\mu_+ \perp \mu_-$ and $\nu_+ \perp \nu_-$. Then $\exists$ 2 Hahn decompositions for μ: $X = P \amalg P^c$ $[\mu_+(P^c) = 0 = \mu_-(P)]$, and $X = Q \amalg Q^c$ $[\nu_+(Q^c) = 0 = \nu_-(Q)]$. Thus $P \Delta Q$, $P^c \Delta Q^c$ are μ-null. Then $\forall E \in \mathcal{M}$,

$$\mu_+(E) = \mu_+(E \cap P) = \mu(E \cap P) = \mu(E \cap P \cap Q) + \mu(E \cap \underbrace{P \cap Q^c}_{\mu\text{-null}})$$

$$= \mu(E \cap P \cap Q) = \mu(E \cap P \cap Q) + \mu(E \cap \underbrace{P^c \cap Q}_{\mu\text{-null}}) = \mu(E \cap Q)$$

$$= \nu_+(E \cap Q) = \nu_+(E).$$

Hence $\mu_+ = \nu_+$. Similarly, $\mu_- = \nu_-$.

Def: For a signed measure μ on $(X, \mathcal{M})$, $\mathcal{L}^1(\mu) := \mathcal{L}^1(\mu_+) \cap \mathcal{L}^1(\mu_-)$ and $\int f \, d\mu := \int f \, d\mu_+ - \int f \, d\mu_-$.

Def: The total variation of $\mu = \mu_+ - \mu_-$ is $|\mu| := \mu_+ + \mu_-$.

Observe $|\mu(E)| = |\mu_+(E) - \mu_-(E)| \leq \mu_+(E) + \mu_-(E) = |\mu|(E)$ $\forall E \in \mathcal{M}$.

Hence μ finite $\iff$ $|\mu|$ finite.

Lemma: Suppose μ_1, μ_2 are measures on X w/ at least one of μ_1, μ_2 finite s.t. $\mu = \mu_1 - \mu_2$. Then

$$|\mu|(X) \leq \mu_1(X) + \mu_2(X).$$

Pf: Let $\mu = \mu_+ - \mu_-$ be the Jordan decomp, and let $X = P \amalg P^c$ be a Hahn decomp. Then

$$0 \leq \mu_+(X) = \mu(X \cap P) = \mu(P) = \mu_1(P) - \mu_2(P) \leq \mu_1(P) \leq \mu_1(X)$$

$$0 \leq \mu_-(X) = -\mu(X \cap P^c) = -\mu(P^c) = \mu_2(P^c) - \mu_1(P^c) \leq \mu_2(X).$$

Hence $|\mu|(X) = \mu_+(X) + \mu_-(X) \leq \mu_1(X) + \mu_2(X)$.

Def: Let $M(X, \mathcal{M}, \mathbb{R}) := \{$finite signed measures on $(X, \mathcal{M})\}$

On $M(X, \mathcal{M}, \mathbb{R})$, define $\|\mu\| := |\mu|(X)$.

Thm: $(M := M(X, \mathcal{M}, \mathbb{R}), \|\cdot\|)$ is Banach

Step 1: $\|\cdot\|$ is a norm on M.

Pf: $\|\mu\| = 0 \iff |\mu|(X) = 0 \iff \mu_+(X) = 0 = \mu_-(X) \iff \mu_\pm = 0 \iff \mu = 0$.

Clearly $\|\alpha\mu\| = |\alpha\mu|(X) = |\alpha| \cdot |\mu|(X) = |\alpha| \cdot \|\mu\|$. Finally, if $\mu, \nu \in M$,

$\|\mu+\nu\| = |\mu+\nu|(X) = |\mu_+ - \mu_- + \nu_+ - \nu_-|(X) = |\mu_+ + \nu_+ - (\mu_- + \nu_-)|(X)$

$\leq (\mu_+ + \nu_+)(X) + (\mu_- + \nu_-)(X) = |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|$.

Step 2: $(\mu_n) \subset M$ Cauchy $\Rightarrow$ $(\mu_n(E)) \subset \mathbb{R}$ uniformly Cauchy $\forall E \in \mathcal{M}$.

Pf: $\forall E \in \mathcal{M}$, $|\mu_m(E) - \mu_n(E)| \leq |\mu_m - \mu_n|(E) \leq \|\mu_m - \mu_n\| \longrightarrow 0$.

Moreover, the rate is independent of $E \in \mathcal{M}$!

Fix a Cauchy seq. (μ_n). By Step 2, define $\mu(E) := \underbrace{\lim_n \mu_n(E)}_{\in \mathbb{R},\, \neq \pm\infty}$

Clearly $\mu(\emptyset) = 0$. Since (μ_n) Cauchy, $(\|\mu_n\|)$ converges.

↳ rev. Δ ineq.

Step 3: μ is finitely additive.

Pf: If $E_1, \dots, E_k$ disjoint, then

$$\mu\left(\coprod_{i=1}^k E_i\right) = \lim_n \mu_n\left(\coprod_{i=1}^k E_i\right) = \lim_n \sum_{i=1}^k \mu_n(E_i) = \sum_{i=1}^k \lim_n \mu_n(E_i) = \sum_{i=1}^k \mu(E_i).$$

Step 4: μ is σ-additive.

Pf: Let (E_i) be disjoint and $\varepsilon > 0$. Pick $N \in \mathbb{N}$ s.t. $\forall n \geq N$ and $F \in \mathcal{M}$, $|\mu_n(F) - \mu(F)| < \frac{\varepsilon}{3}$. Pick $k \in \mathbb{N}$ s.t. $\left|\sum_{i > k} \mu_N(E_i)\right| < \frac{\varepsilon}{3}$.

Then
$$\left|\mu\left(\coprod E_i\right) - \underbrace{\mu\left(\coprod_{i=1}^k E_i\right)}_{\sum_{i=1}^k \mu(E_i)}\right| \leq \left.\begin{aligned} &\left|\mu\left(\coprod E_i\right) - \mu_N\left(\coprod E_i\right)\right| < \tfrac{\varepsilon}{3} \\ +&\left|\mu_N\left(\coprod E_i\right) - \mu_N\left(\coprod_{i=1}^k E_i\right)\right| < \tfrac{\varepsilon}{3} \\ +&\left|\mu_N\left(\coprod_{i=1}^k E_i\right) - \mu\left(\coprod_{i=1}^k E_i\right)\right| < \tfrac{\varepsilon}{3} \end{aligned}\right] = \varepsilon.$$

Hence $\forall \varepsilon > 0$, $\exists k \in \mathbb{N}$ s.t. $\left|\mu\left(\coprod E_i\right) - \sum_{i=1}^k \mu(E_i)\right| < \varepsilon \Rightarrow \mu$ σ-additive.

Step 5: $\sum \mu(E_i)$ converges absolutely when (E_i) disjoint. Hence μ is a finite signed measure.

Pf: Let $\varepsilon > 0$. For $k \in \mathbb{N}$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|\mu(E_j) - \mu_n(E_j)| < \frac{\varepsilon}{k}$ $\forall j = 1, \ldots, k$. Then $\forall n \geq N$,

$$\sum^k |\mu(E_j)| < \sum^k |\mu_n(E_j)| + \frac{\varepsilon}{k} \leq \left[\sum^k |\mu_n|(E_j)\right] + \varepsilon \leq |\mu_n|\left(\coprod^k E_j\right) + \varepsilon$$

$$\leq \underbrace{\|\mu_n\|}_{\text{converges, say to } L} + \varepsilon \longrightarrow L + \varepsilon \quad \underline{\text{independent}} \text{ of } k.$$

Hence $\sum |\mu(E_j)| \leq L + \varepsilon$ is bounded.

<u>Step 6:</u> $\mu_n \to \mu$ in M.

Pf: Let $\varepsilon > 0$. By <u>Step 2</u>, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |\mu(E) - \mu_n(E)| < \frac{\varepsilon}{2}$ $\forall E \in \mathcal{M}$. For $n \geq N$, let $X = P_n \amalg P_n^c$ be a Hahn decomp for $\mu - \mu_n$.
$(\mu - \mu_n)_+(X) = (\mu - \mu_n)(P_n) < \frac{\varepsilon}{2}$ and $(\mu - \mu_n)_-(X) = (\mu - \mu_n)(P_n^c) < \frac{\varepsilon}{2}$.
So $\|\mu - \mu_n\| = |\mu - \mu_n|(X) < \varepsilon$ $\forall n \geq N$.

<u>Def:</u> Let ν be a signed meas and μ a positive measure on $(X, \mathcal{M})$. Say ν is <u>absolutely continuous</u> wrt μ, denoted $\nu \ll \mu$, if $\mu(E) = 0 \Rightarrow \nu(E) = 0$.

<u>Example:</u> Let $f \in L^1(X, \mathbb{R})$ and set $\overbrace{\nu(E) = \int_E f \, d\mu}^{\text{"}\nu = f\,d\mu\text{"}}$. Then $\nu \ll \mu$.

<u>Exercise:</u> 1 TFAE: 1 $\nu \ll \mu$, 2 $\nu_\pm \ll \mu$, 3 $|\nu| \ll \mu$.

2 If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.

<u>Prop:</u> Suppose ν is a <u>finite</u> signed meas on $(X, \mathcal{M})$ and μ is a pos. meas. TFAE:

1. $\nu \ll \mu$
2. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Pf: Since $\nu \ll \mu \Leftrightarrow |\nu| \ll \mu$ and $|\nu(E)| \leq |\nu|(E)$, we may assume that ν is <u>positive</u>. Clearly 2 $\Rightarrow$ 1.

<u>$\neg$2 $\Rightarrow$ $\neg$1:</u> Suppose $\exists \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$, $\exists E_n \in \mathcal{M}$ w/ $\mu(E_n) < 2^{-n}$, and $\nu(E_n) \geq \varepsilon$. Set $F = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n$. Since $\mu\left(\bigcup_{n=k}^\infty E_n\right) < \sum_k^\infty 2^{-k} = 2^{1-k}$, $\mu(F) = 0$. But since ν finite, $\nu(F) = \lim_k \nu\left(\bigcup_{n=k}^\infty E_n\right) \geq \varepsilon$.
Hence $\neg(\nu \ll \mu)$.

Example: $(X, \mathcal{M}) = (\mathbb{N}, P(\mathbb{N}))$, $\mu(E) = \sum_{n\in E} \frac{1}{2^n}$, $\nu(E) = \sum_{n \in E} 2^n$. Then $\nu \ll \mu$ and $\mu \ll \nu$, but ② above fails as ν not finite.

Lemma: Suppose ν, μ finite measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$ or $\exists\, \varepsilon > 0$ and $E \in \mathcal{M}$ s.t. $\mu(E) > 0$ and $\nu \geq \varepsilon\mu$ on E.
[E is positive for $\nu - \varepsilon\mu$]

Pf: Let $X = P_n \amalg P_n^c$ be a Hahn decomp for $\nu - \frac{1}{n}\mu$. Set $P = \cup P_n$, so $P^c = \cap P_n^c$. Then P^c is negative $\forall\, \nu - n^{-1}\mu$. Then $0 \leq \nu(P^c) \leq \frac{1}{n}\mu(P^c)$ $\forall n \in \mathbb{N}$, so $\nu(P^c) = 0$. If $\mu(P) = 0$, then $\nu \perp \mu$. If $\mu(P) > 0$, $\mu(P_n) > 0$ for some n, and P_n positive for $\nu - \frac{1}{n}\mu$.

Thm (Lebesgue-Radon-Nikodym): Let ν be a σ-finite signed measure and μ a σ-finite pos. measure on $(X, \mathcal{M})$. $\exists!$ σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ s.t.

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \text{and} \quad \nu = \lambda + \rho. \quad \Big] \text{ Lebesgue decomp.}$$

Moreover, $\exists!$ extended μ-integrable fct f s.t. $d\rho = f\,d\mu$.
If ν positive / finite, so are λ, ρ, and $f \in \mathcal{L}^+ / \mathcal{L}^1$. ← RN Derivative of ρ wrt μ

Uniqueness: Suppose λ, λ' are σ-finite signed measures s.t. $\lambda, \lambda' \perp \mu$ and $f, f' \in \mathcal{L}^1$ s.t.

$$d\nu = d\lambda + f\,d\mu = d\lambda' + f'\,d\mu.$$

Then $d(\lambda - \lambda') = (f' - f)\,d\mu$ as signed measures.

But $(\lambda - \lambda') \perp \mu$, and $(f' - f)\,d\mu \ll d\mu$, so as signed measures,

exercise!

$$d(\lambda - \lambda') = 0 = (f' - f)\,d\mu.$$

We conclude $\lambda = \lambda'$ and $f = f'$ in $\mathcal{L}^1$.

Existence:

Case 1: μ, ν finite positive measures.

Set $A := \{ f \in L^1(X, \mu, [0,\infty]) \mid \int_E f \, d\mu \le \nu(E) \ \forall E \in \mathcal{M} \}$. Observe:

⓪ $0 \in A$

① $f, g \in A \Rightarrow f \vee g \in A$

Pf: Set $G := \{ g > f \}$. Then $\forall E \in \mathcal{M}$,

$$\int_E f \vee g \, d\mu = \int_{E \cap G} g \, d\mu + \int_{E \setminus G} f \, d\mu \le \nu(E \cap G) + \nu(E \setminus G) = \nu(E).$$

Set $m := \sup \{ \int f \, d\mu \mid f \in A \}$ and note $m \le \nu(X) < \infty$.

Choose $(f_n) \subset A$ s.t. $\int f_n \, d\mu \nearrow m$. Set $g_n := \max\{f_1, \ldots, f_n\} \in A$

and $f := \sup g_n$. Then $\int f_n \, d\mu \le \int g_n \, d\mu \nearrow m$ by Squeeze Thm.

Since $g_n \nearrow f$ ptwise, by MCT, $\int_E f \, d\mu = \lim \int_E g_n \, d\mu \le \nu(E) \ \forall E \in \mathcal{M}$,

so $f \in A$ and $\int f \, d\mu = m$.

Claim: $\lambda(E) := \nu(E) - \underbrace{\int_E f \, d\mu}_{\rho(E) :=} \ge 0$ is singular w.r.t. μ.

So $\lambda \perp \mu$, $\rho \ll \mu$, $\nu = \lambda + \rho$, and $d\rho = f \, d\mu$.

Pf: If not, by lemma on previous page, $\exists E \in \mathcal{M}$ and $\varepsilon > 0$ s.t.

$\mu(E) > 0$ and $\lambda \ge \varepsilon\mu$ on E. But then $\forall F \in \mathcal{M}$,

$$\begin{aligned}
\int_F f + \varepsilon \chi_E \, d\mu &= \int_F f \, d\mu + \varepsilon\mu(E \cap F) \le \int_F f \, d\mu + \lambda(E \cap F) \\
&= \int_F f \, d\mu + \nu(E \cap F) - \int_{E \cap F} f \, d\mu \\
&= \int_{F \setminus E} f \, d\mu + \nu(E \cap F) \\
&\le \nu(F \setminus E) + \nu(E \cap F) = \nu(F).
\end{aligned}$$

Hence $f + \varepsilon \chi_E \in A$, but $\int f + \varepsilon \chi_E \, d\mu = m + \varepsilon\mu(E) > m$, contradiction.

Case 2: μ, ν are σ-finite positive measures.

Write $X = \coprod X_n$ s.t. $\mu(X_n)$ and $\nu(X_n) < \infty$ [Take $X = \coprod Y_i$ s.t. $\mu(Y_i) < \infty$ $\forall i$ and $X = \coprod Z_j$ s.t. $\nu(Z_j) < \infty$ $\forall j$. Get ctble disjoint seq $(Y_i \cap Z_j)$, s.t. $\mu(Y_i \cap Z_j)$ and $\nu(Y_i \cap Z_j) < \infty$].

Set $\mu_n(E) := \mu(E \cap X_n)$ and $\nu_n(E) := \nu(E \cap X_n)$. By Case 1, $\exists$ pos. meas $\lambda_n \perp \mu_n$ and $f_n \in \mathcal{L}^1_+$ s.t. $d\nu_n = d\lambda_n + f_n d\mu_n$. Since $\mu_n(X_n^c) = \nu_n(X_n^c) = 0$, $\lambda(X_n^c) = \nu_n(X_n^c) - \int_{X_n^c} f_n \, d\mu_n = 0$, so we may assume $f_n|_{X_n^c} = 0$. Let $\lambda = \sum \lambda_n$ and $f = \sum f_n \in \mathcal{L}^+$.

Then $\lambda \perp \mu$ [exercise!], λ and $f d\mu$ σ-finite, and

$$d\nu = d\lambda + f d\mu.$$

Case 3: μ is σ-finite positive and ν is σ-finite signed.

Pf: Use Jordan decomp. to set $\nu = \nu_+ - \nu_-$ w/ $\nu_+ \perp \nu_-$.

Apply Case 2 to $\nu_\pm$ separately. Then subtract results.

Remark: If μ σ-finite positive and ν σ-finite signed s.t. $\nu \ll \mu$, $\exists!$ extended μ-integrable f s.t. $d\nu = f d\mu$. Call f the Radon-Nikodym derivative of ν wrt. μ, denoted $\frac{d\nu}{d\mu}$.

Exercise: If μ is a σ-finite signed measure, $\left|\frac{d\mu}{d|\mu|}\right| = 1$ $|\mu|$-a.e.

Def: Let X be LCH. A signed measure μ on X is called a signed Radon measure if $\mu_\pm$ are Radon.

Let $RM \subseteq M$ be the set of finite signed Radon measures.

Exercise: If μ is a positive Radon measure on X, then $C_c(X)$ dense in $\mathcal{L}^1(X)$.

Lusin's Thm: If μ is a Radon measure on $\mathfrak{X}$ (LCH) and $f: \mathfrak{X} \to \mathbb{C}$ is m'ble and vanishes outside a set of finite meas., then $\forall \varepsilon > 0$, $\exists g \in C_c(\mathfrak{X})$ s.t. $g = f$ except on a set of measure less than ε. If $\|f\|_\infty < \infty$, can arrange $\|g\|_\infty \leq \|f\|_\infty$. If $\mathrm{im}(f) \subset \mathbb{R}$, can arrange $\mathrm{im}(g) \subset \mathbb{R}$.

Pf: Discussion.

Thm (Riesz-Rep'n): Suppose $\mathfrak{X}$ LCH. Define $\varphi: RM \to C_0(\mathfrak{X}, \mathbb{R})^*$ by $\mu \mapsto \varphi_\mu$ where $\varphi_\mu(f) := \int f \, d\mu$. Then φ is an isometric isomorphism

pf: We already showed φ is surjective. If $\mu \in RM$, then

$$\left|\int f \, d\mu\right| = \left|\int f \, d\mu_+ - \int f \, d\mu_-\right| \leq \left|\int f \, d\mu_+\right| + \left|\int f \, d\mu_-\right|$$

$$\leq \int |f| \, d\mu_+ + \int |f| \, d\mu_- = \int |f| \, d|\mu| \leq \|f\|_\infty \cdot \|\mu\|.$$

Hence $\|\varphi_\mu\| \leq \|\mu\|$ and φ bdd. Moreover, $\left|\frac{d\mu}{d|\mu|}\right| = 1$ $|\mu|$-a.e. [exercise]. Since $|\mu|$ finite, by Lusin's Thm, $\exists f \in C_c(\mathfrak{X}, \mathbb{R})$ s.t. $\|f\|_\infty = 1$ and $f = \overline{\frac{d\mu}{d|\mu|}}$ except on a set $E \in \mathcal{B}_{\mathfrak{X}}$ s.t. $|\mu|(E) < \frac{\varepsilon}{2}$.

Then $$\|\mu\| = \int d|\mu| = \int \left|\frac{d\mu}{d\mu}\right|^2 d|\mu| = \int \overline{\frac{d\mu}{d|\mu|}} \, \frac{d\mu}{d|\mu|} \, d|\mu|$$

$$= \int \overline{\frac{d\mu}{d|\mu|}} \, d\mu \leq \left|\int f \, d\mu\right| + \left|\int f - \overline{\frac{d\mu}{d|\mu|}} \, d\mu\right|$$

$$\leq \|\varphi_\mu\| \cdot \underbrace{\|f\|_\infty}_{=1} + \underbrace{2|\mu|(E)}_{< \frac{\varepsilon}{2}} \leq \|\varphi_\mu\| + \varepsilon.$$

Hence $\|\mu\| = \|\varphi_\mu\|$ and φ is an isometric iso.

Complex measures

We call a fct $\nu : \mathcal{M} \to \mathbb{C}$ a complex measure if

- $\nu(\emptyset) = 0$
- $\forall$ disjoint seq. $(E_n) \subset \mathcal{M}$, $\nu(\coprod E_n) = \overbrace{\sum \nu(E_n)}^{\text{absolutely convergent}}$

Exercise: If ν a cplx measure on $(X, \mathcal{M})$, $Re(\nu)$, $Im(\nu)$ are finite signed measures on $(X, \mathcal{M})$.

Examples:

① If $\mu_0, \mu_1, \mu_2, \mu_3$ finite measures on $(X, \mathcal{M})$, $\sum_{k=0}^{3} i^k \mu_k$ is a complex measure.

② For $g \in L^1(X, \mathbb{C})$, $\nu(E) := \int_E g \, d\mu$ is a cplx measure.

By Jordan Decomposition Thm, get following corollary:

Cor: If ν is a cplx measure on $(X, \mathcal{M})$, $\exists!$ pairs of mutually singular finite measures $Re(\nu)_\pm, Im(\nu)_\pm$ $[\mu_0, \mu_1, \mu_2, \mu_3]$ s.t. $\nu = Re(\nu)_+ - Re(\nu)_- + i[Im(\nu)_+ - Im(\nu)_-] = \sum_{k=0}^{3} i^k \mu_k$.

Def: For a cplx meas. ν, $\mathcal{L}^1(\nu) := \bigcap_{k=0}^{3} \mathcal{L}^1(\mu_k)$ and $\int f \, d\nu := \sum_{k=0}^{3} i^k \int f \, d\mu_k$.

Warning: The total variation of a cplx meas $\mu = \sum_{k=0}^{3} i^k \mu_k$ is not $\sum_{k=0}^{3} \mu_k$! We must use Radon-Nikodym.

Def: If ν is a complex measure + μ a positive measure on $(X, \mathcal{M})$, say:

- $\nu \perp \mu$ if $Re(\nu) \perp \mu$ and $Im(\nu) \perp \mu$
- $\nu \ll \mu$ if $Re(\nu) \ll \mu$ and $Im(\nu) \ll \mu$.

Thm (Lebesgue–Radon–Nikodym): If ν is a cplx meas on $(X, \mathcal{M})$ and μ is a σ-finite positive meas. on $(X, \mathcal{M})$, $\exists!$ cplx meas λ and $f \in \mathcal{L}^1(\mu)$ s.t. $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$.

[If $\lambda' \perp \mu$ and $f' \in \mathcal{L}^1(\mu)$ s.t. $d\nu = d\lambda' + f'$, then $\lambda = \lambda'$ and $f = f'$.]

Pf: Apply LRN separately to $Re(\nu)$ and $Im(\nu)$ and recombine.

Lemma: Suppose ν is a cplx measure. $\exists!$ pos. measure $|\nu|$ satisfying the following property:

(*) $\forall$ pos. measure μ and $f \in \mathcal{L}^1(\mu)$ s.t. $d\nu = f d\mu$, $d|\nu| = |f| d\mu$.

Call $|\nu|$ the <u>total variation</u> of ν.

Pf: First, consider $\mu := |Re(\nu)| + |Im(\nu)|$. By LRN, $\exists f \in \mathcal{L}^1(\mu)$ s.t. $d\nu = f d\mu$. Define $d|\nu| := |f| d\mu$. This will <u>uniquely define</u> $|\nu|$ if it satisfies (*). Suppose also $d\nu = g d\rho$ for another pos. measure ρ and $g \in \mathcal{L}^1(\rho)$. Consider $\mu + \rho$ on $(X, \mathcal{M})$, and observe $\nu \ll \mu$, $\rho \ll \mu + \rho$. So $d\mu = \frac{d\mu}{d(\mu+\rho)} d(\mu+\rho)$ and $d\rho = \frac{d\rho}{d(\mu+\rho)} d(\mu+\rho)$.

Exercise/discussion: If $\nu \ll \mu \ll \lambda$ w/ μ, λ σ-finite pos. measures and ν either σ-finite signed or complex, then:

1. $\forall f \in \mathcal{L}^1(\nu)$, $f \frac{d\nu}{d\mu} \in \mathcal{L}^1(\mu)$ and $\int f \, d\nu = \int f \frac{d\nu}{d\mu} d\mu$
2. $\nu \ll \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}$ λ-a.e.

Since $f \frac{d\mu}{d(\mu+\rho)} d(\mu+\rho) = f d\mu = d\nu = g \, d\rho = g \frac{d\rho}{d(\mu+\rho)} d(\mu+\rho)$,

$f \underbrace{\frac{d\mu}{d(\mu+\rho)}}_{\geq 0} = g \underbrace{\frac{d\rho}{d(\mu+\rho)}}_{\geq 0}$ a.e. $\Rightarrow |f| \frac{d\mu}{d(\mu+\rho)} = \left| f \frac{d\mu}{d(\mu+\rho)} \right| = \left| g \frac{d\rho}{d(\mu+\rho)} \right| = |g| \frac{d\rho}{d(\mu+\rho)}$

Hence $|f| d\mu = d|\nu| = |g| d\rho$, and $|\nu|$ satisfies (*).

Remarks:

① $\nu \ll |\nu|$, as $|\nu(E)| = \left|\int_E f\,d\mu\right| \le \int_E |f|\,d\mu = |\nu|(E)$ $\forall E \in \mathcal{M}$.

② If ν is finite signed, $d\nu = (\chi_P - \chi_{P^c})\,d|\nu|$, and

$d|\nu| = \underbrace{(\chi_P + \chi_{P^c})}_{1}\,d|\nu|$ for any Hahn decomposition $X = P \amalg P^c$ for ν

$\Rightarrow$ $|\nu|$ agrees w/ old definition for finite signed measures.

③ Observe if $d\nu = f\,d\mu$, then

$$\begin{aligned} d\,\mathrm{Re}(\nu) &= \mathrm{Re}(f)\,d\mu \\ d\,\mathrm{Im}(\nu) &= \mathrm{Im}(f)\,d\mu \end{aligned} \quad \Rightarrow \quad \begin{aligned} d|\mathrm{Re}(\nu)| &= |\mathrm{Re}(f)|\,d\mu \\ d|\mathrm{Im}(\nu)| &= |\mathrm{Im}(f)|\,d\mu \end{aligned}$$

Since $|f|^2 = |\mathrm{Re}(f)|^2 + |\mathrm{Im}(f)|^2$, we have

$$\frac{d|\nu|}{d\mu} = |f| = \left[|\mathrm{Re}(f)|^2 + |\mathrm{Im}(f)|^2\right]^{\frac{1}{2}} = \left[\left[\frac{d|\mathrm{Re}(\nu)|}{d\mu}\right]^2 + \left[\frac{d|\mathrm{Im}(\nu)|}{d\mu}\right]^2\right]^{1/2}.$$

Def: $M := M(X, \mathcal{M}, \mathbb{C}) := \{\text{cplx measures on } (X, \mathcal{M})\}$

Observe $M(X, \mathcal{M}, \mathbb{C}) = M(X, \mathcal{M}, \mathbb{R}) \oplus i\,M(X, \mathcal{M}, \mathbb{R})$ as $\vec{v}$.sp.

Define $\|\nu\| := |\nu|(X)$.

Lemma: $\max\{\|\mathrm{Re}(\nu)\|, \|\mathrm{Im}(\nu)\|\} \le \|\nu\| \le 2\max\{\|\mathrm{Re}(\nu)\|, \|\mathrm{Im}(\nu)\|\}$.

Pf: By ③ above, both $|\mathrm{Re}(f)|, |\mathrm{Im}(f)| \le |f|$, so

$\|\mathrm{Re}(\nu)\| = |\mathrm{Re}(\nu)|(X) = \int |\mathrm{Re}(f)|\,d\mu \le \int |f|\,d\mu = |\nu|(X) = \|\nu\|$

and similarly $\|\mathrm{Im}(\nu)\| \le \|\nu\|$. By the Δ ineq,

$|f| \le |\mathrm{Re}(f)| + |\mathrm{Im}(f)|$, so

$\|\nu\| = |\nu|(X) = \int |f|\,d\mu \le \int |\mathrm{Re}(f)| + |\mathrm{Im}(f)|\,d\mu$

$= |\mathrm{Re}(f)|(X) + |\mathrm{Im}(f)|(X) = \|\mathrm{Re}(f)\| + \|\mathrm{Im}(f)\| \le 2\max.$

Cor: $(M, \|\cdot\|)$ is Banach

Pf: Observe that if $(\mathfrak{X}, \|\cdot\|)$ is Banach, so is $\mathfrak{X} \oplus \mathfrak{X}$ with $\|(x,y)\|_\infty := \max\{\|x\|, \|y\|\}$. Since $\|\cdot\|$ on M is equivalent to $\|\cdot\|_\infty$ on $M(\mathfrak{X}, \mathcal{M}, \mathbb{R}) \oplus M(\mathfrak{X}, \mathcal{M}, \mathbb{R})$, the result follows.

When $\mathfrak{X}$ is LCH, let $RM \subset M(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}}, \mathbb{C})$ be the subspace of $\nu \in M$ s.t. $\mathrm{Re}(\nu)$ and $\mathrm{Im}(\nu)$ are Radon.

Thm (Riesz-Rep): Suppose $\mathfrak{X}$ LCH. Define $\Psi: RM \to C_0(\mathfrak{X})^*$ by $\Psi_\nu(f) := \int f \, d\nu$. Then Ψ is an isometric iso.

Pf: Exercise.