1. Topology

Suppose $f : X \to Y$ is a function. Then f induces functions

$$f : P(X) \to P(Y) \qquad \text{by} \qquad A \mapsto f(A) := \{f(a) | a \in A\}$$
$$f^{-1} : P(Y) \to P(X) \qquad \text{by} \qquad B \mapsto f^{-1}(B) := \{x \in X | f(x) \in B\}$$

Exercise 1.0.1.

(1) Determine the relationship between $f^{-1}(f(A))$ and $A \subset X$. When are they equal?
(2) Determine the relationship between $f(f^{-1}(B))$ and $B \subset X$. When are they equal?
(3) Prove that $A \mapsto f(A)$ preserves unions, but not necessarily intersections or complements. Under what conditions on f does this preserve intersections? complements?
(4) Prove that $B \mapsto f^{-1}(B)$ preserves unions, intersections, and complements.

1.1. **Topology basics.**

Definition 1.1.1. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that:

- $\emptyset, X \in \mathcal{T}$,
- $\mathcal{T}$ is closed under arbitrary unions, and
- $\mathcal{T}$ is closed under finite intersections.

The elements of $\mathcal{T}$ are called *open sets*. An open set containing $x \in X$ is called a *neighborhood* of x. Complements of elements of $\mathcal{T}$ are called *closed sets*.

Definition 1.1.2. Observe that if $\mathcal{S}, \mathcal{T}$ are topologies on X, then so is $\mathcal{S} \cap \mathcal{T}$. This means if $\mathcal{E} \subset P(X)$, there is a *smallest* topology $\mathcal{T}(\mathcal{E})$ which contains $\mathcal{E}$ called the topology *generated by* $\mathcal{E}$.

Definition 1.1.3. Suppose $(X, \mathcal{T})$ is a topological space. A *neighborhood/local base* for $\mathcal{T}$ at $x \in X$ is a subset $\mathcal{B}(x) \subset \mathcal{T}$ consisting of neighborhoods of x such that

- for all $U \in \mathcal{T}$ such that $x \in U$, there is a $V \in \mathcal{B}(x)$ such that $V \subset U$.

A *base* for $\mathcal{T}$ is a subset $\mathcal{B} \subset \mathcal{T}$ which contains a neighborhood base for $\mathcal{T}$ at every point of X.

Example 1.1.4. Given a topological space $(X, \mathcal{T})$, the set $\mathcal{T}(x)$ of all open subsets which contain x is a neighborhood base at x.

Exercise 1.1.5. Show that $\mathcal{B} \subset \mathcal{T}$ is a base if and only if every $U \in \mathcal{T}$ is a union of members of $\mathcal{B}$.

Definition 1.1.6. Suppose $(X, \mathcal{T})$ is a topological space. We call $(X, \mathcal{T})$:

- *first countable* if there is a countable neighborhood base for $\mathcal{T}$ at every $x \in X$
- *second countable* if there is a countable base for $\mathcal{T}$.

Exercise 1.1.7. Show that second countable implies *separable*, i.e., there is a countable dense subset.

Exercise 1.1.8. Suppose X is first countable and $A \subset X$. Then $x \in \overline{A}$ (the smallest closed subset of X containing A) if and only if there is a sequence $(x_n) \subset A$ such that $x_n \to x$ (for every open subset U containing x, (x_n) is *eventually* in U).

Definition 1.1.9. Suppose X, Y are topological spaces. A function $f : X \to Y$ is called *continuous* at $x \in X$ if for every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$. We call f *continuous* if f is continuous at x for all $x \in X$.

Exercise 1.1.10. Show that $f : X \to Y$ is continuous if and only if the preimage of every open set in Y is open in X, i.e., for every $V \in \mathcal{T}_Y$,

$$f^{-1}(V) := \{x \in X | f(x) \in V\} \in \mathcal{T}_X.$$

Exercise 1.1.11. Show that the composite of continuous functions is continuous.

Exercise 1.1.12. Prove the following assertions.

(1) Given $f : X \to Y$ and a topology $\mathcal{T}$ on Y, $\{f^{-1}(U) | U \in \mathcal{T}\}$ is a topology on X. Moreover it is the weakest topology on X such that f is continuous.
(2) Given $f : X \to Y$ and a topology $\mathcal{S}$ on X, $\{U \subset Y | f^{-1}(U) \in \mathcal{S}\}$ is a topology on Y. Moreover it is the strongest topology on Y such that f is continuous.

1.1.1. *Metric spaces.*

Definition 1.1.13. A *metric space* is a set X together with a *distance* function $d : X \times X \to [0, \infty)$ satisfying

- (definite) $d(x, y) = 0$ if and only if $x = y$,
- (symmetric) $d(x, y) = d(y, x)$ for all $x, y \in X$, and
- (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The topology $\mathcal{T}_d$ *induced* by d is generated by the *open balls of radius* r

$$B_r(x) := \{y \in X | d(x, y) < r\} \qquad r > 0.$$

That is, U is open with respect to d if and only if for every $x \in U$, there is an $r > 0$ such that $B_r(x) \subset U$. Observe that every metric space is first countable.

Exercise 1.1.14. Let (X, d) be a metric space. Show tha $(X, \mathcal{T}_d)$ is second countable if and only if $(X, \mathcal{T}_d)$ is separable.

Exercise 1.1.15. Two metrics d_1, d_2 on X are called *equivalent* if there is a $C > 0$ such that

$$C^{-1} d_1(x, y) \leq d_2(x, y) \leq C d_1(x, y) \qquad \forall x, y \in X.$$

Show that equivalent metrics induce the same topology on X. That is, show that $U \subset X$ is open with respect to d_1 if and only if U is open with respect to d_2.

Exercise 1.1.16 (Sarason)**.** Let (X, d) be a metric space.

(1) Let $\alpha : [0, \infty) \to [0, \infty)$ be a continuous non-decreasing function satisfying
    - $\alpha(s) = 0$ if and only if $s = 0$, and
    - $\alpha(s + t) \leq \alpha(s) + \alpha(t)$ for all $s, t \geq 0$.

    Define $\sigma(x, y) := \alpha(d(x, y))$. Show that σ is a metric, and σ induces the same topology on X as d.
(2) Define $d_1, d_2 : X \times X \to [0, \infty)$ by

$$d_1(x, y) := \begin{cases} d(x, y) & \text{if } d(x, y) \leq 1 \\ 1 & \text{otherwise.} \end{cases}$$

$$d_2(x, y) := \frac{d(x, y)}{1 + d(x, y)}.$$

Use part (1) to show that d_1 and d_2 are metrics on X which induce the same topology on X as d.

Exercise 1.1.17. Suppose V is a $\mathbb{F}$-vector space for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A *norm* on V is a function $\|\cdot\| : V \to [0, \infty)$ such that

- (definite) $\|v\| = 0$ if and only if $v = 0$.
- (homogeneous) $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$ for all $\lambda \in \mathbb{F}$ and $v \in V$.
- (subadditive) $\|u + v\| \leq \|u\| + \|v\|$.

(1) Prove that $d(u, v) := \|u, v\|$ defines a metric on V.
(2) Prove that the following conditions are equivalent:
  (a) (V, d) is a complete metric space, i.e., every Cauchy sequence converges.
  (b) For every sequence $(v_n) \subset V$ with $\sum \|v_n\| < \infty$, the sequence $(\sum^k v_n)$ converges.

1.1.2. *Connectedness.*

Definition 1.1.18 (Relative topology)**.** Suppose X is a topological space and $A \subset X$ is a subset. The *relative topology* on A is given by $U \subset A$ is open if and only if there is an open set $V \subset X$ such that $U = V \cap A$.

Exercise 1.1.19. Suppose X is a topological space and $A \subset X$ is a subset. Show that $F \subset A$ is closed if and only if there is a closed set $G \subset X$ such that $F = G \cap A$.

Definition 1.1.20 ((Dis)connected set)**.** Let X be a topological space. We call a subset X *disconnected* if there exist non-empty, disjoint open sets U, V such that $X = U \amalg V$. A subset $A \subset X$ is disconnected if it is disconnected in its relative topology. If a subset is not disconnected, it is called *connected*. That is, $A \subset X$ is connected if and only if whenever $A \subset X$ can be written as the disjoint union $A = U \amalg V$ with U, V relatively open in A, then U or V is empty.

Exercise 1.1.21. Prove that the unit interval $[0, 1] \subset \mathbb{R}$ is connected.

Exercise 1.1.22.

(1) Suppose $f : X \to Y$ is continuous and $A \subset X$ is connected. Prove $f(A) \subset Y$ is connected.
(2) A subset $A \subset X$ is called *path connected* if for every $x, y \in A$, there is a continuous map $\gamma : [0, 1] \to A$ (called a *path*) such that $\gamma(0) = x$ and $\gamma(1) = y$. Prove that a path connected subset is connected.

Exercise 1.1.23. Recall that an *interval* $I \subset \mathbb{R}$ is a subset such that $a < b < c$ and $a, c \in I$ implies $b \in I$.

(1) Show that all intervals in $\mathbb{R}$ are connected.
(2) Prove that if $X \subset \mathbb{R}$ is not an interval, then X is not connected.

Exercise 1.1.24.

(1) Show that every open subset of $\mathbb{R}$ is a countable disjoint union of open intervals.
(2) Show that every open subset of $\mathbb{R}$ is a countable union of open intervals where both endpoints are rational.

1.1.3. *Separation axioms.*

Definition 1.1.25. We have the following separation properties for a topological space $(X, \mathcal{T})$.

- (T_0) For every $x, y \in X$ distinct, there is an open set $U \in \mathcal{T}$ which contains exactly one of x, y.
- (T_1) For every $x, y \in X$ distinct, there is an open set $U \in \mathcal{T}$ which only contains x. (Observe that by swapping x and y, there is also an open set $V \in \mathcal{T}$ which only contains y.)
- (T_2) (a.k.a. Hausdorff) for every $x, y \in X$ distinct, there are disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.
- (T_3) (a.k.a. Regular) (T_1) and for every closed $F \subset X$ and $x \in F^c$, there are disjoint open sets $U, V \in \mathcal{T}$ such that $F \subset U$ and $x \in V$.
- (T_4) (a.k.a. Normal) (T_1) and for every disjoint closed sets $F, G \subset X$, there are disjoint open sets $U, V \in \mathcal{T}$ such that $F \subset U$ and $G \subset V$.

Exercise 1.1.26. Let X be a set. The *finite complement topology* $\mathcal{T}$ has its opens those sets U such that U^c is finite and the empty set. Show $\mathcal{T}$ is (T_1). When is $\mathcal{T}$ Hausdorff?

Exercise 1.1.27. Suppose X is a normal topological space and $F \subset G \subset X$ with F closed and G open. Show there is an open U such that $F \subset U \subset \overline{U} \subset G$.

Lemma 1.1.28. *Suppose X is a normal topological space and $A, B \subset X$ are disjoint non-empty closed sets. Consider the dyadic rationals:*

$$D := \left\{ \frac{k}{2^n} \middle| n \in \mathbb{N},\ k = 1, \ldots, 2^n - 1 \right\} \subset (0,1) \tag{1.1.29}$$

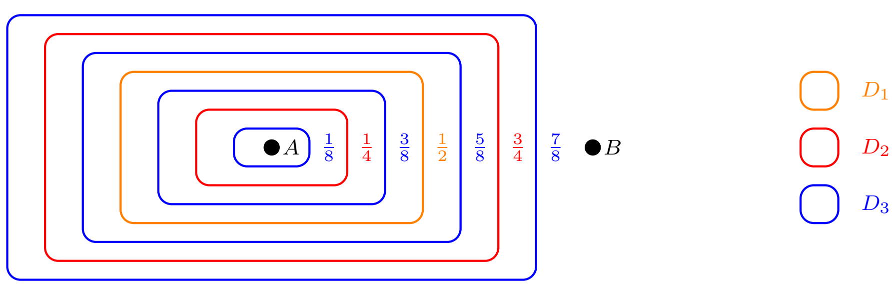
There are open sets $(U_d)_{d \in D}$ such that

- $A \subset U_d \subset \overline{U_d} \subset B^c$ *for all* $d \in D$, *and*
- $\overline{U_d} \subset U_{d'}$ *whenever* $d < d'$.

Proof. For $n \in \mathbb{N}$, set

$$D_n := \left\{ \frac{k}{2^n} \middle| k = 1, \ldots, 2^n - 1 \right\}.$$

We construct U_d for $d \in D_n$ inductively. Here is a cartoon of the main idea:



<u>Base case:</u> Let $U_{1/2}$ be any open set $A \subset U_{1/2} \subset \overline{U_{1/2}} \subset B^c$.
<u>Inductive Step:</u> Suppose that U_d have been defined for all $d \in D_1 \cup \ldots, \cup D_n$. Then, using the convention $U_0 := A$ and $U_1 := B^c$, we define $U_{\frac{2k+1}{2^{n+1}}}$ for $k = 0, 1, \ldots, 2^n - 1$ to be any open set such that

$$\overline{U_{k/2^n}} \subset U_{\frac{2k+1}{2^{n+1}}} \subset \overline{U_{\frac{2k+1}{2^{n+1}}}} \subset U_{\frac{k+1}{2^n}}. \qquad \square$$

Lemma 1.1.30 (Urysohn)**.** *Let X be a normal topological space. If $A, B \subset X$ are disjoint nonempty closed subsets, there is a continuous function $f : X \to [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.*

Proof. For the dyadic rationals $D \subset (0, 1)$ as in (1.1.29), we have open sets $(U_d)_{d\in D}$ satisfying the conditions in Lemma 1.1.28. Define $f : X \to [0, 1]$ by $f(x) := \sup \{d|x \notin U_d\}$. It is clear by construction that $f|_A = 0$ and $f|_B = 1$. Also observe that

(D1) $f(x) > d$ implies that $x \notin \overline{U_d}$, and $f(x) < d'$, then $x \in U_{d'}$.
(D2) If $x \notin \overline{U_d}$, then $f(x) \geq d$, and if $x \in U_{d'}$, then $f(x) \leq d'$.

It remains to prove that f is continuous. Fix $x_0 \in X$ and $\varepsilon > 0$.
Case 1: Suppose $0 < f(x) < 1$. Choose $d, d' \in D$ such that $d < f(x_0) < d'$ and $d' - d < \varepsilon$. By (D1) above, $x_0 \in U_{d'} \setminus \overline{U_d}$. By (D2) above, $|f(x) - f(x_0)| < \varepsilon$ for all $x \in U_{d'} \setminus \overline{U_d}$.
Case 2: $f(x) = 0$ or 1. Similar to above and omitted. □

Theorem 1.1.31 (Tietze Extension)**.** *Suppose X is normal, $A \subset X$ is closed, and $f : A \to [a, b]$ is continuous. Then there is a continuous function $F : X \to [a, b]$ such that $F|_A = f$.*

Proof. Without loss of generality, $[a, b] = [0, 1]$. (Otherwise, replace f with $(f - a)/(b - a)$.) We inductively construct a sequence of continuous functions (g_n) on X such that

- $0 \leq g_n \leq 2^{n-1}/3^n$ for all $n \in \mathbb{N}$, and
- $0 \leq f - \sum_{k=1}^n g_k \leq \left(\frac{2}{3}\right)^n$ on A for all $n \in \mathbb{N}$.

Then by (a), $\sum g_n$ converges uniformly to a continuous limit function F on X, and by (b), $F|_A = f$.
Base case: Set $B := f^{-1}([0, 1/3]) \subset A$ and $C := f^{-1}([2/3, 1]) \subset A$. Since f is continuous on A, $B, C \subset A \subset X$ are closed. By Urysohn's Lemma, there is a continuous function $g_1 : X \to [0, 1/3]$ such that $g_1|_B = 0$ and $g_1|_C = 1/3$. Then

$$f - g_1 \leq \left\{\begin{array}{ll} \frac{1}{3} - 0 = \frac{1}{3} & \text{on } B \subset A \\ \frac{2}{3} - 0 = \frac{2}{3} & \text{on } A \setminus (B \cup C) \\ 1 - \frac{1}{3} = \frac{2}{3} & \text{on } C \subset A \end{array}\right\} \leq \frac{2}{3} \qquad \text{on } A.$$

Inductive Step: Suppose we have constructed $g_1, \ldots, g_{n-1}$. Then there is a continuous function $g_n : X \to [0, 2^{n-1}/3^n]$ such that $g_n = 0$ on $\{f - \sum_{k=1}^{n-1} g_k \leq 2^{n-1}/3^n\}$ and $g_n = 2^{n-1}/3^n$ on $\{f - \sum_{k=1}^{n-1} g_k \geq 2^n/3^n\}$. This implies that $f - \sum_{k=1}^n g_k \leq 2^n/3^n$ on A as in the base case. □

1.2. **Locally compact Hausdorff spaces.**

Definition 1.2.1. A topological space X is called *compact* if every open cover has a finite subcover.

Exercise 1.2.2. A collection of subsets of $(A_i)_{i\in I}$ of X has the *finite intersection property* if for any finite $J \subset I$, we have $\bigcap_{j\in J} A_j \neq \emptyset$. Prove that the following are equivalent.

(1) Every open cover of X has a finite subcover.
(2) For every collection of closed subsets $(F_i)_{i\in I}$ with the finite intersection property, $\bigcap_{i\in I} F_i \neq \emptyset$.

Fact 1.2.3. An interval in $\mathbb{R}$ is compact if and only if it is closed and bounded.

Exercise 1.2.4. In this exercise, you will prove that the half-open interval topology on $(0, 1]$ is *Lindelöf*, i.e., every open cover has a countable sub-cover.

(1) Suppose $U \subset \mathbb{R}$ is open and suppose $((a_j, b_j))_{j\in J}$ is a collection of open intervals which cover U:
$$U \subset \bigcup_{j\in J}(a_j, b_j).$$
Show there is a countable sub-cover, i.e., show that there is a countable subset $I \subset J$ such that
$$U \subset \bigcup_{i\in I}(a_i, b_i).$$
Hint: Use Exercise 1.1.24.

(2) Suppose $((a_j, b_j])_{j\in J}$ is a collection of half-open intervals which cover $(0, 1]$:
$$(0, 1] \subset \bigcup_{j\in J}(a_j, b_j].$$
Show there is a countable sub-cover, i.e., show that there is a countable subset $I \subset J$ such that
$$(0, 1] \subset \bigcup_{i\in I}(a_i, b_i].$$

Exercises 1.2.5. Suppose X is a topological space. Verify the following assertions.

(1) If X is compact and $F \subset X$ is closed, then F is compact.
(2) If X is Hausdorff, $K \subset X$ is compact, and $x \notin K$, then there are disjoint open U, V such that $x \in U$ and $K \subset V$. In particular, K is closed.
(3) If X is compact Hausdorff, then X is normal.
(4) If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.
(5) If X is compact and Y is Hausdorff, and continuous bijection $f : X \to Y$ is automatically a homeomorphism (i.e., f^{-1} is continuous).

Exercise 1.2.6 (Lebesgue Number Lemma)**.** Suppose (X, d) is a compact metric space. Prove that for every open cover $(U_i)_{i\in I}$, there is a $\delta > 0$ such that for every $x_0 \in X$, there is an $i_0 \in I$ such that $B_\delta(x_0) \subset U_{i_0}$.

Exercise 1.2.7. Consider the following conditions:

(1) For every $x \in X$, there is a neighborhood U of x such that $\overline{U}$ is compact.
(2) For every $x \in X$, there is a neighborhood base $\mathcal{B}(x)$ consisting of neighborhoods U of x such that $\overline{U}$ is compact.
(3) For every $x \in X$ and every neighborhood U of x, there is an open V with $x \in V \subset U$ with $\overline{V}$ compact.
(4) For every $x \in X$ and every neighborhood U of x, there is an open V with $x \in V \subset \overline{V} \subset U$ with $\overline{V}$ compact.

Determine which conditions imply which other conditions. Then show all the above conditions are equivalent when X is Hausdorff.

Definition 1.2.8. A Hausdorff space satisfying one (equivalently all) of the conditions in Exercise 1.2.7 is called a *locally compact Hausdorff* (LCH) space.

Exercise 1.2.9. Suppose X is a second countable LCH space. Prove the following assertions.

(1) X is *σ-compact*, i.e., there is a sequence (K_n) of compact subsets of X such that $X = \bigcup K_n$.
(2) Every compact $K \subset X$ is a *G_δ-set*, i.e., a countable intersection of open sets.

Exercise 1.2.10 (Baire Category)**.** Suppose X is either:

(1) a complete metric space, or
(2) an LCH space.

Suppose (U_n) is a sequence of open dense subsets of X. Prove that $\bigcap U_n$ is dense in X.
Hint: Let V_0 be an arbitrary non-empty open set. Inductively construct an increasing sequence $(V_n)_{n\geq 1}$ of non-empty open subsets with $V_n \subset \overline{V_n} \subset U_{n+1} \cap V_n$ such that in the two cases above,

(1) V_n is a ball of radius $1/n$ for all $n \in \mathbb{N}$, or
(2) $\overline{V_n}$ is compact for all $n \in \mathbb{N}$.

Exercise 1.2.11. Suppose X is LCH. Verify the following assertions.

(1) If $K \subset U \subset X$ where K is compact and U is open, there is an open V with $K \subset V \subset \overline{V} \subset U$ with $\overline{V}$ compact.
Hint: Use Exercise 1.2.7(4).
(2) (Urysohn) If $K \subset U \subset X$ as above, there is a continuous $f : X \to [0,1]$ such that $f|_K = 1$ and $f = 0$ outside of a compact subset of U.
(3) (Tietze) Of $K \subset X$ is compact and $f \in C(K)$, there is an $F \in C_c(X)$ such that $F|_K = f$.

Definition 1.2.12. Let X be an LCH space. We define the following function algebras:

- $C(X)$ is the algebra of continuous ($\mathbb{C}$-valued) functions on X.
- $C_c(X)$ is the algebra of continuous functions of compact support, i.e., there is a compact set K such that $f|_{K^c} = 0$. We'll write $\operatorname{supp}(f) := \overline{\{x | f(x) \neq 0\}}$, so f has compact support if and only if $\operatorname{supp}(f)$ is compact.
- $C_0(X)$ is the algebra of continuous functions which *vanish at infinity*, i.e., for all $\varepsilon > 0$, $\{|f| \geq \varepsilon\}$ is compact.
- $C_b(X)$ is the algebra of continuous bounded functions.

We write $C(X, \mathbb{R}), C_c(X, \mathbb{R}), C_0(X, \mathbb{R}), C_b(X, \mathbb{R})$ for the real subalgebras of real-valued functions. Observe that

$$C_c(X) \subset C_0(X) \subset C_b(X) \subset C(X).$$

The *uniform/∞*-norm on $C_b(X)$ is given by

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

Exercise 1.2.13. Show that $C(X), C_c(X), C_0(X), C_b(X)$ are all complex algebras. Moreover, show $C_c(X), C_0(X)$ are unital if and only if X is compact.

Exercise 1.2.14 (Dini's Lemma)**.** Suppose X is a compact topological space and $(f_n) \subset C(X, [0,1])$. Show that if $f_n(x) \searrow 0$ pointwise, then $f_n \searrow 0$ uniformly.

Theorem 1.2.15. *Suppose X is LCH.*

(1) $\|\cdot\|_\infty$ is a norm on $C_b(X)$.

(2) $C_b(X)$ *is complete with respect to* $\|\cdot\|_\infty$*.*
(3) $C_0(X) \subset C_b(X)$ *is closed (and thus complete).*
(4) $\overline{C_c(X)}^{\|\cdot\|_\infty} = C_0(X)$*.*

Proof.
(1) Exercise.
(2) Suppose (f_n) is uniformly Cauchy. Then $(f_n(x))$ is Cauchy in $\mathbb{C}$ for every $x \in X$. Define $f(x) := \lim f_n(x)$, which is continuous (use $\varepsilon/3$ argument). Then one shows $\|f_n\|_\infty \subset [0, \infty)$ is bounded. Finally, you can show $f_n \to f$ uniformly, and $\sup |f(x)| \leq \sup \|f_n\| < \infty$.
(3) Suppose $(f_n) \subset C_0(X)$ such that $f_n \to f$ in $C_b(X)$. Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that $n \geq N$ implies $\|f - f_n\|_\infty < \varepsilon/2$. Since $f_N \in C_0(X)$, $\{|f_N| \geq \varepsilon/2\}$ is compact. Then $\{|f| \geq \varepsilon\} \subset \{|f_N| \geq \varepsilon/2\}$ is compact as a closed subset of a compact set.
(4) It suffices to prove that we can uniformly approximate any function in $C_0(X)$ by a function in $C_c(X)$. Let $f \in C_0(X)$ and $\varepsilon > 0$ so that $K := \{|f| \geq \varepsilon\}$ is compact. By the LCH Urysohn Lemma (Exercise 1.2.11(2)), there is a continuous function $g : X \to [0, 1]$ such that $g|_K = 1$ and g has compact support. Then $fg \in C_c(X)$, and $\|f - fg\|_\infty < \varepsilon$. □

Exercise 1.2.16. Suppose $(X, \mathcal{T})$ is a locally compact topological space and (f_n) is a sequence of continuous $\mathbb{C}$-valued functions on X. Show that the following are equivalent:

(1) There is a continuous function $f : X \to \mathbb{C}$ such that $f_n|_K \to f|_K$ uniformly on every compact $K \subset \mathbb{C}$.
(2) For every compact $K \subset X$, $(f_n|_K)$ is uniformly Cauchy.

Deduce that $C(X)$ is complete in the topology of local uniform convergence.

Exercise 1.2.17. Suppose X is a locally compact Hausdorff space, $K \subset X$ is compact, and $\{U_1, \ldots, U_n\}$ is an open cover of K. Prove that there are $g_i \in C_c(X, [0, 1])$ for $i = 1, \ldots, n$ such that $g_i = 0$ on U_i^c and $\sum_{i=1}^n g_i = 1$ everywhere on K.

1.3. **Convergence in topological spaces.** Let $(X, \mathcal{T})$ be a topological space. Recall that a sequence (x_n) converges to x, denoted $x_n \to x$ if for every open $U \in \mathcal{T}$ with $x \in U$, there is an $N \in \mathbb{N}$ such that $n > N$ implies $x_n \in U$ (x_n is *eventually* in U for every open neighborhood U of x). Not all spaces are first countable, so sequences do not suffice to describe the topology!

1.3.1. *Nets.*

Definition 1.3.1. A *directed set* is a set I equipped with a *preorder* (reflexive and transitive binary relation) $\leq$ satisfying

- for all $i, j \in I$, there is a $k \in I$ such that $i \leq k$ and $j \leq k$.

Examples 1.3.2.

(1) $\mathbb{N}, \mathbb{R}$, or any linearly ordered set.
(2) $\mathbb{R} \setminus \{a\}$ where $x \leq y$ if and only if $|x - a| \geq |y - a|$ (y is closer to a than x is).
(3) Any neighborhood base $\mathcal{T}(x)$ at $x \in X$, ordered by reverse inclusion ($U \leq V$ iff $V \subseteq U$).
(4) If X is any infinite set, $\{F \subset X | F \text{ is finite}\}$ ordered by inclusion.

Definition 1.3.3. Let X be a nonempty set and I a directed set. A *net in X based on I* (or an *I-net in X*) is a function $x : I \to X$, where write $x_i := x(i)$ and $x = (x_i)_{i \in I}$.

Given an I-net $(x_i)_{i \in I}$ and a subset $S \subset X$, we say

- (x_i) is *eventually* in S if there is some $j \in I$ such that for all $i \geq j$, $x_i \in S$.
- (x_i) is *frequently* in S if for every $j \in I$, there is an $i \geq j$ such that $x_i \in S$.

We say (x_i) *converges* to $x \in X$ if (x_i) is eventually in every neighborhood of x. We say x is a *cluster point* of (x_i) if (x_i) is frequently in every neighborhood of x.

Proposition 1.3.4. *Suppose X is a topological space and $A \subset X$. The following are equivalent for $x \in X$:*

(1) x is an accumulation/limit point of A (for all open U such that $x \in U$, $A \cap (U \setminus \{x\})$ is not empty), and
(2) there is a net in $A \setminus \{x\}$ that converges to x.

Proof.
(1) $\Rightarrow$ (2): Let $\mathcal{B}(x)$ be any neighborhood base at x, ordered by reverse inclusion. (For example, we can take $\mathcal{T}(x)$, the set of all open sets which contain x.) For every $U \in \mathcal{B}(x)$, pick $x_U \in U \cap (A \setminus \{x\})$. (Observe this requires the Axiom of Choice!) Then observe that $(x_U)_{U \in \mathcal{B}(x)}$ converges to x.
(2) $\Rightarrow$ (1): Exercise. $\square$

Corollary 1.3.5. *A subset $A \subset X$ is closed if and only if every convergent net in A only converges to points in A.*

Proposition 1.3.6. *X is Hausdorff if and only if every convergent net has a unique limit.*

Proof.
$\Rightarrow$: If there is a net without a unique limit, any 2 distinct limit points of the same net cannot be separted by disjoint open sets.
$\Leftarrow$: We'll prove the contrapositive. Suppose X is not Hausdorff, so there are $x, y \in X$ such that for every neighborhoods U, V of x, y respectively, $U \cap V$ is nonempty. Let $\mathcal{B}(x), \mathcal{B}(y)$ be a neighborhood base for $\mathcal{T}$ at x, y respectively, both ordered by reverse inclusion. Direct $\mathcal{B}(x) \times \mathcal{B}(y)$ by $(U_1, V_1) \geq (U_2, V_2)$ if and only if $U_1 \subset U_2$ and $V_1 \subset V_2$. Then for all $(U, V) \in \mathcal{B}(x) \times \mathcal{B}(y)$, choose a point $x_{(U,V)} \in U \cap V$. (Again, this uses the Axiom of Choice!) This net converges to both x and y. $\square$

Proposition 1.3.7. *A function $f : X \to Y$ is continuous if and only if for every convergent net $x_i \to x$ in X, $f(x_i) \to f(x)$ in Y.*

Proof.
$\Rightarrow$: Suppose $f : X \to Y$ is continuous. Let (x_i) be a convergent net with $x_i \to x$ in X. We need to show that $f(x_i) \to f(x)$ in Y. Let V be an open neighborhood of $f(x)$ in Y. Observe that $f^{-1}(V)$ is open in X, and $x \in f^{-1}(V)$. Since $x_i \to x$, (x_i) is eventually in $f^{-1}(V)$. Hence $f(x_i)$ is eventually in V.
$\Leftarrow$: We'll show that the preimage of every closed set is closed. Let $F \subset Y$ be closed. We may assume F is non-empty. By Corollary 1.3.5, it suffices to prove that every convergent net (x_i) in $f^{-1}(F)$ only converges to points of $f^{-1}(F)$. So suppose (x_i) is a convergent net in $f^{-1}(F)$, and say $x_i \to x$. Then $f(x_i) \in F$ for all i, and $f(x_i) \to f(x)$ by assumption. Since F is closed, by Corollary 1.3.5, $f(x) \in F$, and thus $x \in f^{-1}(F)$. $\square$

Definition 1.3.8. A *subnet* of an I-net (x_i) consists of a J-net (y_j) together with a function $f : J \to I$ which need not be injective such that

- $y_j = x_{f(j)}$ for all $j \in J$, i.e., $y = x \circ f : J \to X$.
- for all $i \in I$, there is a $j_0 \in J$ such that $f(j) \geq i$ for all $j \geq j_0$, i.e., for every $i \in I$, $(f(j))$ is eventually greater than i.

Observe that if $x_i \to x$, then $y_j \to x$ for any subnet (y_j) of (x_i).

Proposition 1.3.9. *Suppose (x_i) is a net in X and $x \in X$. The following are equivalent:*

(1) x is a cluster point of (x_i).
(2) there is a subnet (y_j) of (x_i) such that $y_j \to x$.

Proof.
(1) $\Rightarrow$ (2): Choose a neighborhood base $\mathcal{B}(x)$ at x. Define $J := I \times \mathcal{B}(x)$ where $(i_1, U_1) \leq (i_2, U_2)$ iff $i_1 \leq i_2$ and $U_1 \supset U_2$. For each $(i, U) \in J$, define $f(i, U) := i'$ to be *any* i' with $i' \geq i$ and $x_{i'} \in U$. Then if $(i_1, U_1) \leq (i_2, U_2)$, $i_1 \leq i_2 \leq f(i_2, U_2)$, and $x_{f(i_2,U_2)} \in U_2 \subset U_1$. This means $(x_{f(i,U)})$ is a subnet of (x_i) converging to x.
(2) $\Rightarrow$ (1): Exercise. □

Exercise 1.3.10. When $(X, \mathcal{T})$ is first countable, then Propositions 1.3.4, 1.3.6, 1.3.7, and 1.3.9 and Corollary 1.3.5 all hold with sequences instead of nets.

Exercise 1.3.11. Suppose (X, d) is a metric space. Prove that the following are equivalent:

(1) X is compact.
(2) X is sequentially compact (every sequence has a convergent subsequence).
(3) X is complete and totally bounded.

Deduce that if in addition X is complete and $A \subset X$, then $\overline{A}$ is compact if and only if A is totally bounded.

Theorem 1.3.12. *Suppose X is a topological space. The following are equivalent:*

(1) X is compact.
(2) For every family of closed sets (F_i) with the finite intersection property, $\bigcap F_i$ is nonempty.
(3) Every net in X has a cluster point.
(4) Every net in X has a convergent subnet.

Proof.
(1) $\Leftrightarrow$ (2): This is Exercise 1.2.2.
(3) $\Leftrightarrow$ (4): This follows by Proposition 1.3.9.
(2) $\Rightarrow$ (3): Let (x_i) be a net in X. For $i \in I$, define $A_i := \{x_j | j \geq i\}$. Observe $\bigcap \overline{A_i}$ is the set of cluster points of (x_i). Moreover, (A_i) has the finite intersection property, so $(\overline{A_i})$ also has the finite intersection property. We conclude by (2) that $\bigcap \overline{A_i}$ is nonempty, and thus (x_i) has a cluster point.
(3) $\Rightarrow$ (2): We'll prove the contrapositive. If (2) fails, then there is a family of closed sets (F_i) with the finite intersection property such that $\bigcap F_i = \emptyset$. Define J to be the set of non-empty finite intersections of (F_i) ordered by reverse inclusion. Since (F_i) has the finite intersection property, for every $F \in J$, F is nonempty. Use the Axiom of Choice to pick $x_F \in F$ for every $F \in J$. Then any cluster point of (x_F) lies in $\bigcap_{F \in J} F = \bigcap F_i = \emptyset$. □

1.3.2. *Filters.*

Exercise 1.3.13 (Pedersen *Analysis Now*, E1.3.4 and E1.3.6)**.** A *filter* on a set X is a collection $\mathcal{F}$ of non-empty subsets of X satisfying

- $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, and
- $A \in \mathcal{F}$ and $A \subset B$ implies $B \in \mathcal{F}$.

Suppose $\mathcal{T}$ is a topology on X. We say a filter $\mathcal{F}$ *converges* to $x \in X$ if every open neighborhood U of x lies in $\mathcal{F}$.

(1) Show that $A \subset X$ is open if and only if $A \in \mathcal{F}$ for every filter $\mathcal{F}$ that converges to a point in A.
(2) Show that if $\mathcal{F}$ and $\mathcal{G}$ are filters and $\mathcal{F} \subset \mathcal{G}$ ($\mathcal{G}$ is a *subfilter* of $\mathcal{F}$), then $\mathcal{G}$ converges to x whenever $\mathcal{F}$ converges to x.
(3) Suppose (x_λ) is a net in X. Let $\mathcal{F}$ be the collection of sets A such that (x_λ) is eventually in A. Show that $\mathcal{F}$ is a filter. Then show that $x_\lambda \to x$ if and only if $\mathcal{F}$ converges to x.
(4) Show that $(X, \mathcal{T})$ is Hausdorff if and only if every convergent filter has a unique limit.

Exercise 1.3.14 (Pedersen *Analysis Now*, E1.3.5)**.** A filter $\mathcal{F}$ on a set X is called an *ultrafilter* if it is not properly contained in any other filter.

(1) Show that a filter $\mathcal{F}$ is an ultrafilter if and only if for every $A \subset X$, we have either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.
(2) Use Zorn's Lemma to prove that every filter is contained in an ultrafilter.

Exercise 1.3.15. Let X and Y be sets and $f : X \to Y$ a function. Let $\mathcal{F}$ be an ultrafilter on X. Prove that $f^*(\mathcal{F}) := \{A \subset Y | f^{-1}(A) \in \mathcal{F}\}$ is an ultrafilter on Y.

Exercise 1.3.16. Given a filter $\mathcal{F}$ on X, show that $\mathcal{F}$ is an ultrafilter if and only if $\bigcup_{i=1}^n A_i \in \mathcal{F}$ implies that $A_i \in \mathcal{F}$ for some $i \in \{1, \ldots, n\}$.

Exercise 1.3.17. Let X be a nonempty set and let $\mathcal{U}$ be a collection of subsets of X.
Note: It is not assumed that $\mathcal{U}$ is a filter!
Show that the following two statements are equivalent.

(1) $\mathcal{U}$ is an ultrafilter on X.
(2) Whenever X can be partitioned into three disjoint sets $X = A_1 \amalg A_2 \amalg A_3$, there is a unique $i \in \{1, 2, 3\}$ such that $A_i \in \mathcal{U}$.
Hint: The A_i's need not be distinct nor non-empty.

Exercise 1.3.18. Let $(X, \mathcal{T})$ be a topological space. A net $(x_\lambda)_{\lambda \in \Lambda}$ is called *universal* if for every subset $Y \subset X$, (x_λ) is either eventually in Y or eventually in Y^c. Show that every net has a universal subnet.
Hint: Let (x_λ) be a net in X. We say a filter $\mathcal{F}$ on X is associated to *(x_λ) if (x_λ) is frequently in every $F \in \mathcal{F}$.*

(1) Show that the set of filters associated to (x_λ) is non-empty.
(2) Order the set of filters associated to (x_λ) by inclusion. Show that if $(\mathcal{F}_j)$ is a totally ordered set of filters for (x_λ), then $\cup \mathcal{F}_j$ is also a filter for (x_λ).
(3) Use Zorn's Lemma to assert there is a maximal filter $\mathcal{F}$ associated to (x_λ).
(4) Show that $\mathcal{F}$ is an ultrafilter.
(5) Find a subnet of (x_λ) that is universal.

Exercise 1.3.19. Let $(X, \mathcal{T})$ be a topological space. Prove that the following are equivalent:

(1) $(X, \mathcal{T})$ is compact
(2) every ultrafilter converges
(3) every universal net converges.

1.4. **Categories, universal properties, and product topology.**

Definition 1.4.1. A *category* $\mathcal{C}$ is a collection of *objects* together with a set of *morphisms* $\mathcal{C}(a \to b)$ for every ordered pair of objects $a, b \in \mathcal{C}$ and a composition operation $- \circ_{\mathcal{C}} - : \mathcal{C}(b \to c) \times \mathcal{C}(a \to b) \to \mathcal{C}(a \to c)$, i.e., $f : a \to b$ and $g : b \to c$, then $g \circ f : a \to c$ such that

- composition is associative, i.e., $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : a \to b$, $g : b \to c$, and $h : c \to d$.
- every object has an identity morphism, i.e., for every $b \in \mathcal{C}$, there is a $\mathrm{id}_b : b \to b$ such that $\mathrm{id}_b \circ f = f$ for all $f : a \to b$ and $g \circ \mathrm{id}_b = g$ for all $g : b \to c$

Definition 1.4.2. Suppose $(X_i)_{i \in I}$ is a family of sets. The (categorical) *product* is the Cartesian product

$$\prod_{i \in I} X_i := \left\{ x : I \to \bigcup_{i \in I} X_i \,\middle|\, x_i := x(i) \in X_i \right\}$$

together with the canonical projection maps $\pi_j : \prod X_i \to X_j$ given by $\pi_j(x) = x_j$. It satisfies the following *universal property* :

- (product) for any set Z and functions $f_i : Z \to X_i$ for $i \in I$, there is a unique function $\prod f_i : Z \to \prod X_i$ such that $\pi_j \circ \prod f_i = f_j$ for all $j \in I$.

$$\begin{tikzcd}
Z \arrow[d, dashed, "\exists! \prod f_i"'] \arrow[dr, "f_j"] & \\
\prod_{i \in I} X_i \arrow[r, "\pi_j"] & X_j
\end{tikzcd}$$

Exercise 1.4.3. Suppose Y is another set together with functions $\theta_i : Y \to X_i$ for all $i \in I$ satisfying the universal property of the product. Show there is a unique bijection between Y and $\prod X_i$ which is compatible with the projection maps. In this sense, we say that the product is *unique up to unique isomorphism*.

Exercise 1.4.4. A set $\coprod X_i$ together with maps $\iota_j : X_j \to \coprod X_i$ for each $j \in I$ is called the *coproduct* of $(X_i)_{i \in I}$ if it satisfies the following universal property:

- (coproduct) for any set Z and functions $f_i : X_i \to Z$ for $i \in I$, there is a unique function $\coprod f_i : \coprod X_i \to Z$ such that $(\coprod f_i) \circ \iota_j = f_j$.

$$\begin{tikzcd}
X_j \arrow[r, "\iota_j"] \arrow[dr, "f_j"'] & \coprod_{i \in I} X_i \arrow[d, dashed, "\exists! \coprod f_i"] \\
& Z
\end{tikzcd}$$

(1) Show that the coproduct, if it exists, is unique up to unique isomorphism.
(2) What is the coproduct in the category of sets?

Definition 1.4.5. Suppose $(X_i)_{i\in I}$ is a family of topological spaces. The (categorical) *product* is the Cartesian product $\prod_{i\in I} X_i$ equipped with the *weakest* topology such that the canonical projection maps $\pi_j : \prod X_i \to X_j$ are continuous for every $j \in I$. We call this topology the *product topology*.

Exercise 1.4.6. Prove that the open sets $\prod U_i$ with $U_i \subset X$ open where only finitely many of the U_i are not equal to X_i form a base for the product topology.

Exercise 1.4.7. Prove that $\prod X_i$ with the product topology together with the canonical projection maps $\pi_j : \prod X_i \to X_j$ is the categorical product in the category of topological spaces with continuous maps. That is, prove the product satisfies the universal property in Definition 1.4.2 subject to the additional condition that all functions are continuous.

Exercise 1.4.8. What is the categorical coproduct of topological spaces?

Theorem 1.4.9 (Tychonoff)**.** *Suppose $(X_i)_{i\in I}$ is a family of compact topological spaces. Then the product $\prod X_i$ is compact in the product topology.*

Proof. Discussion section. □

Definition 1.4.10. Suppose $\mathcal{C}, \mathcal{D}$ are categories. A (covariant) *functor* $F : \mathcal{C} \to \mathcal{D}$ assigns to each object $c \in \mathcal{C}$ an object $F(c) \in \mathcal{D}$ and to each morphism $f \in \mathcal{C}(a \to b)$ a morphism $F(f) \in \mathcal{D}(F(a) \to F(b))$ such that

- $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for all objects $c \in \mathcal{C}$, and
- $F(g \circ f) = F(g) \circ F(f)$ for all $f \in \mathcal{C}(a \to b)$ and $g \in \mathcal{C}(b \to c)$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is similar to a functor, but instead of the second bullet point above, we have $F(g \circ f) = F(f) \circ F(g)$ for composable f, g.

Exercise 1.4.11. Let Set denote the category of sets and functions.

(1) For a function $f : X \to Y$, define $P(f) : P(X) \to P(Y)$ by $P(f)(A) = f(A) = \{f(a) | a \in A\}$. Show that $P\mathsf{Set} \to \mathsf{Set}$ is a functor.
(2) For a set X, define $P^{-1}(X) := P(X) = \{A \subset X\}$. For a function $f : X \to Y$ and $B \subset Y$, define $P^{-1}(f)(B) := f^{-1}(B) = \{x \in X | f(x) \in B\}$. Show that $P^{-1} : \mathsf{Set} \to \mathsf{Set}$ is a contravariant functor.

Exercise 1.4.12. Let Top denote the category topological spaces and continuous maps.

(1) There is a forgetful functor $\mathrm{Forget} : \mathsf{Top} \to \mathsf{Set}$ which forgets the topology.
(2) Given a set X, we can endow it with the discrete topology $\mathcal{T}_{\mathrm{disc}} := P(X)$. This gives a functor $L : \mathsf{Set} \to \mathsf{Top}$. Show that if Y is any topological space, then every function $X \to Y$ is continuous with respect to the discrete topology on X. In other words,
$$\mathsf{Top}(L(X) \to Y) = \mathsf{Set}(X \to \mathrm{Forget}(Y)).$$
(3) Given a set Y, we can endow it with the trivial topology $\mathcal{T}_{\mathrm{triv}} := \{\emptyset, Y\}$. This gives a functor $R : \mathsf{Set} \to \mathsf{Top}$. Show that if X is any topological space and Y is a set, then every function $X \to Y$ is continuous with respect to the trivial topology on Y. In other words,
$$\mathsf{Set}(\mathrm{Forget}(X) \to Y) = \mathsf{Top}(X \to R(Y)).$$

Exercise 1.4.13. Let CptHsd denote the category of compact Hausdorff topological spaces and continuous maps. Let $\mathsf{Alg_u}$ denote the category of unital complex algebras and unital algebra homomorphisms. Show that $X \mapsto C(X)$ and $f : X \to Y$ maps to $- \circ f : C(Y) \to C(X)$ gives a contravariant functor $\mathsf{CptHsd} \to \mathsf{Alg_u}$.

Exercise 1.4.14.

(1) Given LCH spaces X, Y and a continuous function $f : X \to Y$, when does the image of the map $- \circ f : C_0(Y) \to C(X)$ lie in $C_0(X)$?
(2) Show that on the correct category LCH of locally compact Hausdorff topological spaces, the assignments $X \mapsto C_0(X)$ and $f \mapsto - \circ f$ define a contravariant functor to Alg, the category of non-unital complex algebras and algebra homomorphisms

1.5. **The Stone-Weierstrass Theorem.** Weierstrass' original theorem from 1885:

(1) The polynomials are dense in $C[a, b]$ where $-\infty < a < b < \infty$.
(2) A continuous function on $\mathbb{R}$ with period 2π can be uniformly approximated by a finite linear combination of functions of the form $\sin(nx), \cos(nx)$ for $n \in \mathbb{N}$, i.e., a trigonometric polynomial.

Theorem 1.5.1 ($\mathbb{R}$-Stone-Weierstrass)**.** *Suppose X is compact Hausdorff and $A \subset C(X, \mathbb{R})$ is a closed $\mathbb{R}$-subalgebra which separates points (for all distinct $x, y \in X$, there is an $f \in A$ such that $f(x)$ and $f(y)$ are distinct).*

- *If A contains a non-vanishing function, then $A = C(X, \mathbb{R})$.*
- *If every $f \in A$ has a zero, then there exists a unique $x_0 \in X$ such that*

$$A = \{f \in C(X, \mathbb{R}) | f(x_0) = 0\}.$$

Exercise 1.5.2. Suppose X is compact Hausdorff and $A \subset C(X, \mathbb{F})$ is a subalgebra where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. Prove that $\overline{A}$ is also a subalgebra. Deduce that if A separates points, then so does $\overline{A}$.

Lemma 1.5.3. *On any compact $K \subset \mathbb{R}$, the function $x \mapsto |x|$ on $\mathbb{R}$ can be uniformly approximated on K by a polynomial which vanishes at zero.*

Proof. We give a proof of Sarason. We'll show for $R > 0$, there is a sequence of polynomials (p_n) which converges uniformly to $|\cdot|$ on $[-R, R]$ such that $p_n(0) = 0$ for all n. Without loss of generality, $R = 1$. It suffices to find a sequence (q_n) of polynomials converging to $q(t) := 1 - |t|$ on $[-1, 1]$ such that $q_n(0) = 1$ for all n. Observe that

$$q \text{ takes values in } [0,1] \text{ and } (1 - q(t))^2 = t^2 \text{ for all } |t| \leq 1. \qquad (*)$$

For a given $t \in [-1, 1]$, consider the equation $(1 - s)^2 = t^2$. It has 2 solutions, namely $s = 1 \pm |t|$, and exactly one of these values of s lies in $[0, 1]$. Hence $q(t)$ is unique function on $[-1, 1]$ satisfying $(*)$. We can rewrite $(*)$ as

$$q \text{ takes values in } [0,1] \text{ and } q(t) = \frac{1}{2}(1 - t^2 + q(t)^2). \qquad (**)$$

We define (q_n) inductively by

- $q_0(t) = 1$, and
- $q_{n+1}(t) = \frac{1}{2}(1 - t^2 + q_n(t)^2)$.

By induction, for all $n \geq 0$, we have q_n takes values in $[0,1]$, $q_n(0) = 1$, and

$$q_n - q_{n+1} = \frac{1}{2}(q_{n-1}^2 - q_n^2) = \frac{1}{2}(q_{n-1} - q_n)(q_{n-1} + q_n) \geq 0.$$

(Indeed, observe that $q_1(t) = 1 - \frac{1}{2}t^2$, so $q_0 - q_1 = \frac{1}{2}t^2 \geq 0$.) This means that (q_n) is monotone decreasing by construction. Let $\widetilde{q}$ be the pointwise limit. Observe that $\widetilde{q}$ satisfies $(**)$ by construction, so $\widetilde{q} = q$ by uniqueness! Now as $q_n \searrow q$ on $[-1,1]$ pointwise, $q_n \to q$ uniformly by Dini's Lemma (Exercise 1.2.14). $\square$

Lemma 1.5.4. *If $A \subset C(X, \mathbb{R})$ is a closed $\mathbb{R}$-subalgebra, then A is a lattice (for all $f, g \in A$, the functions $f \vee g := \max\{f, g\}$ and $f \vee g := \min\{f, g\}$ belong to A).*

Proof. Suppose $a \in A$ and $a \neq 0$. Then $\frac{a}{\|a\|_\infty} : X \to [-1,1]$. By Lemma 1.5.3, for all $\varepsilon > 0$, there is a polynomial p on $[-1,1]$ with $p(0) = 0$ and $|\,|t| - p(t)| < \varepsilon$ for all $t \in [-1,1]$. Hence

$$\left| \frac{|a(x)|}{\|a\|_\infty} - p\left(\frac{a(x)}{\|a\|_\infty}\right) \right| < \varepsilon \qquad\qquad \forall\, x \in X.$$

In other words,

$$\left\| \frac{|a|}{\|a\|_\infty} - \underbrace{p\left(\frac{a}{\|a\|_\infty}\right)}_{\in A} \right\|_\infty < \varepsilon.$$

Since $p(0) = 0$, $p(a/\|a\|_\infty) \in \operatorname{span}\{a^n | n \in \mathbb{N}\} \subset A$. Since the algebra A is closed and $\varepsilon > 0$ was arbitrary, $|a|/\|a\|_\infty \in A$, and thus $|a| \in A$. Hence for all $a, b \in A$,

$$\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$$
$$\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$$

are both elements of A. $\square$

Lemma 1.5.5. *Suppose $A \subset C(X, \mathbb{R})$ is a $\mathbb{R}$-vector space which is also a lattice. Suppose $f \in C(X, \mathbb{R})$ satisfies*

- *for all $\varepsilon > 0$ and all distinct $x, y \in X$, there is an $a_{x,y} \in A$ such that*
$$|f(x) - a_{x,y}(x)| < \varepsilon \qquad \text{and} \qquad |f(y) - a_{x,y}(y)| < \varepsilon.$$

Then $f \in \overline{A}$.

Proof. For every $\varepsilon > 0$ and $x, y \in X$, pick $a_{x.y} \in A$ such that $|f(x) - a_{x,y}(x)| < \varepsilon$ and $|f(y) - a_{x,y}(y)| < \varepsilon$. Then x, y are both in:

$$U_{x,y} = \{z \in X | f(z) < a_{x,y}(z) + \varepsilon\}$$
$$V_{x,y} = \{z \in X | a_{x,y}(z) < f(z) + \varepsilon\}.$$

Fix $x \in X$. Then sets $(U_{x,y})_{y \in X}$ are an open cover of X. Since X is compact, $X \subset \bigcup_{i=1}^n U_{x,y_i}$ for some $y_1, \ldots, y_n \in X$. Then $a_x := \bigvee_{i=1}^n a_{x,y_i} \in A$, and $f(z) < a_x(z) + \varepsilon$ for all $z \in X$ in construction. Also, $a_x(z) < f(z) + \varepsilon$ for all $z \in W_x := \bigcap_{i=1}^n V_{x,y_i}$, which is some open neighborhood of x. Varying over $x \in X$, $(W_x)_{x \in X}$ are an open cover, so there are finitely many $x_1, \ldots, x_k \in X$ such that $X \subset \bigcup_{i=1}^k W_{x_i}$ by compactness. Setting $a_\varepsilon := \bigwedge_{i=1}^k a_{x_i}$ satisfies $\|f - a_\varepsilon\|_\infty < \varepsilon$. Since ε was arbitrary, we conclude that $f \in \overline{A}$. $\square$

Proof of the $\mathbb{R}$-Stone-Weierstrass Theorem 1.5.1. Suppose $x \neq y$ in X. Since point evaluation is an $\mathbb{R}$-algebra homomorphism $A \to \mathbb{R}$, then

$$A_{x,y} := \{(f(x), f(y)) | f \in A\} \subset \mathbb{R}^2$$

is a $\mathbb{R}$-subalgebra. The only $\mathbb{R}$-subalgebras of $\mathbb{R}^2$ are:

$$(0,0) \qquad \mathbb{R} \times \{0\} \qquad \{0\} \times \mathbb{R} \qquad \Delta = \{(x,x) | x \in \mathbb{R}\} \qquad \mathbb{R}^2.$$

Since A separates points, $A_{x,y} \neq (0,0)$ or Δ for all $x \neq y$.

Claim. *$A_{x,y} = \mathbb{R}^2$ for all $x \neq y$ except for when x, y are equal to one possible $x_0 \in \mathbb{R}$.*

Proof. If there are $x \neq y$ such that $A_{x,y} \neq \mathbb{R}^2$, then without loss of generality, $A_{x,y} = \{0\} \times \mathbb{R}$. Thus $f(x) = 0$ for all $f \in A$. Since A separates points, $f(x') = 0$ for all $f \in A$ implies $x' = x$. So $A_{y,z} = \mathbb{R}^2$ for all $y \neq x \neq z$. □

Claim. *$A_{x,y} = \mathbb{R}^2$ for all $x \neq y$ if and only if A contains a non-vanishing function.*

Proof of Claim. If A contains a non-vanishing function, then $A_{x,y} = \mathbb{R}^2$ for all $x \neq y$. Conversely, suppose $A_{x,y} = \mathbb{R}^2$ for all $x \neq y$. Then for all $x \in X$, choose a continuous function $a_x \in A$ such that $a_x(x) \neq 0$. Observe that the sets $(U_x := \{a_x \neq 0\})_{x \in X}$ form an open cover of X, so by compactness, there are $x_1, \ldots, x_n$ such that $X \subset \bigcup_{i=1}^n U_{x_i}$. By Lemma 1.5.4, A is a lattice, so

$$a := \max\{a_{x_1}, \ldots, a_{x_n}, -a_{x_1}, \ldots, -a_{x_n}\} = \max\{|a_{x_1}|, \ldots, |a_{x_n}|\} \in A.$$

Since $|a_{x_i}| > 0$ on U_{x_i} for all $i = 1, \ldots, n$, we have $a(x) > 0$ for all $x \in X$ by construction. □

From these claims, we see that either A contains a non-vanishing function, in which case $A_{x,y} = \mathbb{R}^2$ for all $x \neq y$, or every function in A vanishes at some point of X, in which case there is a unique $x_0 \in X$ such that $a(x_0) = 0$ for all $a \in A$.

Case 1: For all $x \neq y$ in X and $f \in C(X, \mathbb{R})$, there is an $a_{x,y} \in A$ such that $f(x) = a_{x,y}(x)$ and $f(y) = a_{x,y}(y)$. By Lemma 1.5.4, A is a lattice, and by Lemma 1.5.5, $f \in A$.

Case 2: For all $x_0 \neq x \neq y \neq x_0$ and $f \in \{g \in C(X, \mathbb{R}) | g(x_0) = 0\}$ (which is a closed subalgebra/ideal of $C(X, \mathbb{R})$), there is an $a_{x,y} \in A$ such that $f(x) = a_{x,y}(x)$ and $f(y) = a_{x,y}(y)$. By Lemma 1.5.4, A is a lattice, and by Lemma 1.5.5, $f \in A$. □

Theorem 1.5.6 ($\mathbb{C}$-Stone-Weierstrass)**.** *Suppose X is a compact Hausdorff space. Let $A \subset C(X)$ be a closed subalgebra that separates points of X and is closed under complex conjugation.*

- *If A contains a non-vanishing function, then $A = C(X)$.*
- *If every $f \in A$ has a zero, then there exists a unique $x_0 \in X$ such that*

$$A = \{f \in C(X) | f(x_0) = 0\}.$$

Proof. Note that $A_{\text{sa}} := \left\{f \in A \middle| f = \overline{f}\right\}$ is an $\mathbb{R}$-subalgebra of A. (Here, 'sa' stands for *self-adjoint*.) Since A is closed under complex conjugation, for all $f \in A$, $\operatorname{Re}(f), \operatorname{Im}(f) \in A$, and thus $A = A_{\text{sa}} \oplus iA_{\text{sa}}$. Moreover, $C(X) = C(X, \mathbb{R}) \oplus iC(X, \mathbb{R})$ by similar reasoning. Hence the strategy is to apply the $\mathbb{R}$-Stone-Weierstrass Theorem 1.5.1 to $A_{\text{sa}} \subset C(X, \mathbb{R})$.

First, observe A_{sa} separates points, since if $f \in A$ separates x, y, then one of $\text{Re}(f), \text{Im}(f) \in A_{\text{sa}}$ separates x, y. Second, observe that A_{sa} is closed, since if $(f_n) \subset A_{\text{sa}}$ converges uniformly, then its limit lies in A as A is closed, and since (f_n) must converge pointwise, its limit only takes real values and thus lies in A_{sa}.

We now check the two cases in the statement of the theorem.

Case 1: If A contains a non-vanishing function f, then $|f|^2 = \overline{f} f \in A_{\text{sa}}$ does not vanish. By the $\mathbb{R}$-Stone-Weierstrass Theorem 1.5.1, $A_{\text{sa}} = C(X, \mathbb{R})$, and thus

$$A = A_{\text{sa}} \oplus i A_{\text{sa}} = C(X, \mathbb{R}) \oplus i C(X, \mathbb{R}) = C(X).$$

Case 2: If every element of A vanishes somewhere, then so does every element of $A_{\text{sa}} \subset A$. By the $\mathbb{R}$-Stone-Weierstrass Theorem 1.5.1, $A_{\text{sa}} = \{f \in C(X, \mathbb{R}) | f(x_0) = 0\}$, and thus

$$\begin{aligned} A &= A_{\text{sa}} \oplus i A_{\text{sa}} \\ &= \{f \in C(X, \mathbb{R}) | f(x_0) = 0\} \oplus i \{f \in C(X, \mathbb{R}) | f(x_0) = 0\} \\ &= \{f \in C(X) | f(x_0) = 0\}. \end{aligned}$$

□

Exercise 1.5.7. Suppose X is LCH and $A \subset C_0(X)$ is a closed subalgebra that separates points and is closed under complex conjugation. Then either $A = C_0(X)$ or $A = \{f \in C_0(X) | f(x_0) = 0\}$ for some $x_0 \in X$.
Hint: Use the one point (Alexandroff) compactification discussed in §1.6 below.

Exercise 1.5.8. Show the following collections of functions are uniformly dense in the appropriate algebras:

(1) For $a < b$ in $\mathbb{R}$, the polynomials $\mathbb{R}[t] \subset C([a, b], \mathbb{R})$.
(2) For $a < b$ in $\mathbb{R}$, the piece-wise linear functions $PWL \subset C([a, b], \mathbb{R})$.
(3) For $K \subset \mathbb{C}$ compact, the polynomials $\mathbb{C}[z, \overline{z}] \subset C(K)$.
(4) For $\mathbb{R}/\mathbb{Z}$, the trigonometric polynomials $\text{span}\{\sin(2\pi n x), \cos(2\pi n x) | n \in \mathbb{N} \cup \{0\}\} \subset C(\mathbb{R}/\mathbb{Z}, \mathbb{R})$.

Exercise 1.5.9.

(1) Use the difference quotient to show that complex cojugation $\overline{\cdot} : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto \overline{z}$ is nowhere complex differentiable.
(2) Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk $\{|z| < 1\}$. Describe the uniform closure of $\mathbb{C}[z]$, the polynomials in z, in $C(\overline{\mathbb{D}})$.
Hint: You may use without proof Morera's Theorem from Complex Analysis which states on any open domain $U \subset \mathbb{C}$, the local uniform limit of complex differentiable functions is complex differentiable.
(3) Discuss your answer in the context of the Stone-Weierstrass Theorem.

Exercise 1.5.10. Let X, Y be compact Hausdorff spaces. For $f \in C(X)$ and $g \in C(Y)$, define $(f \otimes g)(x, y) := f(x) g(y)$. Prove that $\text{span}\{f \otimes g | f \in C(X) \text{ and } g \in C(Y)\}$ is uniformly dense in $C(X \times Y)$.

Exercise 1.5.11 (Sarason)**.** Suppose $f \in C([0, 1], \mathbb{R})$ such that $\int_0^1 x^n f(x)\, dx = 0$ for all $n \geq 2020$. Prove that $f = 0$.
Hint: Consider $A := \text{span}\{x^n | n \geq 2020\} \subset C([0, 1], \mathbb{R})$.

Exercise 1.5.12 (Sarason)**.** Find a sequence of polynomials in $\mathbb{R}[t] \subset C(\mathbb{R}, \mathbb{R})$ that simultaneously converges to 1 uniformly on every compact subinterval of $(0, \infty)$ and to -1 uniformly on every compact subinterval of $(-\infty, 0)$.

1.6. **One point (Alexandroff) and Stone-Čech compactification.**

Definition 1.6.1. Suppose X is a topological space. An *embedding* $\varphi : X \to Y$ is a continuous injection which is a homeomorphism onto its image, i.e., $\varphi^{-1} : \varphi(X) \to X$ is continuous with respect to the relative topology.

A *compactification* of a topological space X consists of a compact space K and an embedding $\varphi : X \to K$ such that $\varphi(X)$ is dense in K.

Example 1.6.2. Consider the map $[0, 1) \to S^1 := \{z \in \mathbb{C} | |z| = 1\}$ by $r \mapsto \exp(2\pi i r)$. This map is a continuous bijection, but not a homeomorphism onto its image.

Examples 1.6.3. Compactifications of $\mathbb{R}$ include:

(1) the extended real numbers $\overline{\mathbb{R}} = [-\infty, \infty]$
(2) the 'one point' compactification $\mathbb{R} \cup \{\infty\} \cong S^1$
(3) You can add $(0, 0)$ and S^1 in $\mathbb{R}^2$ to an embedding $\mathbb{R} \hookrightarrow \mathbb{R}^2$ as a spiral.
(4) You can add a circle S^1 embedded in a 2-toruse $\mathbb{T}^2 \subset \mathbb{R}^3$ to an embedding $\mathbb{R} \hookrightarrow \mathbb{T}^2$ which coils $\mathbb{R}$ around the torus from either side.

Definition 1.6.4. Suppose X is an LCH space, and choose any object $\infty \notin X$. Define $X^\bullet := X \amalg \{\infty\}$, where $\amalg$ denotes disjoint union (coproduct in Set). We say $U \subset X^\bullet$ is open if and only if either

- $U \subset X$ is open in X, or
- $\infty \in U$, and U^c is compact.

Due to the next theorem, we call $X^\bullet$ the (Alexandroff) *one point compactification* of X.

Theorem 1.6.5. *If X is LCH, then the space $X^\bullet$ is compact Hausdorff, and the inclusion $X \hookrightarrow X^\bullet$ is an embedding.*

Proof. The inclusion $X \hookrightarrow X^\bullet$ is obviously an embedding.

Compact: Suppose (U_i) is an open cover of $X^\bullet$. Then there is some U_0 such that $\infty \in U_0$ and U_0^c is compact. Then $(U_i \cap X)$ is an open cover of U_0^c, which is compact. So pick a finite subcover.

Hausdorff: Since X is Hausdorff, it suffices to separate $x \in X$ from $\infty \in X^\bullet$. Since X is LCH, there is an open neighborhood $U \subset X$ of x such that $\overline{U} \subset X$ is compact. Set $V := \overline{U}^c$ in $X^\bullet$, which is an open neighborhood of ∞ disjoint from U. $\square$

Definition 1.6.6. A topological space X is *completely regular* if for every closed $F \subset X$ and $x \in F^c$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f|_F = 0$.

We call X *Tychonoff* if X is completely regular and T_1.

Exercises 1.6.7.

(1) X Tychonoff implies X is Hausdorff.
(2) Every normal space is Tychonoff by Urysohn's Lemma.
(3) Any subspace of a Tychonoff space is Tychonoff.

Lemma 1.6.8 (Embedding)**.** *Suppose X is a topological space $\Phi \subset C(X,[0,1])$ is a family of continuous functions. Define $e : X \to [0,1]^\Phi := \{f : \Phi \to [0,1]\} = \prod_{f\in\Phi}[0,1]$ (which is compact in the product topology!) by $x \mapsto (f(x))_{f\in\Phi}$.*

(1) e is continuous.
(2) e is injective if and only if Φ separates points, i.e., for all $x \neq y$ in X, there is an $f \in \Phi$ such that $f(x) \neq f(y)$.
(3) If Φ separates points from closed sets (for all $F \subset X$ closed and $x \in F^c$, there is an $f \in \Phi$ such that $f(x) \notin \overline{f(F)}$), then e is an open map of X onto $e(X)$.
(4) If Φ separates points and Φ separates points from closed sets, then e is an embedding.

Proof.

(1) Observe that $\pi_f \circ e = f$ is continuous for all $f \in \Phi$. Thus e is continuous by the universal property defining the product in Top.
(2) $e(x) \neq e(y)$ if and only if there is an $f \in \Phi$ such that

$$f(x) = (\pi_f \circ e)(x) \neq (\pi_f \circ e)(y) = f(y).$$

(3) Suppose Φ separates points from closed sets. Let $U \subset X$ be open. Suppose $x \in U$. We want to find an open set $V \subset [0,1]^\Phi$ such that $e(x) \in V \cap e(X) \subset e(U)$. There is an $f \in \Phi$ such that $f(x) \notin \overline{f(U^c)}$. Then $W := [0,1] \setminus \overline{f(U^c)}$ is an open set containing $f(x)$, so $e(x) \in \pi_f^{-1}(W)$, which is open in $[0,1]^\Phi$. Observe that

$$e(y) \in \pi_f^{-1}(W) \cap e(X) \quad \Longleftrightarrow \quad f(y) \notin \overline{f(U^c)} \quad \Longrightarrow \quad y \in U.$$

Setting $V := \pi_f^{-1}(W)$, we have $e(x) \in V \cap e(X) \subset e(U)$ as desired.
(4) By (1) and (2), $e : X \to [0,1]^\Phi$ is a continuous injection. By (3), e^{-1} on $e(X)$ is continuous. So e is a homeomorphism onto its image. $\square$

Corollary 1.6.9. *X is Tychonoff if and only if there exists an embedding $X \hookrightarrow [0,1]^I$ for some set I.*

Definition 1.6.10. Suppose X is Tychonoff and set $\Phi := C(X,[0,1])$. Consider the embedding $e : X \hookrightarrow [0,1]^\Phi$ by $e(x)_f := f(x)$. The *Stone-Čech compactification* of X is $\beta X := \overline{e(X)}$, with $X \to \beta X$ being the corestriction of e, still denoted e.

Suppose $f : X \to Y$ is any continuous map between Tychonoff spaces. Define $F : [0,1]^{\Phi_X} \to [0,1]^{\Phi_Y}$ componentwise for $g \in \Phi_Y = C(Y,[0,1])$ by $\pi_g(F(p)) := \pi_{g\circ f}(p)$. Then F is continuous, since $\pi_g \circ F = \pi_{g\circ f} : [0,1]^{\Phi_X} \to [0,1]$ is continuous for all $g \in \Phi_Y$. Moreover, for all $x \in X$,

$$\pi_g(F(e_X(x))) = \pi_{g\circ f}(e_X(x)) = g(f(x)) = \pi_g(e_Y(f(x))).$$

This means that $F \circ e_X = e_Y \circ f : X \to [0,1]^{\Phi_Y}$. Hence $\operatorname{im}(F|_{\beta X}) \subset \overline{e_Y(Y)} = \beta Y$. Define $\beta f := F|_{\beta X} : \beta X \to \beta Y$. Observe we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook, "e_X"] \arrow[d, "f"] & \beta X \arrow[r, hook] \arrow[d, "\beta f"] & {[0,1]^{\Phi_X}} \arrow[d, "F"] \\ Y \arrow[r, hook, "e_Y"] & \beta Y \arrow[r, hook] & {[0,1]^{\Phi_Y}}. \end{tikzcd} \tag{1.6.11}$$

Remark 1.6.12. Suppose X, Y are Tychonoff and $f : X \to Y$ is continuous. We note for future use that if every $h \in \Phi_X$ factorizes as $h = g \circ f$ for some $g \in \Phi_Y$, then F from Definition 1.6.10 is injective. Indeed, if $p, p' \in [0,1]^{\Phi_X}$, we have

$$\begin{aligned}
F(p) = F(p') &\iff \pi_g(F(p)) = \pi_g(F(p')) && \forall\, g \in \Phi_Y \\
&\iff \pi_{g\circ f}(p) = \pi_{g\circ f}(p') && \forall\, g \in \Phi_Y \\
&\iff \pi_h(p) = \pi_h(p') && \forall\, h \in \Phi(X) \\
&\iff p = p'.
\end{aligned}$$

Theorem 1.6.13. *The Stone-Čech compactification $(\beta X, e)$ satisfies the universal property*

- *For every compact Hausdorff space Z and continuous function $f : X \to Z$, there exists a unique continuous function $\beta f : \beta X \to Z$ such that $\widetilde{f} \circ e = f$.*

$$\begin{tikzcd}
\beta X \arrow[dr, dashed, "\exists! \widetilde{f}"] & \\
X \arrow[u, hook, "e"] \arrow[r, "f"'] & Z.
\end{tikzcd}$$

Proof. First, given any compactification $\varphi : X \to K$, compact Hausdorff Z, and continuous map $f : X \to Z$, there exists *at most one* continuous function $g : K \to Z$ such that $g \circ \varphi = f$. So it suffices to prove existence of $\widetilde{f}$. Just observe that since Z is compact, $e_Z(Z) \subset \beta Z$ is dense and compact, so $e_Z(Z) = \beta Z$. Hence $e_Z : Z \to \beta Z$ is a continuous bijection from a compact space to a Hausdorff space, and is thus a homeomorphism. So the map $\widetilde{f} : \beta X \to Z$ given by

$$\beta X \xrightarrow{\beta f} \beta Z \xrightarrow{e_Z^{-1}} Z$$

satisfies $\widetilde{f} \circ e_X = f$ by the commutative diagram (1.6.11). □

Exercise 1.6.14. If $\varphi : X \hookrightarrow Y$ is any compactification of X satisfying the universal property in Theorem 1.6.13, then $\widetilde{\varphi} : \beta X \to Y$ is a homeomorphism.

Corollary 1.6.15. *Let X be Tychonoff and $\varphi : X \to K$ a compactification.*

(1) The unique lift $\widetilde{\varphi} : \beta X \to K$ is surjective.
(2) Suppose for all $f \in C_b(X)$ there is a $g \in C(K)$ such that $f = g \circ \varphi$. Then $\widetilde{\varphi} : \beta X \to K$ is a homeomorphism.

Proof.

(1) Since $\widetilde{\varphi} \circ e_X = \varphi$ and $\varphi(X)$ is dense in K, $\widetilde{\varphi}(\beta X)$ is dense in K. But βX is compact and $\widetilde{\varphi}$ is continuous, so $\widetilde{\varphi}(\beta X)$ is compact. Since K is compact Hausdorff, compact subsets are closed, and thus $\widetilde{\varphi}(\beta X) = K$.
(2) By (1), it suffices to prove that $\widetilde{\varphi} : \beta X \to K$ is injective. Then since βX is compact and K is Hausdorff, the continuous bijection $\widetilde{\varphi}$ is automatically a homeomorphism. Injectivity follows by Remark 1.6.12. Indeed, every $f \in \Phi_X \subset C_b(X)$ factorizes as $f = g \circ \varphi$ for some $g \in \Phi_K$. □

Proposition 1.6.16. *Stone-Čech compactification is a functor $\beta : \mathsf{Tych} \to \mathsf{CptHsd}$.*

Proof.

id: Since

$$\beta\,\mathrm{id}_X \circ e_X \underset{(1.6.11)}{=} e_X \circ \mathrm{id}_X = e_X = \mathrm{id}_{\beta X} \circ e_X$$

we must have $\beta\,\mathrm{id}_X = \mathrm{id}_{\beta X}$ as they agree on the dense subset $X \subset \beta X$.

$-\circ-$: Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous with all spaces Tychonoff. Since

$$\beta(g \circ f) \circ e_X \underset{(1.6.11)}{=} e_Z \circ g \circ f \underset{(1.6.11)}{=} \beta g \circ e_Y \circ f \underset{(1.6.11)}{=} \beta g \circ \beta f \circ e_X,$$

$\beta(g \circ f) = \beta g \circ \beta f$ as the agree on the dense subset $X \subset \beta X$. □

Exercise 1.6.17 (Adapted from Folland §4.8, #74)**.** Consider $\mathbb{N}$(with the discrete topology) as a subset of its Stone-Čech compactification $\beta\mathbb{N}$.

(1) Prove that if A, B are non-empty disjoint subsets of $\mathbb{N}$, then their closures in $\beta\mathbb{N}$ are disjoint.
(2) Suppose $(x_n) \subset \mathbb{N}$ is a sequence which is not eventually constant. Show there exist non-empty disjoint subsets $A, B \subset \mathbb{N}$ such that (x_n) is frequently in A and frequently in B.
(3) Deduce that no sequence in $\mathbb{N}$ converges in $\beta\mathbb{N}$ unless it is eventually constant.

Exercise 1.6.18 (Adapted from `http://u.cs.biu.ac.il/~tsaban/RT/Book/Chapter3.pdf`)**.** Let $\mathcal{U}\mathbb{N}$ be the set of ultrafilters on $\mathbb{N}$. For a subset $S \subset \mathbb{N}$, define $[S] := \{\mathcal{F} \in \mathcal{U}\mathbb{N} | S \in \mathcal{F}\}$. Show that the function $S \mapsto [S]$ satisfies the following properties:

(1) $[\emptyset] = \emptyset$ and $[\mathbb{N}] = \mathcal{U}\mathbb{N}$.
(2) For all $S, T \subset \mathbb{N}$,
    (a) $[S] \subset [T]$ if and only if $S \subset T$.
    (b) $[S] = [T]$ if and only if $S = T$.
    (c) $[S] \cup [T] = [S \cup T]$.
    (d) $[S] \cap [T] = [S \cap T]$.
    (e) $[S^c] = [S]^c$.
(3) Find a sequence of subsets (S_n) of $\mathbb{N}$ such that $[\bigcup S_n] \neq \bigcup [S_n]$.
(4) Find a sequence of subsets (S_n) of $\mathbb{N}$ such that $[\bigcap S_n] \neq \bigcap [S_n]$.

Exercise 1.6.19 (Adapted from `http://u.cs.biu.ac.il/~tsaban/RT/Book/Chapter3.pdf`)**.** Assume the notation of Exercise 1.6.18.

(1) Show that $\{[S] | S \subset \mathbb{N}\}$ is a base for a topology on $\mathcal{U}\mathbb{N}$.
(2) Show that all the sets $[S]$ are both closed and open in $\mathcal{U}\mathbb{N}$.
(3) Show that $\mathcal{U}\mathbb{N}$ is compact.
(4) For $n \in \mathbb{N}$, let $\mathcal{F}_n = \{S \subset \mathbb{N} | n \in S\}$. Show $\mathcal{F}_n$ is an ultrafilter on $\mathbb{N}$.
    Note: Each $\mathcal{F}_n$ is called a principal *ultrafilter on $\mathbb{N}$.*
(5) Show that $\{\mathcal{F}_n | n \in \mathbb{N}\}$ is dense in $\mathcal{U}\mathbb{N}$.
(6) Show that for every compact Hausdorff space K and every function $f : \mathbb{N} \to K$, there is a continuous function $\widetilde{f} : \mathcal{U}\mathbb{N} \to K$ such that $\widetilde{f}(\mathcal{F}_n) = f(n)$ for every $n \in \mathbb{N}$. Deduce that $\mathcal{U}\mathbb{N}$ is homeomorphic to the Stone-Čech compactification $\beta\mathbb{N}$.
    Hint: Given $f : \mathbb{N} \to K$, use Exercise 1.3.15 to get an ultrafilter on K from an ultrafilter on $\mathbb{N}$. Then use Exercises 1.3.13(4) and 1.3.19(2) to define $\widetilde{f}(\mathcal{F})$ for $\mathcal{F} \in \mathcal{U}\mathbb{N}$.