We'll go over some important results in general topology.

Def: Suppose (X, τ) a topological space. A neighborhood base for τ at $x \in X$ is a subset $\mathcal{B}(x) \subset \tau$ s.t.

① $\forall V \in \mathcal{B}(x)$, $x \in V$

② $\forall \mathcal{U} \in \tau$ s.t. $x \in \mathcal{U}$, $\exists V \in \mathcal{B}(x)$ s.t. $V \subseteq \mathcal{U}$.

A base for τ is a subset $\mathcal{B} \subset \tau$ which contains a neighborhood base for τ at every $x \in X$.

Exercise: $\mathcal{B} \subset \tau$ is a base $\iff$ every $\mathcal{U} \in \tau$ is a union of members of $\mathcal{B}$.

Def: (X, τ) is

- first countable if $\exists$ a countable neighborhood base for τ at every $x \in X$.
- second countable if $\exists$ a countable base.

Exercise: Second countable $\Rightarrow$ separable.

Exercise: Suppose X is first countable and $A \subset X$. Then $x \in \overline{A} \iff \exists (x_j) \subset A$ s.t. $x_j \to x$.

Def: A topological space (X, τ) is called:

- T_1 if $\forall x, y \in X$ distinct, $\exists$ open sets $\mathcal{U}, V \in \tau$ s.t. $x \in \mathcal{U} \cap V^c$ and $y \in V \cap \mathcal{U}^c$. [Equivalently, points are closed.]
- Hausdorff (or T_2) if $\forall x, y \in X$ distinct, $\exists$ disjoint open sets $\mathcal{U}, V \in \tau$ s.t. $x \in \mathcal{U}$ and $y \in V$.
- Regular (or T_3) if (X, τ) is T_1 and $\forall F \subset X$ closed and $x \in F^c$, $\exists$ disjoint open sets $\mathcal{U}, V \in \tau$ s.t. $x \in \mathcal{U}$ and $F \subseteq V$
- Normal (or T_4) if (X, τ) is T_1 and $\forall$ disjoint closed $F, G \subset X$, $\exists$ disjoint open sets $\mathcal{U}, V \in \tau$ s.t. $F \subset \mathcal{U}$ and $G \subseteq V$.

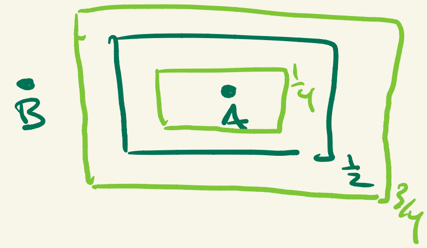
Urysohn's Lemma: (X,τ) normal space. If $A, B \subset X$ are disjoint, nonempty closed subsets, $\exists$ cts $f: X \to [0,1]$ s.t. $f|_A = 0$ and $f|_B = 1$.

Observe: If $F \subset G \subset X$ w/ F closed and G open, then $\exists$ open $\mathcal{U}$ s.t. $F \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset G$ [Take disjoint open $\mathcal{U}, V$ s.t. $F \subseteq \mathcal{U}$, $X \setminus G \subseteq V \Rightarrow \overline{\mathcal{U}} \subseteq X \setminus V$.]

Lemma: Let $D = \left\{ \frac{k}{2^n} \,\middle|\, n \in \mathbb{N},\ k = 1, \dots, 2^n - 1 \right\} \subset (0,1)$ [Dyadic rationals]

$\exists$ open sets $(\mathcal{U}_d)_{d \in D}$ s.t.

- $A \subset \mathcal{U}_d$ and $\overline{\mathcal{U}_d} \subset X \setminus B$ $\forall d \in D$
- $\overline{\mathcal{U}_d} \subseteq \mathcal{U}_{d'}$ $\forall d < d'$.



Pf: Set $D_n = \left\{ \frac{k}{2^n} \,\middle|\, k = 1, \dots, 2^n - 1 \right\}$. Construct $\mathcal{U}_d$ inductively.

n=1: Let $\mathcal{U}_{\frac{1}{2}}$ be any open set $A \subseteq \mathcal{U}_{\frac{1}{2}} \subseteq \overline{\mathcal{U}_{\frac{1}{2}}} \subseteq B$.

Ind. Step: Suppose $\mathcal{U}_d$ defined for all $d \in D_1 \cup \dots \cup D_n$. Then we choose $\mathcal{U}_{\frac{2k+1}{2^{n+1}}}$ for $k = 0, 1, \dots, 2^n - 1$ by

k=0: $A \subseteq \mathcal{U}_{\frac{1}{2^{n+1}}} \subseteq \overline{\mathcal{U}_{\frac{1}{2^{n+1}}}} \subseteq \mathcal{U}_{\frac{1}{2^n}}$

$k=1, \dots, 2^{n-1}-2$: $\overline{\mathcal{U}_{\frac{k}{2^n}}} \subseteq \mathcal{U}_{\frac{2k+1}{2^{n+1}}} \subseteq \overline{\mathcal{U}_{\frac{2k+1}{2^{n+1}}}} \subseteq \mathcal{U}_{\frac{k+1}{2^n}}$

$k = 2^{n-1} - 1$: $\overline{\mathcal{U}_{\frac{2^n-1}{2^n}}} \subseteq \mathcal{U}_{\frac{2^{n+1}-1}{2^{n+1}}} \subseteq \overline{\mathcal{U}_{\frac{2^{n+1}-1}{2^{n+1}}}} \subseteq X \setminus B$.

Proof of UsL: Define $f: X \to [0,1]$ by $f(x) := \sup\{d \mid x \notin \mathcal{U}_d\}$.

Clear $f|_A = 0$ and $f|_B = 1$. Also

(i) $f(x) > d \Rightarrow x \notin \overline{\mathcal{U}_d}$, $f(x) < d' \Rightarrow x \in \mathcal{U}_{d'}$.

(ii) $x \notin \overline{\mathcal{U}_d} \Rightarrow f(x) \geq d$, $x \in \mathcal{U}_{d'} \Rightarrow f(x) \leq d'$.

Show f is cts: Fix $x_0 \in X$ and $\varepsilon > 0$.

Case 1: Suppose $0 < f(x) < 1$. Choose $d, d' \in D$ s.t. $d < f(x_0) < d'$ and $d' - d < \varepsilon$. By (i), $x_0 \in \mathcal{U}_{d'} \setminus \overline{\mathcal{U}_d}$. By (ii), $|f(x) - f(x_0)| < \varepsilon$ $\forall x_0 \in \mathcal{U}_{d'} \setminus \overline{\mathcal{U}_d}$.

Case 2: $f(x) = 0$ or 1. Similar and omitted.

Tietze Extension Thm: Suppose X is normal. If $A \subset X$ closed and $f : A \to [a,b]$ is cts, $\exists$ $F : X \to [a,b]$ cts s.t. $F|_A = f$.

Pf: WLOG, $[a,b] = [0,1]$ [Else replace f w/ $\frac{f-a}{b-a}$].

We'll inductively build a seq. of fcts (g_n) on X s.t.

- $0 \le g_n \le \frac{2^{n-1}}{3^n}$ $\forall n \in \mathbb{N}$ and $\quad (\frac{2}{3})^{n-1} - \frac{2^{n-1}}{3^n} = \frac{2^{n-1}}{3^{n-1}}[1 - \frac{1}{3}]$
- $0 \le f - \sum^n g_k \le (\frac{2}{3})^n$ on A $\forall n \in \mathbb{N}$.

Then $\sum g_n$ converges uniformly to a cts limit fct F, and $\forall n \in \mathbb{N}$, $0 \le f - F \le f - \sum^n g_k \le (\frac{2}{3})^n$ on A, so $F|_A = f$.

Base case: Set $B := f^{-1}([0, \frac{1}{3}])$ and $C := f^{-1}([\frac{2}{3}, 1])$. Since f is cts, $B, C \subset X$ closed. By Urysohn's lemma, $\exists$ cts $g_1 : X \to [0, \frac{1}{3}]$ s.t. $g_1|_B = 0$ and $g_1|_C = \frac{1}{3}$. Then

$$f - g_1 \le \left\{ \begin{array}{lll} \frac{1}{3} - 0 = \frac{1}{3} & \text{on} & B \cap A \\ \frac{2}{3} - 0 = \frac{2}{3} & \text{on} & (B \cup C)^c \cap A \\ 1 - \frac{1}{3} = \frac{2}{3} & \text{on} & C \cap A \end{array} \right\} \le \frac{2}{3} \text{ on } A.$$

Inductive step: If we have $g_1, \ldots, g_{n-1}$, $\exists$ cts $g_n : X \to [0, \frac{2^{n-1}}{3^n}]$ s.t. $g_n = 0$ on $\{f - \sum^{n-1} g_k \le \frac{2^{n-1}}{3^n}\}$ and $g_n = \frac{2^{n-1}}{3^n}$ on $\{\underbrace{f - \sum^{n-1} g_k}_{\le (\frac{2}{3})^{n-1}} \ge (\frac{2}{3})^n\}$.

$\leadsto$ $f - \sum^n g_k \le (\frac{2}{3})^n$ on A as before. $[(\frac{2}{3})^{n-1} - \frac{2^{n-1}}{3^n} = (\frac{2}{3})^n]$

Convergence in topological spaces:

Recall a seq. $x_n \to x$ if $\forall$ open $\mathcal{U} \in \tau$ s.t. $x \in \mathcal{U}$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow x_n \in \mathcal{U}$. [$(x_n)$ eventually in $\mathcal{U}$.]

Unfortunately, not all spaces are first countable, so sequences do not suffice to describe the topology.

Def: A directed set is a set I equipped with a preorder [reflexive + transitive] binary relation $\leq$ satisfying

- $\forall\, i, j \in I$, $\exists\, k \in I$ s.t. $i \leq k$ and $j \leq k$.

Examples:

① $\mathbb{N}$, $\mathbb{R}$, or any linearly ordered set.

② $\mathbb{R} \setminus \{a\}$ where $x \leq y \iff |x-a| \geq |y-a|$.

③ Any neighborhood base for (X, τ) at $x \in X$, ordered by reverse inclusion [$\mathcal{U} \leq V \iff V \subseteq \mathcal{U}$]

④ If X is any infinite set, $\{F \subset X \mid F \text{ finite}\}$ ordered by inclusion.

Def: Let X be a nonempty set and I a directed set. A net in X based on I (an I-net in X) is a fct $x : I \to X$ where we write $x_i = x(i)$ and $x = (x_i)_{i \in I}$.

Given an I-net $(x_i)_{i \in I}$ and a subset $S \subset X$, we say

- (x_i) is eventually in S if $\exists\, j \in I$ s.t. $j \leq i \Rightarrow x_i \in S$.
- (x_i) is frequently in S if $\forall\, j \in I$, $\exists\, j \leq i$ s.t. $x_i \in S$.

We say (x_i) converges to $x \in X$ if (x_i) is eventually in every neighborhood of x. We say x is a cluster pt. of (x_i) if (x_i) is frequently in every neighborhood of x.

Prop 1: For $A \subset X$, TFAE:

① x is an accumulation/limit pt of A

$\hookrightarrow$ $\forall$ open $\mathcal{U} \in \tau$ s.t. $x \in \mathcal{U}$, $A \cap (\mathcal{U} \setminus \{x\}) \neq \emptyset$

② $\exists$ a net in $A \setminus \{x\}$ that converges to x.

Pf of ①$\Rightarrow$②: Let I be a nbhd base at x, ordered by reverse inclusion. For all $V \in I$, pick $x_V \in V \cap (A \setminus \{x\})$. [This uses the Axiom of Choice!] Then $(x_V)_{V \in I}$ converges to x.

Cor: $A \subseteq X$ closed $\Leftrightarrow$ every convergent net in A only converges to points in A.

Prop 2: X is Hausdorff $\Leftrightarrow$ every convergent net has a $!$ limit.

Pf of $\Leftarrow$: we'll prove the contrapositive. Suppose X is not Hausdorff, so $\exists x, y \in X$ s.t. $\forall$ nbhds $\mathcal{U}$ of x and V of y, $\mathcal{U} \cap V \neq \emptyset$. Let I be a nbhd base for τ at x and J be a nbhd base for τ at y, ordered by reverse inclusion. Direct $I \times J$ by $(\mathcal{U}_1, V_1) \geqslant (\mathcal{U}_2, V_2)$ if $\mathcal{U}_1 \subseteq \mathcal{U}_2$ and $V_1 \subseteq V_2$. $\forall (\mathcal{U}, V) \in I \times J$, choose a pt $x_{(\mathcal{U},V)} \in \mathcal{U} \cap V$. [AoC!] This net converges to both x and y.

Prop 3: $f: X \to Y$ cts $\Leftrightarrow$ $\forall$ convergent net $x_i \to x$ in X, $f(x_i) \to f(x)$ in Y.

Pf: Exercise.

Subnets: The net $(y_j)_{j \in J}$ is a subnet of $(x_i)_{i \in I}$ if $\exists$ a fct $f: J \to I$ [which need not be injective!] s.t.

- $y_j = x_{f(j)}$ $\forall j \in J$ and
- $\forall i \in I$, $\exists j_0 \in J$ s.t. $f(j) \geqslant i$ $\forall j \geqslant j_0$.

Remark: If $x_i \to x$, then $\forall$ subnet (y_j), $y_j \to x$.

Prop 4: Suppose $(x_i) \subset X$ is a net and $x \in X$. TFAE:

1. x is a cluster pt of (x_i)
2. $\exists$ subnet (y_j) of (x_i) s.t. $y_j \to x$.

Pf of ① $\Rightarrow$ ②: Choose a nbhd base $\mathcal{B}(x)$ at x. Define $J := I \times \mathcal{B}(x)$ where $(i_1, \mathcal{U}_1) \le (i_2, \mathcal{U}_2)$ if $i_1 \le i_2$ and $\mathcal{U}_1 \supseteq \mathcal{U}_2$.

For each $(i, \mathcal{U}) \in J$, define $f(i, \mathcal{U}) := i'$ to be any i' s.t. $i' \geq i$ and $x_{i'} \in \mathcal{U}$. [i' exists since (x_i) frequently in $\mathcal{U}$.]

Then if $(i_1, \mathcal{U}_1) \le (i_2, \mathcal{U}_2)$, $i_1 \le i_2 \le f(i_2, \mathcal{U}_2)$ and $x_{f(i_2,\mathcal{U}_2)} \in \mathcal{U}_2 \subseteq \mathcal{U}_1$. Thus $(x_{f(i,\mathcal{U})})_{(i,\mathcal{U}) \in J}$ is a subnet of (x_i) converging to x.

Remark: When (X, τ) is first countable, Props 1–4 hold w/ sequences instead of nets.

(Locally) Cpt Spaces:

Exercises:

- If X cpt and $F \subset X$ closed, then F is cpt.
- If X is Hausdorff, $F \subset X$ cpt, and $x \notin F$, then $\exists$ opn $\mathcal{U}, V \subseteq X$ s.t. $x \in \mathcal{U}$ and $F \subset V$.
- If X is Hausdorff and $F \subset X$ cpt, then F is closed.
- If X is cpt Hausdorff, then X is normal.
- If X cpt and $f: X \to Y$ cts, $f(X)$ is cpt.
- If X cpt, Y Hausdorff, and $f: X \to Y$ a cts bijection, then f is a homeomorphism [$f^{-1}: Y \to X$ is cts]

Thm: Suppose X is a topological space. TFAE:

① X is cpt
② $\forall$ family of closed subsets $(F_i)_{i\in I}$ of X w/ $\underline{\text{FIP}}$, $\bigcap F_i \neq \emptyset$.
③ Every net in X has a cluster pt
④ Every net in X has a convergent subnet.

Pf: ① $\Leftrightarrow$ ② was HW1. ③ $\Leftrightarrow$ ④ was Prop 4.

② $\Rightarrow$ ③: Let $(x_i)_{i\in I}$ be a net in X. For $i \in I$, define $F_i := \{x_j \mid j \geq i\}$. Then

- $\bigcap \overline{F_i}$ is the set of cluster pts of (x_i), and
- (F_i) has $\underline{\text{FIP}} \Rightarrow (\overline{F_i})$ has $\underline{\text{FIP}} \Rightarrow \bigcap \overline{F_i} \neq \emptyset$.

③ $\Rightarrow$ ②: we'll prove the contrapositive. If ② fails, $\exists$ $(F_i)_{i\in I}$ closed sets in X w/ FIP, but $\bigcap F_i = \emptyset$. Define $J := \{\text{(nonempty) finite intersections of } (F_i)\}$, ordered by reverse inclusion. Since (F_i) has FIP, $\forall F \in J$, $F \neq \emptyset$. Use AoC to pick $x_F \in F$ $\forall F \in J$. Then any cluster pt of $(x_F)_{F\in J}$ lies in $\bigcap_{F\in J} F = \bigcap_{i\in I} F_i = \emptyset$.

Will be HW on $\underline{\text{ctbly cpt}}$ and $\underline{\text{sequentially cpt}}$ spaces.

Thm: If (X, ρ) is a metric space, TFAE:

① X cpt
② X sequentially cpt
③ X complete + totally bdd.

Cor: Let (X, ρ) be a $\underline{\text{complete}}$ metric space and $A \subset X$.

$\overline{A}$ is cpt $\Leftrightarrow$ A is totally bdd.

Def: X is <u>locally cpt</u> if $\forall x \in X$, $\exists$ open $\mathcal{U} \ni x$ s.t. $\overline{\mathcal{U}}$ cpt.

<u>Notation:</u> LCH means <u>locally cpt Hausdorff</u>.

<u>Exercises:</u> Suppose X is LCH.

- $\forall$ open $\mathcal{U} \subset X$ and $x \in \mathcal{U}$, $\exists$ open $x \in V \subset \overline{V} \subset \mathcal{U}$ s.t. $\overline{V}$ cpt.
- If $K \subset \mathcal{U} \subset X$ where K cpt and $\mathcal{U}$ open, $\exists$ open V w/ $K \subset V \subset \overline{V} \subset \mathcal{U}$ s.t. $\overline{V}$ is cpt.
- (Urysohn) If $K \subset \mathcal{U} \subset X$ as above, $\exists$ cts $f: X \to [0,1]$ s.t. $f|_K = 0$ and $f = 1$ outside a cpt subset of $\mathcal{U}$.
- (Tietze) If $K \subset X$ cpt and $f \in C(K)$, $\exists F \in C_c(X)$ s.t. $f|_K = F$.

Def: Suppose X is LCH. A fct $f \in C(X)$ <u>vanishes at ∞</u> if $\forall \varepsilon > 0$, $\{ f \geq \varepsilon \}$ is cpt.

$C_0(X) = \{ \text{cts } f: X \to \mathbb{C} \mid f \text{ vanishes at } \infty \}$

$C_b(X) = \{ \text{cts bdd } f: X \to \mathbb{C} \}$

Observe $C_c(X) \subset C_0(X) \subset C_b(X)$.

The <u>uniform/∞-norm</u> on $C_b(X)$ is given by

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

Prop: Suppose X is LCH.

1. $C_b(X)$ is complete w.r.t. $\|\cdot\|_\infty$.
2. $C_0(X) \subset C_b(X)$ is closed.
3. $\overline{C_c(X)}^{\|\cdot\|_\infty} = C_0(X)$.

Pf:

① If (f_n) is uniformly Cauchy, then

- $(f_n(x)) \subset \mathbb{C}$ is Cauchy $\forall x \in X$
- $f(x) := \lim f_n(x)$ is cts (uniform limit of cts fcts)
- $(\|f_n\|) \subset [0,\infty)$ is Cauchy
- $\sup |f(x)| \leq \sup \|f_n\| < \infty$.

② Suppose $(f_n) \subset C_0(X)$ s.t. $f_n \to f \in C_b(X)$. Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ s.t. $n \geq N \implies \|f - f_n\| < \frac{\varepsilon}{2}$. Pick $K \subset X$ cpt s.t. $\{|f_N| \geq \frac{\varepsilon}{2}\}$ cpt. Then $\underbrace{\{|f| \geq \varepsilon\}}_{\text{closed}} \subset \underbrace{\{|f_N| \geq \frac{\varepsilon}{2}\}}_{\text{cpt}}$ is cpt.

$$\left[\varepsilon \leq |f(x)| \leq \underbrace{|f(x) - f_N(x)|}_{< \frac{\varepsilon}{2}} + \underbrace{|f_N(x)|}_{\geq \frac{\varepsilon}{2}!} \implies x \in \{|f_N| \geq \tfrac{\varepsilon}{2}\}\right]$$

③ Let $f \in C_0(X)$ and $\varepsilon > 0$ so $A := \{|f| \geq \varepsilon\}$ cpt. By the LCH Urysohn's Lemma, $\exists$ cts $g : X \to [0,1]$ s.t. $g|_A = 1$. Then $gf \in C_c(X)$ and $\|f - gf\|_\infty < \varepsilon$.

$$\left[|f(x) - g(x)f(x)| = |f(x)| \cdot |1 - g(x)| \leq \chi_{A^c}(x) \cdot |f(x)| < \varepsilon \quad \forall x \in X\right]$$

Exercise: Prove $C(X)$ closed in topology of local uniform convergence: $f_n \to f \iff f_n|_K \to f|_K$ uniformly $\forall$ cpt K.

Tychonoff's Thm: Suppose $(X_i)_{i \in I}$ is a family of cpt top. spaces. Then $\prod_{i \in I} X_i$ is cpt in the product topology: weakest topology on $\prod X_i$ s.t. the canonical projection maps are cts.

Pf: In Discussion Section.

Def: Let X be a top. space. A subset $F \subset C(X)$ is called <u>equicts at x_0</u> if $\forall \varepsilon > 0$, $\exists$ open $x_0 \in \mathcal{U} \subset X$ s.t. $\forall x \in \mathcal{U}$, $\forall f \in F$, $|f(x) - f(x_0)| < \varepsilon$. We call F <u>equicts</u> if it is equicts at x_0 $\forall x_0 \in X$.

<u>Thm (Arzelà-Ascoli)</u> Suppose X is cpt Hausdorff. For $F \subset C(X)$, TFAE:

1. $\overline{F}$ is cpt
2. F is totally bdd
3. F is equicts and <u>ptwise bdd</u> $[\{f(x) \mid f \in F\}$ bdd $\forall x \in X]$

<u>Pf:</u> ① $\Leftrightarrow$ ② follows as $C(X) = C_b(X)$ is complete.

<u>② $\Rightarrow$ ③:</u> <u>Equicts:</u> Let $\varepsilon > 0$. Pick $f_1, \ldots, f_n \in F$ s.t. $F \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i)$.
Let $x_0 \in X$. Pick an open $x_0 \in \mathcal{U} \subset X$ s.t. $|f_i(x) - f_i(x_0)| < \frac{\varepsilon}{3}$ $\forall x \in \mathcal{U}$, $\forall i = 1, \ldots, n$. Now $\forall f \in F$, $\exists i \in \{1, \ldots, n\}$ s.t. $\|f - f_i\| < \frac{\varepsilon}{3}$.
Then $\forall x \in \mathcal{U}$, $|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_i(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_i(x) - f_i(x_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_i(x_0) - f(x_0)|}_{<\frac{\varepsilon}{3}}$.

<u>Ptwise bdd:</u> Observe that $\forall x \in X$, $ev_x : C(X) \to \mathbb{C}$ by $f \mapsto f(x)$ is <u>cts</u>. Then $\{f(x) \mid f \in F\} = ev_x(F) \subset \underbrace{ev_x}_{\text{cts}}(\underbrace{\overline{F}}_{\text{cpt}})$ which is cpt and thus bdd.

<u>③ $\Rightarrow$ ②:</u> Discussion!

<u>Stone-Weierstrass Thm:</u> Suppose X cpt Hausd, so $C(X)$ is a Banach alg. Let $A \subset C(X)$ be a closed subalg. that separates pts of X $[\forall x \neq y, \exists f \in A$ s.t. $f(x) \neq f(y)]$ and is closed under complex conjugation.

- If A contains a non-vanishing fct, $A = C(X)$.
- If every $f \in A$ has a zero, $\exists x_0 \in X$ s.t. $A = \{f \in C(X) \mid f(x_0) = 0\}$.

<u>Step 1:</u> The fct $x \mapsto |x|$ on $\mathbb{R}$ can be uniformly approx. by a polynomial which vanishes at 0 on any cpt $K \subset \mathbb{R}$.

<u>Pf:</u> (Sarason) We'll show for $R > 0$, $\exists$ seq (p_n) of poly's <u>conv. unif</u> to $|\cdot|$ on $[-R, R]$ s.t. $p_n(0) = 0$ $\forall n$. WLOG, $R = 1$. Define $q(t) := 1 - |t|$ on $[-1, 1]$. It suffices to find a seq. (q_n) of poly's converging to q unif s.t. $q_n(0) = 1$ $\forall n$. Observe

$(*)$ q takes values in $[0,1]$ and $(1 - q(t))^2 = t^2$ $\forall |t| \leq 1$.

For a given $t \in [-1, 1]$, consider the eq. $(1-s)^2 = t^2$. It has 2 sol'ns: $s = 1 \pm |t|$, and exactly one of these $1 - |t| \in [0,1]$. Hence $q(t)$ is the <u>! fct</u> on $[-1,1]$ satisfying $(*)$. Rewrite $(*)$ as:

$(**)$ q takes values in $[0,1]$ and $q(t) = \frac{1}{2}(1 - t^2 + q(t)^2)$.

We'll define (q_n) inductively by:

- $q_0(t) = 1$
- $q_{n+1}(t) = \frac{1}{2}(1 - t^2 + q_n(t)^2)$

e.g. $q_1(t) = 1 - \frac{1}{2}t^2$ $\rightsquigarrow$ $q_0 - q_1 = \frac{1}{2}t^2 \geq 0$.

By induction, $\forall n \geq 0$,

- q_n takes values in $[0,1]$,
- $q_n(0) = 1$,
- $q_n - q_{n+1} = \frac{1}{2}(q_{n-1}^2 - q_n^2) = \frac{1}{2}(q_{n-1} - q_n)(q_{n-1} + q_n) \geq 0$

Thus (q_n) is <u>monotone decreasing</u> by construction. Let $\tilde{q}$ be its ptwise limit, which takes values in $[0,1]$. Observe $\tilde{q}$ satisfies $(**)$ by construction, so $\tilde{q} = q$ by !ness. Finally, as $q_n \searrow q$ on $[-1,1]$, $q_n \to q$ <u>uniformly</u> by <u>Dini's Lemma</u>.

Step 2: If $A \subset C(X, \mathbb{R})$ is a closed subalg., then A is a lattice, i.e., $\forall f,g \in A$, $\max\{f,g\}, \min\{f,g\} \in A$.

Pf: Suppose $f \in A$ and $f \neq 0$. Then $\frac{f}{\|f\|_\infty} : X \to [-1,1]$. By Step 1, $\exists$ poly p on $[-1,1]$ s.t. $p(0)=0$ and $||t| - p(t)| < \varepsilon$ $\forall t \in [-1,1]$. Hence $\left| \frac{|f(x)|}{\|f\|_\infty} - p\left[\frac{f(x)}{\|f\|_\infty}\right] \right| < \varepsilon$ $\forall x \in X$, i.e., $\left\| \frac{|f|}{\|f\|_\infty} - p\left(\frac{f}{\|f\|_\infty}\right) \right\|_\infty < \varepsilon$. Since $p(0)=0$, $p\left(\frac{f}{\|f\|_\infty}\right) \in \overbrace{\operatorname{span}\{f^n \mid n \in \mathbb{Z}^{\geq 1}\}}^{\subset A}$. Since A is closed and $\varepsilon > 0$ was arbitrary, $\frac{|f|}{\|f\|_\infty} \in A$, and thus $|f| \in A$. Then if $f,g \in A$,

$$\left.\begin{aligned} \max\{f,g\} &= \tfrac{1}{2}[f+g+|f-g|] \\ \min\{f,g\} &= \tfrac{1}{2}[f+g-|f-g|] \end{aligned}\right] \in A.$$

Step 3: Suppose $A \subset C(X, \mathbb{R})$ is a $\mathbb{R}$-vector space and also a lattice. If $f \in C(X, \mathbb{R})$ can be approximated by $a \in A$ at every 2 pts of X, then $f \in \overline{A}^{\|\cdot\|_\infty}$.

Pf: $\forall \varepsilon > 0$ and $x, y \in X$, pick $a_{xy} \in A$ s.t. $|f(x) - a_{xy}(x)| < \varepsilon$ and $|f(y) - a_{xy}(y)| < \varepsilon$. Then x, y are both in

$$\mathcal{U}_{xy} := \{z \in X \mid f(z) < a_{xy}(z) + \varepsilon\} \quad \text{and}$$
$$V_{xy} := \{z \in X \mid a_{xy}(z) < f(z) + \varepsilon\}.$$

Fix $x \in X$. The sets $(\mathcal{U}_{xy})_{y \in X}$ are an open cover of X. Since X is cpt, $X \subset \bigcup_{i=1}^m \mathcal{U}_{xy_i}$. Then $a_x := \bigvee_{i=1}^m a_{xy_i} \in A$, and $f(z) < a_x(z) + \varepsilon$ $\forall z \in X$ by construction. Also, $a_x(z) < f(z) + \varepsilon$ $\forall z \in W_x := \bigcap_{i=1}^m V_{xy_i}$, an open nbhd of x. Varying $x \in X$, (W_x) is an open cover, so $X \subset \bigcup_{j=1}^n W_{x_j}$ as X cpt., and $a_\varepsilon := \bigwedge_{j=1}^n a_{x_j}$ satisfies $\|f - a_\varepsilon\|_\infty < \varepsilon$.

Sep 4: Suppose $A \subset C(X, \mathbb{R})$ is a subalg which separates pts.

- If A contains a non-vanishing fct, $\overline{A} = C(X, \mathbb{R})$.
- If every $f \in A$ has a zero, $\exists x_0 \in X$ s.t. $\overline{A} = \{f \mid f(x_0) = 0\}$.

Pf: Suppose $x \neq y$ in X. Then since point evaluation is an $\mathbb{R}$-algebra hom $A \to \mathbb{R}$, $A_{xy} := \{(f(x), f(y)) \mid f \in A\} \subset \mathbb{R}^2$ is a subalgebra. The only subalgebras of $\mathbb{R}^2$ are:

$$(0,0), \quad \mathbb{R} \times \{0\}, \quad \{0\} \times \mathbb{R}, \quad \Delta = \{(x,x) \mid x \in \mathbb{R}\}, \quad \mathbb{R}^2.$$

Since A separates pts, $A_{xy} \neq (0,0)$ or Δ $\forall x \neq y$.

Claim: $A_{xy} = \mathbb{R}^2$ $\forall x \neq y$ except for at most one possible $x_0 \in \mathbb{R}$.

Pf: If $\exists x \neq y$ s.t. $A_{xy} \neq \mathbb{R}^2$, then WLOG $A_{xy} = \{0\} \times \mathbb{R}$, so $f(x) = 0$ $\forall f \in A$. Since A separates pts, $f(x') = 0$ $\forall f \in A \Rightarrow x' = x$. So $A_{yz} = \mathbb{R}^2$ $\forall y \neq x \neq z$.

Case 1: Suppose A contains a non-vanishing fct. By the claim, $A_{xy} = \mathbb{R}^2$ $\forall x \neq y$. So $\forall f \in C(X, \mathbb{R})$, $\exists a_{xy} \in A$ s.t. $f(x) = a_{xy}(x)$ and $f(y) = a_{xy}(y)$. By Step 2, $\overline{A}$ is a lattice, and by Step 3, f can be uniformly approximated by $\overline{A}$, so $f \in \overline{A}$.

Case 2: Suppose $\exists x_0 \in X$ s.t. $a(x_0) = 0$ $\forall a \in A$. Then the argument of Case 1 applies $\forall f \in \{f \in C(X, \mathbb{R}) \mid f(x_0) = 0\}$, which is a closed subalgebra (ideal) in $C(X, \mathbb{R})$. We conclude $\overline{A} = \{f \in C(X, \mathbb{R}) \mid f(x_0) = 0\}$.

Step 5: $A \subset C(X, \mathbb{C})$ separates pts + is closed under complex conjugation. Similar statements as in Step 4 hold.

Pf: Apply Step 4 to $A_{sa} := \{a \in A \mid a = \bar{a}\}$. Since $A = A_{sa} \oplus i A_{sa}$ and $C(X, \mathbb{C}) = C(X, \mathbb{R}) \oplus i\, C(X, \mathbb{R})$, the result follows.

Exercise: Suppose X is LCH, $A \subset C_0(X, \mathbb{C})$ separates points and is closed under complex conjugation. Then either $\bar{A} = C_0(X, \mathbb{C})$ or $\{f \in C_0(X, \mathbb{C}) \mid f(x_0) = 0\}$ for some $x_0 \in X$.

- use 1pt compactification

Def: An embedding $\varphi: X \longrightarrow Y$ is a cts injection which is a homeomorphism onto its image [$\varphi^{-1}: \varphi(X) \longrightarrow X$ is cts].

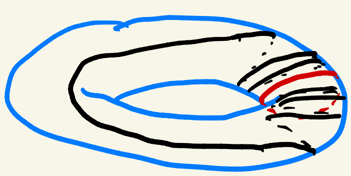
Def: A compactification of a topological space X is a cpt space K and an embedding $\varphi: X \longrightarrow K$ s.t. $\varphi(X)$ is dense in K.

Examples: $\mathbb{R}$ can be compactified by:

① $\overline{\mathbb{R}} = [-\infty, +\infty]$

② add one pt to get S^1

③  add $(0,0)$ and S^1 to $\mathbb{R}$ embedded in $\mathbb{R}^2$ in $\mathbb{D}$.

④  add a circle to $\mathbb{R}$ in $\mathbb{R}^3$ in $\mathbb{T}^2$.

Alexandroff (one pt) compactification: Suppose X is LCH. Choose an object ∞ not in X. Define $X^* := X \cup \{\infty\}$. Say $\mathcal{U} \subset X^*$ open if $\mathcal{U} \subset X$ open, or $\infty \in \mathcal{U}$ and $\mathcal{U}^c$ is cpt.

Thm: X^* is cpt Hausdorff, and $X \hookrightarrow X^*$ is an embedding.

Pf: We'll prove X^* is cpt Hausdorff.

cpt: Suppose $\{\mathcal{U}_i\}_{i\in I}$ is an open cover. Then $\exists\, \mathcal{U}_0$ s.t. $\infty \in \mathcal{U}_0$ and $\mathcal{U}_0^c$ is cpt. Then $\{\mathcal{U}_i \cap X\}_{i \in I}$ covers $X \setminus \mathcal{U}_0^c$, cpt. Pick a finite subcover.

Hausdorff: It suffices to separate $x \in X$ and ∞. Since X is LCH, $\exists$ open $\mathcal{U} \subset X$ s.t. $x \in \mathcal{U}$ and $\overline{\mathcal{U}}$ cpt. Set $V := \overline{\mathcal{U}}^c$, which is an open nbhd of ∞ disjoint from $\mathcal{U}$.

Def: X is completely regular if $\forall$ closed $F \subset X$ and $x \in F^c$, $\exists$ cts $f: X \longrightarrow [0,1]$ s.t. $f(x) = 1$ and $f|_F = 0$.
X is called Tychonoff if X is completely regular + T_1.

Facts:

1. Tychonoff $\Rightarrow$ Hausdorff
2. every normal space is Tychonoff by Tietze Extension.
3. Any subspace of a Tychonoff space is Tychonoff.

Embedding Lemma: Suppose $\Phi \subset C(X, [0,1])$ is a family of fcts. Define $e: X \to [0,1]^\Phi$ [cpt!] by $x \mapsto (f(x))_{f\in\Phi}$.

1. e is cts.
2. e is injective $\Leftrightarrow$ Φ separates pts
3. If Φ separates pts from closed sets [$\forall F \subset X$ closed and $x \in F^c$, $\exists f \in \Phi$ s.t. $f(x) \notin \overline{f(F)}$], e is open.
4. If separates pts and separates pts + closed sets, e is an embedding.

Pf: ① Observe $\pi_f \circ e = f$ is cts $\forall f \in \Phi$.

② $e(x) \neq e(y) \iff \exists f \in \Phi$ s.t. $\underbrace{[\pi_f \circ e](x)}_{f(x)} \neq \underbrace{[\pi_f \circ e](y)}_{f(y)}$

③ Suppose Φ separates pts from closed sets. Let $\mathcal{U} \subset X$ be open. Let $x \in \mathcal{U}$. Find an open set $V \subset [0,1]^{\Phi}$ s.t. $e(x) \in V \cap e(X) \subset e(\mathcal{U})$. $\exists f \in \Phi$ s.t. $f(x) \notin \overline{f(\mathcal{U}^c)}$. Then $W := [0,1] \setminus \overline{f(\mathcal{U}^c)}$ is an open set containing $f(x)$, so $e(x) \in \pi_f^{-1}(W)$, open set in $[0,1]^{\Phi}$. Observe that

$$e(y) \in \pi_f^{-1}(W) \cap e(X) \iff f(y) \notin \overline{f(\mathcal{U}^c)} \implies y \in \mathcal{U}.$$

Hence $e(x) \in \underbrace{\pi_f^{-1}(W)}_{=:V} \cap e(X) \subset e(\mathcal{U})$ as desired.

④ By ②, $e : X \hookrightarrow [0,1]^{\Phi}$ is a cts injection. By ③, e^{-1} on $e(X)$ is cts, so e is a homeom. onto $e(X)$.

Cor: X is Tychonoff $\iff \exists$ embedding $X \hookrightarrow [0,1]^I$.

Stone-Čech compactification: Suppose X is Tychonoff. Let $\Phi := C(X, [0,1])$, and consider the embedding $e : X \hookrightarrow [0,1]^{\Phi}$. Define the compactification $\beta X := \overline{e(X)}$.

Thm: The compactification $(\beta X, e)$ of X satisfies:

①
$$\begin{tikzcd} \beta X \arrow[dr, dashed, "\exists \beta f"] & \\ X \arrow[u, "e"] \arrow[r, "f"'] & Z \end{tikzcd}$$
$\forall$ cpt Hausdorff Z and cts $f : X \to Z$, $\exists$ cts $\beta f : \beta X \to Z$ s.t. $\beta f \circ e = f$.

② The map βf above is unique cts $g : \beta X \to Z$ s.t. $g \circ e = f$.

③ βX is uniquely characterized by the universal property ①.

④ β is a functor $\{\text{Tychonoff spaces}\} \longrightarrow \{\text{cpt Hausdorff spaces}\}$

Strategy: We'll prove the above in the following order: ②, ③, ④, ①.

② Suppose $\varphi: X \to Y$ is a compactification and $f: X \to Z$ is cts. $\exists$ at most one cts fct $g: Y \to Z$ s.t. $g \circ \varphi = f$.

Pf: If g_1, g_2 both satisfy $g_i \circ \varphi = f$, $g_1 = g_2$ on $\varphi(X) \subset Y$, dense. Hence $g_1 = g_2$.

③ $\exists$ at most one compactification $\varphi: X \to Y$ s.t.:

$$\begin{tikzcd} Y \arrow[dr, dashed, "\exists \tilde{f}"] & \\ X \arrow[u, hook, "\varphi"] \arrow[r, "f"'] & Z \end{tikzcd}$$

$\forall$ cpt Hausdorff Z and cts $f: X \to Z$, $\exists$ cts $\tilde{f}: Y \to Z$ s.t. $f = \tilde{f} \circ \varphi$.

Pf: Suppose (φ, Y) and (ψ, Z) are such embeddings. Then

$$\begin{tikzcd} Y \arrow[r, "\exists f"] & Z \arrow[r, "\exists g"] & Y \arrow[r, "\exists f"] & Z \\ & X \arrow[ul, "\varphi"] \arrow[u, "\psi"] \arrow[ur, "\varphi"] \arrow[urr, "\psi"'] & & \end{tikzcd}$$

$[g \circ f] \circ \varphi = \varphi \Rightarrow g \circ f = \mathrm{id}_Z$ by !ness in ②

$[f \circ g] \circ \psi = \psi \Rightarrow f \circ g = \mathrm{id}_Y$ by !ness in ②

④ Suppose $f: \underset{\text{Tychonoff}}{X \longrightarrow Y}$ is cts. Define $F: [0,1]^{\Phi_X} \to [0,1]^{\Phi_Y}$ componentwise: for $g \in \Phi_Y = C(Y, [0,1])$, define $\pi_g[F(p)] := \pi_{g \circ f}(p)$.

Then F is cts since $\pi_g \circ F = \pi_{g \circ f}: [0,1]^{\Phi_X} \to [0,1]$ cts $\forall g \in \Phi_Y$.

Moreover, $\forall x \in X$,

$$\pi_g(F(e_X(x))) = \pi_{g \circ f}(e_X(x)) = g(f(x)) = \pi_g(e_Y(f(x))).$$

Hence $F \circ e_X = e_Y \circ f: X \to [0,1]^{\Phi_Y}$, so $\mathrm{im}(F|_{\beta X}) \subset \overline{e_Y(Y)} = \beta Y$.

Define $\beta f := F|_{\beta X}: \beta X \to \beta Y$.

$$\begin{tikzcd} X \arrow[r, hook, "e_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \beta X \arrow[r, hook] \arrow[d, "\beta f"] \arrow[dr, phantom, "\circlearrowleft"] & {[0,1]^{\Phi_X}} \arrow[d, "F"] \\ Y \arrow[r, hook, "e_Y"] & \beta Y \arrow[r, hook] & {[0,1]^{\Phi_Y}} \end{tikzcd}$$

Remark: If every $h \in \Phi_X$ factorizes as $h = g \circ f$ for some $g \in \Phi_Y$, then F is injective. Indeed, if $p, p' \in [0,1]^{\Phi}$, we have

$$F(p) = F(p') \iff \pi_g[F(p)] = \pi_g[F(p')] \ \forall g \in \Phi_Y \iff \pi_{g \circ f}(p) = \pi_{g \circ f}(p') \ \forall g$$
$$\implies \pi_h(p) = \pi_h(p') \ \forall h \in \Phi_X \iff p = p'.$$

Observe:
$$\begin{tikzcd} \beta X \arrow[drr, dashed, "\exists \beta f"] & & \\ X \arrow[u, "e_X"] \arrow[r, "f"'] & Y \arrow[r, hook, "e_Y"'] & \beta Y \end{tikzcd}$$
s.t. $\beta f \circ e_X = e_Y \circ f \Rightarrow$ unique by ②!

Functoriality:

id: Since $\beta[id_X] \circ e_X = e_X \circ id_X = e_X = id_{\beta X} \circ e_X$, $\beta[id_X] = id_{\beta X}$ by uniqueness in ②.

$\circ$: Suppose $f : X \to Y$ and $g : Y \to Z$, all Tychonoff. Since

$\beta(g \circ f) \circ e_X = e_Z \circ g \circ f = \beta g \circ e_Y \circ f = \beta g \circ \beta f \circ e_X$,

$\beta(g \circ f) = \beta g \circ \beta f$ by uniqueness in ②.

① Existence of factorization of maps
$$\begin{tikzcd} \beta X \arrow[dr, dashed, "\beta f"] & \\ X \arrow[u, "e"] \arrow[r, "f"'] & Z \end{tikzcd}$$

Pf: Since Z is cpt, $e_Z(Z) \subset \beta Z$ is dense + cpt $\Rightarrow e_Z(Z) = \beta Z$. Then $e_Z : Z \xrightarrow{\sim} \beta Z$ is cts bij cpt $\to$ Hausd $\Rightarrow Z \cong \beta Z$ homeom. So the map $\beta f : \beta X \to \beta Z \cong Z$ satisfies $\beta f \circ e_X = f$.

Cor: Let X be Tychonoff and $\varphi : X \to K$ a compactification.

① The unique lift $\beta\varphi : \beta X \to K$ is surjective.

② Suppose $\forall f \in C_b(X)$, $\exists g \in C(K)$ s.t. $f = g \circ \varphi$. Then $\beta\varphi$ is a homeomorphism $\beta X \cong K$.

Pf: ① Since $\beta\varphi \circ e_X = \varphi$ and $\varphi(X)$ is dense in K, $\beta\varphi(\beta X)$ is dense in K. But βX cpt $\Rightarrow \beta\varphi(\beta X)$ cpt and thus closed. So $\beta\varphi(\beta X) = K$.

② By ①, it suffices to prove $\beta\varphi : \overset{\text{cpt}}{\beta X} \to \overset{\text{Hausd.}}{K}$ is injective. This follows similar to the Remark. Indeed, every $f \in \Phi_X \subset C_b(X)$ factorizes as $f = \lambda(g \circ \varphi)$ for some $\lambda \in \mathbb{R}$ and $g \in \Phi_K$.