Let Γ be a countable discrete group.

Exposition following notes of Ozawa from a 2009 mini-course.

Def: A fct $f : \Gamma \to \mathbb{C}$ is called positive definite if $\forall n \in \mathbb{N}$, $\forall g_1, \dots, g_n \in \Gamma$, the matrix $[f(g_i^{-1} g_j)] \geq 0$ in $M_n(\mathbb{C})$.

Lemma: Suppose $a \in M_n(\mathbb{C})$ is positive and constant along the diagonal. Then $\forall i, j \in \{1, \dots, n\}$, $|a_{ij}| \leq a_{kk}$

Pf: Let $b \in M_n(\mathbb{C})$ s.t. $a = b^* b$. Then $\forall i, j$,

$$|a_{ij}|^2 = |\langle a e_j, e_i \rangle|^2 = |\langle b e_j, b e_i \rangle|^2 \leq \|b e_j\|^2 \|b e_i\|^2 = \langle a e_i, e_i \rangle \langle a e_j, e_j \rangle = a_{ii} a_{jj}.$$

Now $a_{ii} = a_{jj} = a_{kk} \Rightarrow |a_{ij}| \leq a_{kk}$.

Prop: If f is positive definite, then $f \in \ell^\infty \Gamma$ w/ $\|f\|_\infty = f(e)$.

Pf: $\forall g \in \Gamma$, $|f(g)| = |a_{12}| \leq a_{11} = f(e)$ for $a = \begin{bmatrix} f(e) & f(g) \\ f(g^{-1}) & f(e) \end{bmatrix} \geq 0$.

Def: For $f : \Gamma \to \mathbb{C}$, get a multiplier $M_f : \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ by

$$\sum x_g g \longmapsto \sum f(g) x_g g.$$

↳ want to extend to a map $L\Gamma \to L\Gamma$!

Thm: For $f : \Gamma \to \mathbb{C}$, TFAE:

1. f is positive definite.
2. f is a coefficient of a unitary rep'n, i.e., $\exists \pi : \Gamma \to \mathcal{U}(H)$ a unitary Γ-rep'n and $\Omega \in H$ s.t. $\forall g \in \Gamma$, $f(g) = \langle \pi(g) \Omega, \Omega \rangle$.
3. M_f extends to a normal cp map $L\Gamma \to L\Gamma$.

Pf:

1 ⇒ 2: use GNS. Equip $\mathbb{C}\Gamma$ w/ positive sesquilinear form given by $\langle \eta, \xi \rangle_f := \sum f(h^{-1} g) \eta(g) \overline{\xi(h)}$. The usual steps give a Hilbert space H w/ cyclic vector δ_e. For $g \in \Gamma$ and $\xi \in \mathbb{C}\Gamma$, defining $(\pi_g \xi)(h) := \xi(g^{-1} h)$ extends !ly to a unitary $\pi_g \in \mathcal{U}(H)$.

Indeed, $\pi_g^{-1} = \pi_{g^{-1}}$ and π_g is isometric:

$$\|\pi_g \xi\|_H^2 = \sum_{k,h} f(k^{-1} h) \xi(g^{-1} h) \overline{\xi(g^{-1} k)} = \sum_{h,k} f((g^{-1} k)^{-1} (g^{-1} h)) \xi(g^{-1} h) \overline{\xi(g^{-1} k)} = \|\xi\|_H^2.$$

Finally, $\langle \pi_g \delta_e, \delta_e \rangle = f(g)$ $\forall g \in \Gamma$.

② ⇒ ③: We'll use:

Fell's Absorption Principle: If (H,π) any unitary rep of Γ and λ is the left regular rep, then $\lambda\otimes\pi$ is a rep on $\ell^2\Gamma\bar{\otimes}H$ unitarily equivalent to $\lambda\otimes 1$. [HW!]

Define $\tilde{\pi}: L\Gamma \to B(\ell^2\Gamma\otimes H)$ by $\lambda_g \overset{\text{amplification}}{\longmapsto} \lambda_g\otimes 1 \overset{\text{Ad}(u) \text{ from } \circlearrowleft}{\longmapsto} \lambda_g\otimes\pi_g$. As $\tilde{\pi}$ is a composite of normal unital $*$-homs, $\tilde{\pi}$ is normal. Now define $v: \ell^2\Gamma \to \ell^2\Gamma\bar{\otimes}H$ by $\xi \mapsto \xi\otimes\frac{\Omega}{\|\Omega\|}$, which is an isometry.

Claim: $\forall x\in\mathbb{C}\Gamma$, $M_f x = \|\Omega\|^2 v^*\tilde{\pi}(x)v$, which is manifestly normal + cp.

Pf: By linearity of both sides, it suffices to check on $x=\lambda_g\in L\Gamma$.

Then $v^*\tilde{\pi}(\lambda_g)v\delta_h = v^*(\lambda_g\otimes\pi_g)[\delta_h\otimes\frac{\Omega}{\|\Omega\|}] = v^*(\delta_{gh}\otimes\pi_g\frac{\Omega}{\|\Omega\|})$

$= \langle \pi_g\frac{\Omega}{\|\Omega\|}, \frac{\Omega}{\|\Omega\|}\rangle\delta_{gh} = \frac{1}{\|\Omega\|^2}f(g)\lambda_g\delta_h.$

By linearity + continuity, $\|\Omega\|^2 v^*\tilde{\pi}(\lambda_g)v = M_f\lambda_g$.

③ ⇒ ①: Let $g_1,\dots,g_n\in\Gamma$. Then $[\lambda(g_i^{-1}g_j)] = \begin{bmatrix}\lambda(g_1)\\ \vdots\\ \lambda(g_n)\end{bmatrix}^*[\lambda(g_1)\cdots\lambda(g_n)] \geq 0$ in $M_n(L\Gamma)$.

Now, since M_f is cp, $[M_f\lambda(g_i^{-1}g_j)]\geq 0$ in $M_n(L\Gamma)$, so

$$[f(g_i^{-1}g_j)] = \begin{bmatrix}\lambda(g_1) & & \\ & \ddots & \\ & & \lambda(g_n)\end{bmatrix}[M_f\lambda(g_i^{-1}g_j)]\begin{bmatrix}\lambda(g_1) & & \\ & \ddots & \\ & & \lambda(g_n)\end{bmatrix}^* \geq 0$$

in $M_n(L\Gamma)$, and thus is also ≥ 0 in $M_n(\mathbb{C})$.

Example: Suppose $\varphi: L\Gamma\to L\Gamma$ is cp and recall $\mathrm{tr} = \langle\cdot\,\delta_e,\delta_e\rangle$ on $L\Gamma$. Define $f(g) := \mathrm{tr}(\varphi(\lambda_g)\lambda_g^*)$. We claim $M_f: L\Gamma\to L\Gamma$ is cp so that f is pos. def. Indeed, M_f is the composite of the following cp maps:

$$L\Gamma \xrightarrow{\Delta} L\Gamma\bar{\otimes}L\Gamma \xrightarrow{\mathrm{id}\otimes\varphi} L\Gamma\bar{\otimes}L\Gamma \xrightarrow{v^*\cdot v} L\Gamma$$

$\lambda_g \longmapsto \lambda_g\otimes\lambda_g$

$x\otimes y \longmapsto x\otimes\varphi(y)$

$x\otimes y \longmapsto \langle x\Omega, y\Omega\rangle x$?? check!

$v\delta_g = \delta_g\otimes\delta_g$

$v^*[\delta_g\otimes\delta_h] = \delta_{g=h}\delta_g$.

$v^*[\lambda_g\otimes\lambda_h]v = \delta_{g=h}\lambda_g$

Example: If $\Lambda\leq\Gamma$ a subgp, $\chi_\Lambda:\Gamma\to\mathbb{C}$ defined by $\chi_\Lambda(g) := \langle\pi_g\delta_\Lambda,\delta_\Lambda\rangle$ where $\pi:\Gamma\to U(\ell^2\Gamma/\Lambda)$. In this case, $M_{\chi_\Lambda} = E_{L\Lambda}$, the canonical trace-preserving conditional expectation $L\Gamma\to L\Lambda$.

Def: If $N \subseteq M$ is an inclusion of vNa's, a <u>conditional expectation</u> $E: M \to N$ is a cp map s.t. $E|_N = \mathrm{id}_N$.

A conditional expectation is called <u>normal</u> if $x_\lambda \nearrow x \Rightarrow E(x_\lambda) \nearrow E(x)$, which is equivalent to σWOT continuity.

Lemma: $E(axb) = a\,E(x)\,b \quad \forall a,b \in N$ and $x \in M$.

Pf: Use <u>Stinespring dilation</u>: $\exists$ Hilb. space K, a <u>$*$-hom</u> $\pi: M \to B(K)$, and a $v \in B(H,K)$ s.t. $E = v^*\pi(\cdot)v$. Note: E normal $\Leftrightarrow \pi$ normal.

- $E(x^*) = E(x)^*$
- $E(x^*x) - E(x)^*E(x) = v^*\pi(x)^*[1 - vv^*]\pi(x)v \geq 0$
- If $n \in N$, $E(n^*n) - E(n)^*E(n) = 0 \Rightarrow [1 - vv^*]\pi(n)v = 0$

Thus $\forall a \in N, x \in M$,
$$\begin{aligned} E(ax) &= v^*\pi(a)\pi(x)v \\ &= v^*\pi(a)vv^*\pi(x)v + \underbrace{v^*\pi(a)(1-vv^*)\pi(x)v}_{=0} \\ &= E(a)E(x) \\ &= aE(x). \end{aligned}$$

Similarly, $E(xb) = E(x)b \quad \forall b \in N, x \in M$.

Example: If (M, tr) a finite vNa w/ faithful normal tracial state tr and $N \subseteq M$, then $\exists!\ E: M \to N$ conditional expectation preserving tr which is necessarily normal. [HW!]

Amenability: Again, Γ is a countable discrete gp.

Def: Let (H,π) be a unitary rep of Γ. (H,π) contains an <u>invariant vector</u> if $\exists \xi \in H$ s.t. $\pi_g\xi = \xi \quad \forall g \in \Gamma$. (H,π) contains <u>almost invariant vectors</u> if $\forall$ finite $F \subseteq \Gamma$ and $\forall \varepsilon > 0$, $\exists \xi \in H$ s.t. $\|\pi_g\xi - \xi\| < \varepsilon\|\xi\| \quad \forall g \in F$.

Def: A (left) <u>invariant mean</u> on Γ is a finitely additive, (left) Γ-invariant probability measure on 2^Γ, the power set of Γ.

Let $\mathrm{Prob}(\Gamma) = \{\mu \in \ell^1\Gamma \mid \mu \geq 0 \text{ and } \sum_g \mu(g) = 1\}$. We say Γ has an <u>approximate invariant mean</u> if $\forall$ finite $F \subseteq \Gamma$ and $\varepsilon > 0$, $\exists \mu \in \mathrm{Prob}(\Gamma)$ s.t. $\max_{g\in F} \|g\cdot\mu - \mu\|_1 < \varepsilon$, where $(g\cdot\mu)(A) = \mu(g^{-1}A)$ for $A \subseteq \Gamma$.

Def/Thm: For Γ, TFAE. If one/all hold, call Γ amenable.

① $\exists$ state $m \in (\ell^\infty\Gamma)^*$ s.t. $m(g\cdot f) = m(f)$ $\forall g \in \Gamma$ and $f \in \ell^\infty\Gamma$ where $(g\cdot f)(h) := f(g^{-1}h)$.

② $\exists$ a left invariant mean on Γ.

Fact: $\exists$ a left-invariant mean $\iff$ $\exists$ a right invariant mean $\iff$ $\exists$ an invariant mean.

③ $\exists$ an approximate invariant mean on Γ.

④ [Følner sequence] $\exists$ seq. of finite sets $\emptyset \neq F_n \subset \Gamma$ s.t. $\forall g \in \Gamma$, $\dfrac{|gF_n \,\Delta\, F_n|}{|F_n|} \xrightarrow{n\to\infty} 0$ (symmetric difference.)

⑤ The left regular rep'n $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ has almost invariant vectors.

⑥ [The trivial rep'n is weakly contained in the left regular rep'n] $\exists (\xi_n) \subset \ell^2\Gamma$ w/ $\|\xi_n\| = 1$ $\forall n$ s.t. $\|\lambda_g \xi_n - \xi_n\|_2 \xrightarrow{n\to\infty} 0$ $\forall g \in \Gamma$.

⑦ $\exists$ seq. (f_n) of finitely supported pos. def. fcts on Γ s.t. $f_n \to 1$ pointwise.

⑧ The reduced C^*-gp alg. $C_r^*(\Gamma) :=$ norm closure of span $\{\lambda_g \mid g \in \Gamma\} \subseteq B(\ell^2\Gamma)$ is isomorphic to the universal C^*-gp alg $C^*(\Gamma)$, which is the closure of $\mathbb{C}\Gamma$ in $\|x\|_u := \sup\{\|\pi(x)\| \mid \pi : \Gamma \to \mathcal{U}(H)$ unitary rep'n$\}$. $\leftarrow$ well-defined since $\|\pi(g)\| = 1$ $\forall g \in \Gamma$!

⑨ $\exists$ a 1-dim'l rep'n of $C_r^*(\Gamma)$.

⑩ $\forall$ finite $F \subseteq \Gamma$, $\left\| \frac{1}{|F|} \sum_{g \in F} \lambda_g \right\|_{B(\ell^2\Gamma)} = 1$.

⑪ [$L\Gamma$ amenable] $\exists$ a conditional expectation $E : B(\ell^2\Gamma) \longrightarrow L\Gamma$

⑫ $\exists$ a state $\varphi \in B(\ell^2\Gamma)^*$ s.t.
- $\varphi(\lambda_g x) = \varphi(x \lambda_g)$ $\forall g \in \Gamma$, $\forall x \in B(\ell^2\Gamma)$
- $\varphi|_{L\Gamma} = tr_{L\Gamma}$.

Before the proof, examples!

Nonexample: $\mathbb{F}_n$ $n \geq 2$ is not amenable.

Pf for $\mathbb{F}_2$: Suppose $\mathbb{F}_2 = \langle a, b \rangle$. For $x \in \{a, b, a^{-1}, b^{-1}\}$, let W_x be the reduced words beginning w/ x. Then $\mathbb{F}_2$ can be written

① $\mathbb{F}_2 = \{e\} \cup W_a \cup W_b \cup W_{a^{-1}} \cup W_{b^{-1}}$

② $\mathbb{F}_2 = W_a \cup aW_{a^{-1}}$

③ $\mathbb{F}_2 = W_b \cup bW_{b^{-1}}$

Hence no invariant mean on $\mathbb{F}_2$ can exist!

Examples of amenable gps:

① Finite gps: Easy! Can easily do ①–④ above.

② $\mathbb{Z}$: (a) $F_n := [-n,n]$. Then $\forall m \in \mathbb{Z}$, $\frac{|mF_n \Delta F_n|}{|F_n|} \underset{\substack{\text{(eventually)} \\ (n \geq m)}}{=} \frac{2m}{2n+1} \to 0$

(b) Define $f_n : \mathbb{Z} \to \mathbb{C}$ by $f_n(k) = \langle \lambda_k \frac{1}{\sqrt{|F_n|}} \chi_{F_n}, \chi_{F_n} \rangle = \begin{cases} 1 - \frac{|k|}{n} & |k| \leq n \\ 0 & \text{else.} \end{cases}$

Then $f_n \to 1$ pointwise.

Note: $\hat{f}_n = \sum_k f_n(k) e^{ikt} \in C[-\pi,\pi]$ is the Fejér kernel.

③ If Γ is locally finite, i.e., $\Gamma = \bigcup \Gamma_n$ finite subgps. Let m_n be the Haar measure on Γ_n and let ω be a non-principal/free ultrafilter on $\mathbb{N}$, i.e., $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. For $f \in \ell^\infty \Gamma$, define

$$m(f) := \lim_\omega m_n(f|_{\Gamma_n}).$$

④ The class of amenable gps is closed under:

- products
- extensions
- subgps
- quotients
- direct limits

⑤ All abelian gps are amenable. Use ④

Pf of Equivalences of Amenability defns:

①⇔②: If $m \in (\ell^\infty \Gamma)^*$ a left Γ-invariant state, define $\mu : 2^\Gamma \to [0,1]$ by $\mu(A) = m(\chi_A)$. If $\mu : 2^\Gamma \to [0,1]$ is a finitely additive left Γ-invariant prob. measure on Γ, define $m = \int \cdot \, d\mu$, which is a left Γ-invariant state on $\ell^\infty \Gamma$.

①⇒③: [Day]

③⇒④: [Namioka]

HW!

⑤⇔⑥: easy exercise.

④⇒⑥: Suppose (F_n) a Følner sequence for Γ. Set $\xi_n := \frac{1}{\sqrt{|F_n|}} \chi_{F_n}$.

Then $\xi_n \in \ell^2 \Gamma$ is a unit vector, and $\forall g \in \Gamma$,

$$\|\lambda_g \xi_n - \xi_n\|_2^2 = \sum_h |\xi_n(g^{-1}h) - \xi_n(h)|^2 = \sum_h \frac{1}{|F_n|} |\chi_{F_n}(g^{-1}h) - \chi_{F_n}(h)|^2$$

$$= \frac{|gF_n \Delta F_n|}{|F_n|} \to 0 \quad \text{as } n \to \infty.$$

(6) $\Rightarrow$ (4): Let (ξ_n) be a seq. of unit $\vec{v}$'s s.t. $\|\lambda_g \xi_n - \xi_n\|_2 \to 0$ as $n \to \infty$.

Define $\varphi_n(g) := \langle \lambda_g \xi_n, \xi_n \rangle$. Then φ_n is pos. def. [coeff of rep'n]

Moreover, $|\varphi_n(g) - 1| = |\langle \lambda_g \xi_n, \xi_n \rangle - \langle \xi_n, \xi_n \rangle|$

$$= |\langle \lambda_g \xi_n - \xi_n, \xi_n \rangle| \leq \|\lambda_g \xi_n - \xi_n\| \xrightarrow{n\to\infty} 0.$$

Now let $(E_n) \subset \Gamma$ be finite s.t. $E_n \subset E_{n+1}$ and $\bigcup E_n = \Gamma$. Define $\eta_n = \xi_n|_{E_n}$ and $\psi_n(g) := \langle \lambda_g \eta_n, \eta_n \rangle$, also pos. def. Then

$$|\varphi_n(g) - \psi_n(g)| = |\langle \lambda_g \xi_n, \xi_n \rangle - \langle \lambda_g \eta_n, \eta_n \rangle|$$
$$= |\langle \lambda_g \xi_n, \xi_n - \eta_n \rangle - \langle \lambda_g (\eta_n - \xi_n), \eta_n \rangle| \leq 2\|\xi_n - \eta_n\| \to 0.$$

Thus $\psi_n \to 1$ p.twise and ψ_n finitely supported as η_n has finite supp.

(4) $\Rightarrow$ (1): Define $E : B(\ell^2\Gamma) \to L\Gamma$ by $E(x) = \underbrace{\mathrm{Lim} \frac{1}{|F_n|} \sum_{g \in F_n} \rho_g x \rho_g^*}_{\text{manifestly ccp.}}$ where

- (F_n) a Følner sequence
- Lim is any Banach limit extension of $\lim\limits_{n\to\infty}$ on c to a state on ℓ^∞.
- $\rho_g \in B(\ell^2\Gamma)$ by $(\rho_g \xi)(h) = \xi(hg)$, right reg. rep'n.

Claim: $E(x) \in R\Gamma' = L\Gamma$, and $E|_{L\Gamma} = \mathrm{id}_{L\Gamma}$.

Pf: $\forall h \in \Gamma$, $\left\| \rho_h \left[\frac{1}{|F_n|} \sum_{g \in F_n} \rho_g x \rho_g^* \right] \rho_h^* - \frac{1}{|F_n|} \sum_{g \in F_n} \rho_g x \rho_g^* \right\| \leq \frac{|hF_n \Delta F_n|}{|F_n|} \to 0.$

The last claim is obvious as $[\rho_g, \lambda_h] = 0$ $\forall g, h \in \Gamma$.

(1) $\Leftrightarrow$ (2): we'll prove a more general statement.

Thm: Let $M \subseteq B(\mathcal{H})$ be a tracial vNa w/ normal faithful tracial state tr_M.

$\exists E : B(\mathcal{H}) \to M$ cond. exp. $\Longleftrightarrow$ $\exists \varphi \in B(\mathcal{H})^*$ s.t.
- $\varphi(mx) = \varphi(xm)$ $\forall x \in B(\mathcal{H}), m \in M$.
- $\varphi|_M = tr_M$.

Pf: $\Rightarrow$: Set $\varphi = tr_M \circ E$. Then $\varphi(xm) = tr(E(xm)) = tr(E(x)m) = tr(mE(x)) = \cdots = \varphi(mx)$.

$\Leftarrow$: For $x \in B(\mathcal{H})_+$, define φ_x on M by $\varphi_x(m) := \varphi(mx)$. For $m \in M_+$, we have $|\varphi_x(m)|^2 = |\varphi(m^{1/2} x m^{1/2})|^2 \leq \varphi(m^{1/2} x^2 m^{1/2})\, \varphi(m) \leq \|x\|^2 \varphi(m)^2$.

Since $\varphi|_M = tr_M$, $\varphi_x(m) \leq \|x\| \underbrace{tr(m)}_{\omega_\Omega(m),\ \Omega \in L^2M}$ $\forall x \in B(\mathcal{H})_+$, $\forall m \in M_+$.

Claim 1: φ_x is normal.

Pf: If $m_\lambda \nearrow m$, then $\varphi_x(m - m_\lambda) \leq \|x\|\, tr(m - m_\lambda) \searrow 0$.

Claim 2: $\exists E(x) \in M_+$ s.t. $\varphi_x(m) = tr_M(mE(x))$ $\forall m \in M$.

Pf: Since $0 \leq \varphi_x \leq \|x\| \omega_\Omega$, $\Omega \in L^2M$, by prev. HW, $\exists!\ 0 \leq a \leq \|x\|$ in $M' = JMJ$ s.t. $\varphi_x = \langle \cdot\, a\Omega, \Omega \rangle$. Now $a = Jb^*J$ for a $!\, b \in M$, so $\forall m \in M$, $\varphi_x(m) = \langle mJb^*J\Omega, \Omega \rangle = \langle mb\Omega, \Omega \rangle = tr(mb)$. Define $E(x) = b$.

Finally, it is straightforward that $E : B(\mathcal{H}) \to M$ is a cond. exp.

⑫ ⟹ ①: Recall $M: \ell^\infty\Gamma \longrightarrow B(\ell^2\Gamma)$ by $(M_f \xi)(g) = f(g)\xi(g)$. Then if $f \in \ell^\infty\Gamma$ and $g \in \Gamma$, $\lambda_g M_f \lambda_g^* = M_{g.f}$, so $\varphi(M_f) = \varphi(M_{g.f})$. Hence $\varphi|_{\ell^\infty\Gamma} \in (\ell^\infty\Gamma)^*$ is a left Γ-invariant state.

⑦ ⟹ ⑧: First, note $\|\lambda_x\| \le \|x\|_u := \sup\{\|\pi_g\| \mid (H,\pi) \text{ a unitary } \Gamma\text{-rep}\}$ where $\lambda: \mathbb{C}\Gamma \longrightarrow B(\ell^2\Gamma)$. Thus λ extends to $C^*\Gamma \twoheadrightarrow C^*_r\Gamma$. We must show λ is injective. Suppose (φ_n) seq. of pos. def fcts s.t. $\varphi_n \to 1$ ptwise on Γ.

Claim: A pos. def φ defines a cp multiplier on both $C^*\Gamma$ + $C^*_r\Gamma$.

Pf: We saw φ gives a normal cp multiplier on $L\Gamma$ via Fell's absorption principal. Just do this for $C^*\Gamma$ and $C^*_r\Gamma$: Let (H, π, Ω) s.t. $\varphi(g) = \langle \pi(g)\Omega, \Omega\rangle$. The map $\xi \mapsto \xi \otimes \frac{\Omega}{\|\Omega\|}$ is an isometry $\ell^2\Gamma \to \ell^2\Gamma \otimes H$. Now the map $g \xrightarrow{\tilde{\pi}} \lambda_g \otimes \pi_g$ extends to $\mathbb{C}\Gamma$, and is bdd in $\|\cdot\|_u$. Thus it extends to $C^*\Gamma$. So $M_\varphi x := \underbrace{\|\Omega\|^2 v^* \tilde{\pi}(x) v}_{\text{bdd for both } \|\cdot\|_u, \|\cdot\|_{\ell^2\Gamma}}$ works.

Let M_n and $M_{n,r}$ be the multiplier of φ_n on $C^*\Gamma$, $C^*_r\Gamma$ resp. Note that $\lambda \circ M_n = M_{n,r} \circ \lambda$ on $C^*\Gamma$, since both are cts wrt $\|\cdot\|_u$ and agree on $\mathbb{C}\Gamma$. Since $\varphi_n \to 1$, $M_n x \to x$ for $x \in \mathbb{C}\Gamma$. Since $\|\varphi_n\|_\infty$ are uniformly bdd by $\sup \varphi_n(e)$ $[\varphi_n(e) \to 1]$, $M_n x \to x$ $\forall x \in C^*\Gamma$ by density of $\mathbb{C}\Gamma$ in $C^*\Gamma$.

Suppose $x \in C^*\Gamma$ s.t. $\lambda x = 0$. Then $\lambda M_n x = M_{n,r}\lambda x = 0$ $\forall n$. But φ_n is finitely supported, so $M_n x \in \mathbb{C}\Gamma$ and $\lambda M_n x = 0 \Rightarrow M_n x = 0$. Thus $x = \lim M_n x = 0$.

⑧ ⟹ ⑨: $C^*\Gamma$ has a 1-dim'l rep'n as the trivial rep'n of $\mathbb{C}\Gamma$ $\sum x_g g \longmapsto \sum x_g$ on $\mathbb{C}$ is subordinate to $\|\cdot\|_u$.

Lemma: Let A be a unital C^*-alg, $\varphi \in A^*$ a state, and $a \in A$ st. $\varphi(a^*a) = |\varphi(a)|^2$. Then $\forall b \in A$, $\varphi(ab) = \varphi(a)\varphi(b) = \varphi(ba)$.

Pf: Let $(H_\varphi, \pi_\varphi, \Omega_\varphi)$ be the GNS rep of φ. Then note:

$$\|\pi_\varphi(a)\Omega_\varphi\|^2 = \varphi(a^*a) = |\varphi(a)|^2 = |\langle \pi_\varphi(a)\Omega_\varphi, \Omega_\varphi\rangle|^2 \underset{(CS)}{\le} \|\pi_\varphi(a)\Omega_\varphi\|^2.$$

Thus the $\le$ above is an $=$, so $\pi_\varphi(a)\Omega_\varphi$ and Ω_φ must be proportional, i.e., $\exists \lambda \in \mathbb{C}$ s.t. $\pi_\varphi(a)\Omega_\varphi = \lambda \Omega_\varphi$. Then:

- $\pi_\varphi(a^*)\Omega_\varphi = \bar{\lambda}\Omega_\varphi$
- $\varphi(a) = \lambda$ and $\varphi(a^*) = \bar{\lambda}$.

Hence $\forall b \in A$, $\varphi(ab) = \langle \pi_\varphi(ab)\Omega_\varphi, \Omega_\varphi\rangle = \langle \pi_\varphi(b)\Omega_\varphi, \pi_\varphi(a^*)\Omega_\varphi\rangle$

$$= \lambda \langle \pi_\varphi(b)\Omega_\varphi, \Omega_\varphi\rangle = \varphi(a)\varphi(b)$$
$$= \langle \pi_\varphi(b)\pi_\varphi(a)\Omega_\varphi, \Omega_\varphi\rangle = \varphi(ba).$$

(9) $\Rightarrow$ (1): Let $\tau: C^*_r(\Gamma) \to \mathbb{C}$ be a 1-dim'l rep'n. Then τ is a state. Can extend τ to a state $\varphi \in B(\ell^2\Gamma)^*$ by HB.

- Note: $\forall g \in \Gamma$, $\varphi(\lambda_g^*\lambda_g) = \varphi(1) = 1 = |\tau(\lambda_g)|^2 = |\varphi(\lambda_g)|^2$.

Then $\forall f \in \ell^\infty\Gamma$, $g \cdot f = \lambda_g f \lambda_g^*$, and thus by the Lemma:

$$\varphi(s \cdot f) = \varphi(\lambda_g f \lambda_{g^{-1}}) = \varphi(\lambda_g)\varphi(f)\varphi(\lambda_{g^{-1}}) = \varphi(f),$$

so $\varphi|_{\ell^\infty\Gamma} \in (\ell^\infty\Gamma)^*$ is a left Γ-invariant state.

(6) $\Rightarrow$ (10): Let $(\xi_n) \subset \ell^2\Gamma$ be a seq. of unit v's s.t. $\|\lambda_g\xi_n - \xi_n\| \to 0$ $\forall g$.

Then $\forall$ finite $F \subset \Gamma$,

$\lambda_g\xi_n \to \xi_n$ $\forall g$.

$$1 = \lim_n \Big\|\frac{1}{|F|}\sum_{g\in F}\xi_n\Big\|_2 = \lim_n \Big\|\frac{1}{|F|}\sum_{g\in F}\lambda_g\xi_n\Big\|_2 \le \Big\|\frac{1}{|F|}\sum_{g\in F}\lambda_g\Big\|_\infty \le 1.$$

(10) $\Rightarrow$ (5): Let $F \subseteq \Gamma$ be finite s.t. $F = F^{-1}$. Then $x = \frac{1}{|F|}\sum_{g\in F}\lambda_g$ is s.a. and has norm 1. Let $\varepsilon > 0$. $\exists \xi \in \ell^2\Gamma$ s.t. $|\langle x\xi, \xi\rangle| > 1 - \boxed{\varepsilon'}$ TBD! Let $|\xi| \in \ell^2\Gamma$ be the pointwise absolute value of ξ: $|\xi|(g) := |\xi(g)|$ $\forall g \in \Gamma$.

$$1 - \varepsilon' < |\langle x\xi, \xi\rangle| = \Big|\sum_{g\in\Gamma} x\xi(g)\overline{\xi(g)}\Big| \le \sum_{g\in\Gamma}|x\xi(g)|\cdot|\xi(g)| \le \sum_{g\in\Gamma} x|\xi|(g)\,|\xi|(g)$$

$$= \langle x|\xi|, |\xi|\rangle = \frac{1}{|F|}\sum_{g\in F}\underbrace{\langle \lambda_g|\xi|, |\xi|\rangle}_{\le 1\ \forall g}.$$

Choose $\varepsilon' < \min\{\frac{\varepsilon^2}{2|F|}, \frac{1}{|F|}\}$.

Thus $\forall g \in F$, $\langle \lambda_g|\xi|, |\xi|\rangle > 1 - |F|\varepsilon'$!

$$\|\lambda_g|\xi| - |\xi|\|^2 = \|\lambda_g|\xi|\|^2 + \||\xi|\|^2 - \langle \lambda_g|\xi|, |\xi|\rangle - \langle \lambda_{g^{-1}}|\xi|, |\xi|\rangle$$
$$= 1 - \langle \lambda_g|\xi|, |\xi|\rangle + 1 - \langle \lambda_{g^{-1}}|\xi|, |\xi|\rangle < 2|F|\varepsilon' < \varepsilon^2.$$

Def: A vNa $M \subseteq B(H)$ is called amenable if $\exists$ cond. exp. $E: B(H) \to M$.

we saw Γ amenable $\iff L\Gamma$ is amenable.

Consider the category C^*Alg_{ucp} of unital C^*-alg's w/ ucp maps.

Def: An object/C^*alg C in this category is called injective if
$\forall$ injective ucp map $i: A \to B$, every ucp map $\Phi: A \to C$
extends to a ucp map $B \to C$:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\Phi"'] & B \arrow[dl, dashed, "\exists"] \\ & C & \end{tikzcd}$$

injective in category of unital C^*alg's w/ ucp maps.

Remark: $i(A) \subseteq B$ is an operator system: a $*$-closed subspace containing 1_B.

Arveson's Extension Thm: $B(H)$ is injective: If $S \subseteq B$ is an operator subsystem of B, every ucp $\Phi: S \to B(H)$ extends to a ucp map $B \to B(H)$.

Pf: Let $(P_\lambda) \subset B(H)$ be an increasing net of finite rank proj's $\nearrow 1$.
For λ, define $\Phi_\lambda : S \to B(H)$ by $\Phi_\lambda(s) := P_\lambda \Phi(s) P_\lambda \in B(P_\lambda H) \cong M_{n_\lambda}(\mathbb{C})$.

Fact: Any ucp map $S \to M_n(\mathbb{C})$ extends to a ucp map $A \to M_n(\mathbb{C})$.
Pf: HW!

Thus $\exists$ extension to a ucp map $\tilde{\Phi}_\lambda : A \to M_{n_\lambda}(\mathbb{C}) \cong B(P_\lambda H) \subseteq B(H)$. Now the unit ball of $B(A, B(H))$ is cpt in the pt. ultraweak top. Any cluster pt. of $(\tilde{\Phi}_\lambda)$ will be a ucp map extending Φ.

Details on pt. ultraweak top: Suppose X Banach and M a vNa. Then $B(X, M)$ has predual $B(X,M)_* = \overline{\operatorname{span}}\{T \mapsto \varphi(Tx) \mid x \in X, \varphi \in M_*\} \subseteq B(X,M)^*$.

On bdd sets the wk* top on $B(X,M)$ agrees w/ the point ultraweak top.

So for a bdd net $(T_\lambda) \subset B(X, M)$,

$$T_\lambda \to T \text{ pt. ultraweak} \iff \varphi(T_\lambda x) \to \varphi(Tx) \quad \forall x \in M, \varphi \in M_*.$$

Prop: TFAE:

① $\forall$ faithful normal rep'n (K, π) of M, $\exists E: B(K) \to M$ cond. exp.
② M is amenable.
③ M is injective.

Pf: ①⇒②: obvious.

②⇒③: Suppose $M \subseteq B(H)$ and $\exists$ a cond. exp. $E: B(H) \to M$. Let $S \subseteq B$ be an operator subsystem of a unital C*-alg B and let $\Phi: S \to M \subseteq B(H)$ be ucp. Using Arveson's extension thm, $\exists$ ucp extension $\tilde{\Phi}: B \to B(H)$. Then $E \circ \tilde{\Phi}: B \to M$ is ucp and extends Φ.

③⇒①: Let $\pi: M \to B(K)$ be faithful + normal. Apply ③ to $0 \to M \xrightarrow{\pi} B(K)$.

$\exists$ ucp $E: B(K) \to M$ s.t. $E|_{\pi(M)} = id$; so E a cond. exp.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\pi"] \arrow[d, "id"'] & B(K) \arrow[dl, dotted, "\exists E"] \\ & M & \end{tikzcd}$$

Def: A tracial vNa (M, tr) is AFD if $\forall$ finite $F \subseteq M$, $\exists$ a finite dim'l unital $*$-subalg $A \subseteq M$ s.t. $F \underset{\varepsilon,2}{\subset} A$:

- $\forall f \in F$, $\exists a \in A$ s.t. $\|f\Omega - a\Omega\|_2 < \varepsilon$.

Extremely difficult Thm (Connes):

① M is amenable

② M is AFD

Thm (Murray–vN): Let M be a II_1 factor acting on separable H. TFAE.

① M is AFD.

② $\exists$ an increasing seq. (M_n) of finite dim'l unital $*$-subalg's of M s.t. $(\bigcup M_n)'' = M$.

③ $\exists$ an increasing seq. (N_n) of type $I_{2^{k_n}}$ matrix algebras containing 1_M s.t. $(\bigcup N_n)'' = M$.

④ There is a unital $*$-isomorphism $\phi: M \to R := \bar{\bigotimes}_{\mathbb{N}} M_2(\mathbb{C})$, the ∞-tensor product of $M_2(\mathbb{C})$, a.k.a., the hyperfinite II_1 factor.

Proof: Roberto will give in NCGOA seminar.

Thm: The hyperfinite II_1 factor is amenable.

Pf: Let $M_n = \hat{\bigotimes}_{i=1}^{n} M_2(\mathbb{C})$ so that $R = (\bigcup M_n)''$ on L^2R. For $n \geq 1$ and $x \in B(L^2R)$, define $E_n(x) = \int_{\mathcal{U}(JM_nJ)} u x u^* \, du$.

$\leftarrow$ cpt, Haar measure

Note that: ① $(E_n(x))_n$ is norm bdd.

② $\forall r \in R$, $E_n(r) = r$ $\forall n$.

Now define $E(x) = \lim_{n\to\omega} E_n(x)$ $\forall x \in B(L^2R)$, where ω is a free ultrafilter on $\mathbb{N}$. That is, $\exists!$ elt $E(x) \in B(L^2R)$ s.t.

$$\langle \xi, E(x)\eta \rangle = \lim_{n\to\omega} \langle \xi, E_n(x)\eta \rangle \quad \forall \eta, \xi \in L^2R.$$

Since $E_n(x) \in (JM_nJ)' \subseteq (JM_kJ)'$ $\forall n \geq k$, we have $\forall x \in B(L^2R)$,

$E(x) \in \bigcap_n (JM_nJ)' = R$. [If $x \in (JM_nJ)'$ $\forall n$, then x commutes w/ $J(\bigcup M_n)J$, so x commutes w/ JRJ. Hence $x \in R$.]

Also, $E|_R = \mathrm{id}$. It is now straightforward to show E is ucp.

The Haagerup property: Let Γ be a discrete countable gp. Say Γ has the Haagerup property [has (H)] if $\exists$ a seq. (φ_n) of pos. def. c_0 functions on Γ s.t. $\varphi_n \to 1$ ptwise.

Examples:

1. All amenable gps have (H), since finitely supported $\Rightarrow c_0$.
2. Free gps $\mathbb{F}_n$ w/ $n \geq 2$ have (H).
   - We'll prove this once we have a second equivalent characterization.
3. $SL(2,\mathbb{Z}) \supset \mathbb{F}_2$ index 12 subgp.
   Fact: $SL(2,\mathbb{Z}) \cong \mathbb{Z}/4 \ast_{\mathbb{Z}/2} \mathbb{Z}/6$. [$PSL(2,\mathbb{Z}) = \mathbb{Z}/2 \ast \mathbb{Z}/3$ also has (H)]
4. Gps which act on trees [like $\mathbb{F}_n \curvearrowright$ Cayley graph.]
5. Coxeter gps $\langle g_1, \dots, g_n \mid (g_i g_j)^{m_{ij}} = 1$ where $m_{ii} = 1$, $m_{ij} \geq 2$ $i \neq j \rangle$
   $m_{ij} = \infty$ means no rel'n.
6. Class of gps w/ (H) closed under:
   - taking subgps
   - free products
   - direct products
   - $H \leq G$ finite index, H has (H) $\Rightarrow G$ has (H).

Def: A cocycle of Γ is a triple (H, π, β) where (H,π) is a unitary rep'n $\pi: \Gamma \to \mathcal{U}(H)$ and $\beta: \Gamma \to H$ s.t.

$$\beta(hg) = \beta(h) + \pi_h \beta(g) \quad \forall h, g \in \Gamma.$$

A cocycle is called inner if $\exists \xi \in H$ s.t. $\beta(g) = \xi - \pi_g \xi$ $\forall g \in \Gamma$.

Motivation: Let $\mathrm{Aff}(H) = \{$affine invertible isometries of $H\}$
$$= \{\xi \mapsto u\xi + \eta \mid \eta \in H,\ u \in \mathcal{U}(H)\}.$$
Notice $\mathrm{Aff}(H)$ is a gp under composition:
$$\xi \mapsto u_2\xi + \eta_2 \mapsto u_1[u_2\xi + \eta_2] + \eta_1 \quad \text{is} \quad \xi \mapsto u_1u_2\xi + [u_1\eta_2 + \eta_1]$$
Can think of these as $\underline{H \rtimes \mathcal{U}(H)}$: $(\eta_1, u_1)\cdot(\eta_2, u_2) = (\eta_1 + u_1\eta_2, u_1u_2)$

An $\underline{\text{affine isometric action}}$ of Γ on H is a homom $\Gamma \to \mathrm{Aff}(H)$.

Given $\alpha: \Gamma \to \mathrm{Aff}(H)$, get a unitary rep'n $\pi: \Gamma \to \mathcal{U}(H)$ by
$$\pi: \Gamma \xrightarrow{\alpha} H \rtimes \mathcal{U}(H) \twoheadrightarrow \mathcal{U}(H)$$
Note: A cocycle gives an affine isometric action via
$$\alpha_g\xi := \pi_g\xi + \beta(g).$$
The cocycle condition implies $\alpha_g \circ \alpha_h = \alpha_{gh}$.

Conversely: If $\alpha: \Gamma \to \mathrm{Aff}(H)$ is an affine isometric action, $\forall g \in \Gamma$, $\exists! (\pi_g, \beta(g)) \in \mathcal{U}(H) \times H$ s.t. $\alpha_g\xi = \pi_g\xi + \beta(g)$. Then $\beta: \Gamma \to H$ is a cocycle.

Lemma: Let X be a uniformly cvx Banach space and $B \subset X$ bdd. Then $\inf_{x \in X} \sup_{b \in B} \|x - b\|$ is attained at a $!x \in X$.

Pf: HW!

Prop: A cocycle is inner $\iff$ it is $\underline{\text{bdd}}$.

Pf: $\Rightarrow$: clear since $\|\xi - \pi_g\xi\| \le 2\|\xi\|$ $\forall g \in \Gamma$.

$\Leftarrow$: Consider affine action of Γ on H from β. If β bdd, then $\Gamma \cdot 0_H$ is bdd, since $\alpha_g 0_H = \pi_g 0_H + \beta(g) = \beta(g)$. By the lemma, $\exists! \xi \in H$ minimizing $\sup_{g \in \Gamma} \|\beta(g) - \xi\|$. Now $\forall \eta \in \Gamma \cdot 0_H$, $\forall g \in \Gamma$, we have $\|\alpha_g\xi - \underbrace{\alpha_g\eta}_{\in \Gamma\cdot 0_H}\| = \|\pi_g(\xi - \eta)\| = \|\xi - \eta\|$, so by !ness in the lemma, $\alpha_g\xi = \xi$ $\forall g$ which implies $\alpha_g\xi = \pi_g\xi + \beta(g) = \xi \iff \beta(g) = \xi - \pi_g\xi$ $\forall g \in \Gamma$.

Def: A fct $f: X \to Y$ btwn top. spaces is $\underline{\text{proper}}$ if $f^{-1}K$ cpt $\forall K \subseteq Y$ cpt. An (affine) action α of Γ on H is $\underline{\text{proper}}$ if the map $\Gamma \times H \to H \times H$ by $(g, \xi) \mapsto (\alpha_g\xi, \xi)$ is $\underline{\text{proper}}$. A cocycle $\beta: \Gamma \to H$ is called $\underline{\text{proper}}$ if $g \mapsto \|\beta(g)\|$ is proper. HW: Show $\alpha = (\pi, \beta)$ proper $\iff$ β proper.

Thm: For a countable discrete Γ, TFAE:

① Γ has (H).

② $\exists$ a proper cocycle

③ $\exists$ a proper affine isometric action of Γ on a Hilbert space $\mathcal{H}$.

Pf: ② $\Leftrightarrow$ ③: Immediate from the HW problem.

② $\Rightarrow$ ①: We'll use:

Schoenberg's Thm: If $\beta : \Gamma \to \mathcal{H}$ a cocycle, then $\forall r > 0$,
$f_r(g) := \exp(-r\|\beta(g)\|^2)$ is positive definite.

Sketch of pf: First, note that:

- $\beta(e) = \beta(e^2) = \beta(e) + \pi_e \beta(e) = 2\beta(e) \Rightarrow \beta(e) = 0$.
- $0 = \beta(e) = \beta(g^{-1}g) = \beta(g^{-1}) + \pi_{g^{-1}}\beta(g) \Rightarrow \beta(g^{-1}) = -\pi_{g^{-1}}\beta(g)$
- $\|\beta(g^{-1}h)\| = \|\beta(g^{-1}) + \pi_{g^{-1}}\beta(h)\| = \|-\pi_{g^{-1}}\beta(g) + \pi_{g^{-1}}\beta(h)\| = \|\beta(g) - \beta(h)\|$.

$r=1$: $f(g^{-1}h) = \exp(-\|\beta(g) - \beta(h)\|^2) \underset{(*)}{=} \underline{\exp(-\|\beta(g)\|^2)\exp(-\|\beta(h)\|^2)}\,\underline{\exp(2\operatorname{Re}\langle\beta(g), \beta(h)\rangle)}$.

- $\forall g_1, \dots, g_n$, $\left[\exp(-\|\beta(g_i)\|^2)\exp(-\|\beta(g_j)\|^2)\right] = \begin{bmatrix} \exp(-\|\beta(g_1)\|^2) \\ \vdots \\ \exp(-\|\beta(g_j)\|^2) \end{bmatrix} \left[\ \cdots\ \right] \geq 0$.
- $\forall g_1, \dots, g_n$, $[\langle\beta(g_i), \beta(g_j)\rangle] \geq 0$.
- $\forall a \in M_n(\mathbb{C})_+$, $\operatorname{Re}[a] \geq 0$ where $\operatorname{Re}[a]_{ij} := \operatorname{Re}(a_{ij})$. $[\operatorname{Re}[a] = \frac{1}{2}(a + \bar{a}) \geq 0]$
- $\exp(2\operatorname{Re}\langle\beta(g), \beta(h)\rangle) = \sum \frac{(2\operatorname{Re}\langle\beta(g), \beta(h)\rangle)^n}{n!}$, and $\forall g_1, \dots, g_n \in \Gamma$,

  $[\operatorname{Re}\langle\beta(g_i), \beta(g_j)\rangle] \geq 0 \Rightarrow [\exp(2\operatorname{Re}\langle\beta(g_i), \beta(g_j)\rangle)] \geq 0$.
- Now the <u>Schur product</u> $(a * b)_{ij} := a_{ij} b_{ij}$ of 2 positive matrices is pos.!

  Hence the RHS of $(*)$ above is the entry of Schur prod. $\Rightarrow f$ pos. def.

Now, as $r \to 0^+$, $f_r \to 1$ pointwise. $f_r \to 1$ uniformly $\Leftrightarrow \beta$ bdd.

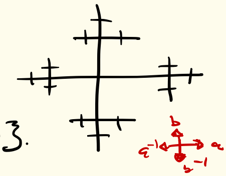
$f_n \in C_0(\Gamma) \Leftrightarrow \beta$ is proper. Thus $\exists \beta$ proper $\Rightarrow \exists f_n \in C_0(\Gamma)$ pos. def.

such that $f_n \to 1$ pointwise.

① $\Rightarrow$ ②: omitted.

Theorem: If Γ acts on a tree preserving the distance of vertices, then Γ has the Haagerup property.

Example: The Cayley graph T_n of $\mathbb{F}_n$ is a tree. eg: $\mathbb{F}_2$:



vertices $= \mathbb{F}_2$; edges $= \{ (x, xa), (x, xb), (x, xa^{-1}), (x, xb^{-1}) \}$.

Now $\mathbb{F}_n \curvearrowright T_n$ on the left. Hence $\mathbb{F}_n$ has (H).

Pf: Let $H := \ell^2$(oriented edges of T) [Each edge appears twice w/ opposite orientations.] For vertices u, v in T, define:

① $d(u,v)$ = length of geodesic u to v, denoted $[u,v]$.

② $\xi(u,v) \in H$ by $\xi(u,v)(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon \text{ not in } [u,v] \\ +1 & \text{if } \varepsilon \in [u,v] \\ -1 & \text{if } \varepsilon \in [v,u] \end{cases}$] orientation matters!

Claim 1: $\forall$ vertices u,v,w, $\xi(u,v) + \xi(v,w) = \xi(u,w)$. [Chasles' relation]

Pf: All triangles in a tree are degenerate. Any common part of $[u,v]$ and $[v,w]$ must be travelled in both directions once.

Claim 2: $\forall$ vertices u,v, $\|\xi(u,v)\|^2 = 2d(u,v)$.

Pf: $\|\xi(u,v)\|^2$ counts twice # edges in geodesic $[u,v]$.

Now note that $\Gamma \curvearrowright T$ gives a unitary rep'n $\pi : \Gamma \to B(H)$ by left translation, so $\forall$ vertices u,v, and $g \in \Gamma$, we have $\pi_g \xi(u,v) \underset{(*)}{=} \xi(gu, gv)$. Now fix $t_0 \in T$, and define $\beta : \Gamma \to H$ by $\beta(g) = \xi(gt_0, t_0)$ $\forall g \in \Gamma$.

Claim 3: β is a proper cocycle, so Γ has (H).

Pf: $\forall g,h \in \Gamma$, $\beta(gh) = \xi(ght_0, t_0) = \xi(ght_0, gt_0) + \xi(gt_0, t_0)$ [by Claim 1]

$= \pi_g \xi(ht_0, t_0) + \xi(gt_0, t_0)$ [by $(*)$]

$= \beta(g) + \pi_g \beta(h)$.

Moreover, $\|\beta(g)\|^2 = 2d(gt_0, t_0) \to \infty$ as $g \to \infty$.

Def: Let (M, tr) be a finite vNa w/ normal faithful tracial state. Say M has the Haagerup property if $\exists$ seq. (φ_n) of (normal) trace-preserving CP maps $M \to M$ s.t. ① $\varphi_n \to \mathrm{id}$ ptwise σ-WOT and ② on L^2M, $\hat{\varphi}_n(x\Omega) := \varphi_n(x)\Omega$ is cpt in $B(L^2M)$. [like the $C_0(\Gamma)$ condition.]

Thm: Γ has (H) $\iff$ $L\Gamma$ has (H).

Pf: $\Rightarrow$: Let (f_n) be seq. of pos def. fcts $\Gamma \to \mathbb{C}$ in $c_0(\Gamma)$. Then $M_{f_n} : L\Gamma \to L\Gamma$ witness that $L\Gamma$ has (H).

$\Leftarrow$: We showed that if $\varphi : L\Gamma \to L\Gamma$ cp, then $f(g) := tr(\varphi(\lambda_g)\lambda_g^*)$ is positive definite. From $(\varphi_n) \rightsquigarrow (f_n)$.

Kazhdan's property (T): Let Γ be a discrete countable gp and $\Lambda \le \Gamma$ a subgroup. We say Γ has property (T) relative to Λ if whenever (f_n) is a seq. of pos. def. fcts $\Gamma \to \mathbb{C}$ s.t. $f_n \to 1$ pointwise, $f_n|_\Lambda \to 1$ uniformly on Λ. We say Γ has property (T) if Γ has property (T) relative to Γ.

Examples:

① All finite gps have (T).

② Γ has (H) and (T) $\iff$ Γ is finite. $\boxed{\text{HW!}}$

③ If Γ has (T) relative to Λ, and Γ has (H), then Λ finite.

④ $SL(2,\mathbb{Z})$ has (H) as $\mathbb{F}_2 \le SL(2,\mathbb{Z})$ w/ index 12. But $SL(n,\mathbb{Z})$ has (T) $\forall n \ge 3$.

⑤ $\mathbb{Z}^2 \trianglelefteq \underline{\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})}$ has relative (T)

$$\left[\left(\begin{array}{cc|c} 1 & & * \\ & 1 & * \\ \hline & & 1 \end{array}\right) \le \left(\begin{array}{c|c} SL(2,\mathbb{Z}) & * \\ & * \\ \hline & 1 \end{array}\right) \right]$$

→ does not have (T) by ④

But: rel (T) wrt ∞ subgp $\Rightarrow$ not (H)! $\rightsquigarrow$ (H) not closed under extensions.

Equivalent definitions of (T):

① $\forall$ seq. (f_n) of pos def fcts w/ $f_n \to 1$ ptwise, $f_n \to 1$ unif.

② $\forall$ unitary rep'n (H, π) of Γ w/ seq. of unit v's (ξ_n) s.t. $\forall g \in \Gamma$, $\|\pi_g \xi_n - \xi_n\| \to 0$, $\exists \xi \in H \setminus \{0\}$ s.t. $\pi_g \xi = \xi$ $\forall g \in \Gamma$.

③ $\underline{\forall \varepsilon > 0}$, $\exists \delta > 0$ and $F \subseteq \Gamma$ finite s.t. $\forall (H, \pi)$ unitary rep and $\xi \in (H)_1$ s.t. $\|\pi_g \xi - \xi\| < \delta$ $\forall g \in F$, $\exists \xi_0 \in (H)_1$ w/ $\pi_g \xi_0 = \xi_0$ $\forall g \in \Gamma$ $\underline{\text{and } \|\xi - \xi_0\| < \varepsilon}$.

④ Same as ③, but omit what is underlined in red.

⑤ $\forall \varepsilon > 0$, $\exists \delta > 0$, $F \subset \Gamma$ finite s.t. $\forall f : \Gamma \to \mathbb{C}$ pos. def. w/ $|f(g) - 1| < \delta$ $\forall g \in F$, we have $|f(g) - 1| < \varepsilon$ $\forall g \in \Gamma$.

⑥ Every cocycle $\beta:\Gamma \to \mathcal{H}$ is inner [$\Leftrightarrow$ bdd]

⑦ Every affine Γ-action has a fixed pt.

Pf:

⑥ $\Leftrightarrow$ ⑦: Observe that for $\alpha \in \mathrm{Aff}(\mathcal{H})$,

$$\alpha_g \xi = \pi_g \xi + \beta(g) = \xi \iff \beta(g) = \xi - \pi_g \xi \qquad \forall g \in \Gamma.$$

Hence α has a fixed pt $\iff$ β is inner.

① $\Rightarrow$ ②: Suppose (H,π) a unitary Γ-rep'n and $(\xi_n) \subset (\mathcal{H})_1$ s.t. $\|\pi_g \xi_n - \xi_n\| \to 0$ $\forall g \in \Gamma$. Define $f_n(g) := \langle \pi(g)\xi_n, \xi_n\rangle$ pos.-def. The rest is HW!

$\neg$④ $\Rightarrow$ $\neg$①: Let $\Gamma = \{g_1, g_2, \ldots\}$ and $F_n := \{g_1, \ldots, g_n\} \subseteq \Gamma$. Set $\delta_n = \frac{1}{n}$. Then $\forall n, \exists$ unitary rep'n (H_n, π_n, ξ_n) s.t. $\|\xi_n\| = 1$, $\|\pi(g)\xi_n - \xi_n\| < \frac{1}{n}$ $\forall g \in F_n$, but $\nexists\, \xi_{n,0} \in (H_n)_1$ w/ $\pi_g \xi_{n,0} = \xi_{n,0}$ $\forall g \in \Gamma$, i.e., $\underbrace{H_n^{\pi_n(\Gamma)}}_{\text{fixed pts}} = (0)$. Now set $(H,\pi) = \bigoplus_n (H_n, \pi_n)$. Then (ξ_n) a seq. of almost inv. $\vec{v}$'s, but w/ no invariant $\vec{v}$.

③ $\Rightarrow$ ④: Trivial, just take an arbitrary ε.

④ $\Rightarrow$ ③: Let $\varepsilon > 0$. Take $\delta > 0$, $F \subset \Gamma$ finite as in ④. Set $\tilde{\delta} = \varepsilon' \delta$ and $\tilde{F} = F$. Suppose (H, π) is a unitary Γ-rep'n w/ ξ a $(\tilde{\delta}, \tilde{F})$-almost invariant unit vector. Note that $[H^\pi]^\perp \subseteq H$ does not contain any nonzero invariant vectors. $\pi|_{[H^\pi]^\perp}$ is a Γ-rep'n, so $\forall \eta \in [H^\pi]^\perp_1$ $\|\pi_g \eta - \eta\| \geq \delta$ for some $g \in F$ $\Rightarrow$ $\|\pi_g p^\perp \xi - p^\perp \xi\| \geq \delta \|p^\perp \xi\|$ where $p^\perp \in B(\mathcal{H})$ is proj. onto $[H^\pi]^\perp$. But:

$$\varepsilon' \delta \geq \|\pi_g \xi - \xi\| = \|\pi_g p^\perp \xi - p^\perp \xi\| \geq \delta \|p^\perp \xi\| \quad \Rightarrow \quad \|p^\perp \xi\| \leq \varepsilon'.$$

Thus: $(1 - p^\perp)\xi$ is close to ξ! Indeed, setting $p = 1 - p^\perp$, if $\boxed{\varepsilon' < 1}$, then $\|p\xi\|^2 \geq 1 - \varepsilon'$. Set $\xi_0 = \dfrac{p\xi}{\|p\xi\|} \in H^\pi$. Then:

$$\xi_0 - p\xi = \frac{p\xi}{\|p\xi\|} - p\xi = \frac{1 - \|p\xi\|}{\|p\xi\|} p\xi \Rightarrow \|\xi_0 - p\xi\| \leq 1 - \|p\xi\| \leq \boxed{\varepsilon' < \frac{\varepsilon}{2}}$$

$$\Rightarrow \|\xi_0 - \xi\| \leq \|\xi_0 - p\xi\| + \|p\xi - \xi\| < \frac{\varepsilon}{2} + \underbrace{\|(1-p^\perp)\xi\|}_{\leq\, \varepsilon' < \frac{\varepsilon}{2}} < \varepsilon.$$

$\varepsilon' < \min\{1, \frac{\varepsilon}{2}\}$.

③ ⇒ ⑤: Let $\varepsilon > 0$. Choose (F, δ) as in ③ for $\varepsilon' = \varepsilon'(\varepsilon)$ TBD! Set $\tilde{F} = F \cup F^{-1} \cup \{e\}$, and $\tilde{\delta}(f, \varepsilon)$ TBD!

Let $f : \Gamma \to \mathbb{C}$ be pos. def. s.t. $|f(g) - 1| < \tilde{\delta}$ $\forall g \in \tilde{F}$. NB: $|f(e) - 1| < \tilde{\delta}$.

$\exists$ (H, π, Ω) s.t. $f(g) = \langle \pi_g \Omega, \Omega \rangle$ $\forall g \in \Gamma$. Since $e \in \tilde{F}$, $|\|\Omega\|^2 - 1| < \tilde{\delta}$.

Let $\xi = \frac{\Omega}{\|\Omega\|}$. Then $\|\pi_g \xi - \xi\|^2 = 2 - \langle \pi_g \xi, \xi \rangle - \langle \pi_{g^{-1}} \xi, \xi \rangle$ $\forall g$

For $g \in \tilde{F}$, $|\langle \pi_g \xi, \xi \rangle - 1| \leq \underbrace{|\langle \pi_g \xi, \xi \rangle - f(g)|}_{|1 - \|\Omega\|^2| \cdot |\langle \pi_g \xi, \xi \rangle| < \tilde{\delta} \cdot 1} + |f(g) - 1| < 2\tilde{\delta}$.

Thus $\forall g \in F$, $\|\pi_g \xi - \xi\|^2 < 4\tilde{\delta} < \delta$ if $\boxed{\tilde{\delta} < \delta/4}$. By ③, $\exists \xi_0 \in (H)_1$ s.t.

$\pi_g \xi_0 = \xi_0$ $\forall g \in \Gamma$ and $\|\xi - \xi_0\| < \varepsilon'$. Then $\forall g \in \Gamma$,

$$\begin{aligned} |1 - f(g)| &= |\langle \pi_g \xi_0, \xi_0 \rangle - \langle \pi_g \Omega, \Omega \rangle| \\ &\leq |\langle \pi_g \xi_0, \xi_0 - \xi \rangle| + |\langle \pi_g (\xi_0 - \xi), \xi \rangle| + |\langle \pi_g \xi, \xi \rangle - \langle \pi_g \Omega, \Omega \rangle| \\ &< \quad \varepsilon' \quad + \quad \varepsilon' \quad + \quad \tilde{\delta} \\ &< \varepsilon. \end{aligned}$$

provided we chose $\boxed{\varepsilon' < \frac{\varepsilon}{3}}$ and $\boxed{\tilde{\delta} < \min\{\delta/4, \varepsilon/3\}}$.

⑤ ⇒ ①: very easy. Suppose (f_n) seq. of pos. def. fcts. s.t. $f_n \to 1$ pointwise on Γ. Let $\varepsilon > 0$. Choose (F, δ) as in ⑤. Since F is finite and $f_n \to 1$ pointwise, eventually $|f_n(g) - 1| < \delta$ $\forall g \in F$. Then $|f_n(g) - 1| < \varepsilon$ $\forall g \in \Gamma$. Hence $f_n \to 1$ unif.

① ⇒ ⑥: Let $\beta : \Gamma \to H$ be a cocycle. By Schoenberg's thm, $\forall r > 0$, $f_r(g) := \exp(-r \|\beta(g)\|^2)$ is pos. def. and $f_r(g) \to 1$ pointwise as $r \to 0^+$. By ①, $f_{1/n} \to 0$ uniformly, so β is bdd $\iff$ inner.

⑥ ⇒ ②: HW!

Def: Let (M, tr) be a finite vNa w/ faithful normal tracial state. We say (M, tr) has property (T) if $\forall$ seq. $(\varphi_n : M \to M)$ of (normal) trace-preserving ucp. maps s.t. $\varphi_n \to id$ ptwise-$\|\cdot\|_2$

$\varphi_n \to id$ uniformly in $\|\cdot\|_2$ on $(M)_1 = \{m \in M \mid \|m\|_\infty \leq 1\}$.

note these are different!

Our goal is to prove:

Thm: Γ has (T) $\iff$ $(L\Gamma, tr)$ has (T).

Preliminary results: (M, tr_M), (N, tr_N) finite vNa's w/ normal faithful tracial states.

An M-N bimodule is a Hilbert space H w/ commuting normal $*$-hom's $\lambda: M \to B(H)$ and $\rho: N^{op} \to B(H)$ $\Leftrightarrow [\lambda(a), \rho(b)] = 0$

we'll suppress λ, ρ and simply write $a \cdot \eta \cdot b$ for $\lambda(a)\rho(b)\eta$.

Lemma: There is a canonical bijection btwn:

① trace-preserving normal ucp maps $\phi: M \to N$

② pointed M-N bimodules (H, ξ) w/ trace vector ξ:

$$\langle m\xi, \xi \rangle = tr_M(m) \ \forall m \in M, \quad \langle \xi n, \xi \rangle = tr_N(n) \ \forall n \in N.$$

Pf ① ⇒ ②: Given ϕ, define $\langle \cdot, \cdot \rangle_\phi$ on $M \otimes N$ by

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle := tr_N(b_2^* \phi(a_2^* a_1) b_1) = \langle \phi(a_2^* a_1) b_1 \Omega, b_2 \Omega \rangle_{L^2 N}.$$

Do usual procedure (mod out by left kernel, complete) to get H_ϕ.

Define $a \cdot (x \otimes y) \cdot b := ax \otimes yb$. These actions are bdd:

- $\| a \cdot \sum_i (x_i \otimes y_i) \|_\phi^2 = \sum_{i,j} \langle \phi(x_j^* a^* a x_i) y_i \Omega, y_j \Omega \rangle_{L^2 N} \le \|a^* a\| \, \| \sum_i x_i \otimes y_i \|_\phi^2$

  $[\phi(x_j^* a^* a x_i)] \le \|a^* a\| \, [\phi(x_j^* x_i)]$ in $M_n(N)$.

- bddness of R-action is easier.

These actions are normal since ϕ is normal. [exercise]

Remark: If $N \subseteq M$ vN subalg and $\phi = E_N$, trace-preserving cond. exp., then $H_{E_N} \cong {}_M L^2 M_N$! $a \otimes b \mapsto ab$ descends to iso!

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{E_N} = tr_N(b_2^* E_N(a_2^* a_1) b_1) = \underbrace{tr_N \circ E_N}_{tr_M}(b_2^* a_2^* a_1 b_1) = \langle a_1 b_1, a_2 b_2 \rangle_{L^2 M}.$$

Now let ξ be image of $1_M \otimes 1_N$ in H_ϕ. Then $\forall m \in M, n \in N$,

$$\|m\xi\|_\phi^2 = tr_N(\phi(m^* m)) = tr_M(m^* m); \quad \|\xi n\|_\phi^2 = tr_N(n^* \phi(1) n) = tr_N(n^* n).$$

② ⇒ ①: Given $({}_M H_N, \xi)$, define $L_\xi: L^2 N \to H$ as ! extension of $n\Omega \mapsto \xi n$.

isometry: $\|\xi n\|_H^2 = tr(n^* n) = \|n\Omega\|^2$, clearly normal too.

Now $\phi: M \to N$ given by $\phi(m) = L_\xi^* \lambda(m) L_\xi$ is normal ucp

where $\lambda: M \to B(H)$ is the left M-action. Finally, $\forall m \in M$,

$$tr_N(\phi(m)) = \langle \phi(m)\Omega, \Omega \rangle = \langle \lambda(m) L_\xi \Omega, L_\xi \Omega \rangle = \langle m\Omega, \Omega \rangle = tr_M(m).$$

Hence ϕ is trace-preserving. Exercise: Show mutually inverse!

Lemma: For (M, tr) a finite vNa w/ faithful normal tracial state, TFAE:

① (M, tr) has (T)

② $\forall \varepsilon > 0$, $\exists$ finite $F \subseteq M$ and $\delta > 0$ s.t. $\forall$ $({}_M H_M, \xi)$ bimodule w/ trace $\vec{v}$ satisfying $\max_{f \in F} \| f\xi - \xi f \| < \delta$, $\exists$ an M-central vector $\xi_0 \in H$ s.t. $\| \xi - \xi_0 \| < \varepsilon$.

Pf: ② ⟹ ①: Suppose (φ_n) a seq. of (normal) trace-preserving ucp maps s.t. $\varphi_n \to id$ ptwise $\|\cdot\|_2$. Let $\tilde{\varepsilon} > 0$, and pick (F, δ) for $\varepsilon = \varepsilon(\tilde{\varepsilon})_{\text{TBD}}$ as in ②. Let (H_n, ξ_n) be the M-M bimod w/ tr. $\vec{v}$ corresp. to φ_n. Since $\varphi_n \to id$ ptwise $\|\cdot\|_2$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $\forall f \in F$, $\|\varphi_n(f)\Omega - f\Omega\|_2 < \tilde{\delta}$ where $\tilde{\delta} = \tilde{\delta}(\delta, F, \varepsilon)$ TBD. Then $\forall n > N$, $\forall f \in F$,

$$\begin{aligned}
\|f\xi_n - \xi_n f\|^2 &= \|f\xi_n\|^2 + \|\xi_n f\|^2 - 2\operatorname{Re}\langle f\xi_n, \xi_n f\rangle \\
&= tr_M(\varphi_n(f^*f)) + tr_M(f^*f) - 2\operatorname{Re} tr_M(\varphi_n(f) f^*) \\
\text{φ_n trace-preserving} \to \quad &= 2 tr_M(f^*f) - 2\operatorname{Re} tr_M(\varphi_n(f) f^*) \\
&= 2\operatorname{Re} tr_M[(\varphi_n(f) - f) f^*] \\
&\leq 2 |\langle (\varphi_n(f) - f)\Omega, f\Omega\rangle| \\
CS \to \quad &\leq 2 \|\varphi_n(f)\Omega - f\Omega\|_{L^2 M} \cdot \|f\Omega\|_{L^2 M} \\
&< 2\tilde{\delta} \cdot K \quad \text{where } K := \max_{f \in F} \|f\Omega\|_{L^2 M} < \infty.
\end{aligned}$$

Now if we had $\boxed{\tilde{\delta} < \frac{\delta^2}{2K}}$, then by ②, $\forall n > N$, $\exists$ M-central $\vec{v}$. $\xi_{n,0} \in H_n$ s.t. $\|\xi_{n,0} - \xi_n\| < \tilde{\varepsilon}$. Then $\forall n > N$, $\forall x \in (M)_1$,

$$\begin{aligned}
\|\varphi_n(x)\Omega - x\Omega\|_2 &= \|\varphi_n(x)\Omega\|_2^2 + \|x\Omega\|_2^2 - 2\operatorname{Re} tr_M(\varphi_n(x) x^*) \\
&= tr_M(\varphi_n(x)^*\varphi_n(x)) + tr_M(x^*x) - 2\operatorname{Re} tr_M(\varphi_n(x) x^*) \\
&\leq tr_M(\varphi_n(x^*x)) + tr_M(x^*x) - 2\operatorname{Re} tr_M(\varphi_n(x) x^*) \\
&= 2 tr_M(x^*x) - 2\operatorname{Re}\langle x\xi_n, \xi_n x\rangle \\
&= \|x\xi_n - \xi_n x\| \\
&= \|x\xi_n - x\xi_{n,0}\| + \|\xi_{n,0} x - \xi_n x\| \leq 2\varepsilon < \tilde{\varepsilon}.
\end{aligned}$$

$\boxed{\text{Pick } \varepsilon < \frac{\tilde{\varepsilon}}{2}}$

① ⟹ ②: Omitted.

[It would be good to check this! I'm a little unsure about the choices of exp. in the def of (T) I gave for (M, tr).]

$\Leftarrow$: Suppose $(L\Gamma, tr)$ has (T). Let (f_n) be a seq. of pos. def. fcts on Γ s.t. $f_n \to 1$ pointwise. Then (M_{f_n}) is a seq. of cp. maps on $L\Gamma$ s.t. $M_{f_n} \to id$ pointwise $\|\cdot\|_2$. Since $L\Gamma$ has (T), $M_{f_n} \to id$ uniformly in $\|\cdot\|_2$ on $(L\Gamma)_1$. In particular, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $\forall g \in \Gamma$, $\varepsilon > \|M_{f_n}(\lambda_g)\Omega - \lambda_g \Omega\|_{L^2 L\Gamma}$

$$= \|f_n(g)\lambda_g \Omega - \lambda_g \Omega\|_{L^2 L\Gamma}$$

$$= |f_n(g) - 1|.$$

Hence $f_n \to 1$ uniformly as $n \to \infty$.

$\Rightarrow$: Suppose Γ has (T). Let $\varepsilon > 0$. Choose $(\tilde{F}, \tilde{\delta})$ as in ③ for the equiv. defs of (T) for this ε. Pick $F = \{\lambda_g \mid g \in \tilde{F}\} \subset L\Gamma$ and set $\delta = \tilde{\delta}$. Let (H, ξ) be an $L\Gamma$-$L\Gamma$ bimodule w/ trace vector ξ satisfying $\max_{g \in \tilde{F}} \|\lambda_g \xi - \xi \lambda_g\| < \tilde{\delta}$. Note that ${}_{L\Gamma}H_{L\Gamma}$ induces a unitary rep'n $\pi : \Gamma \to B(H)$ by $\pi_g \eta := \lambda_g \eta \lambda_g^*$. Then we have $\xi \in (H)_1$ and $\forall g \in F$, $\|\pi_g \xi - \xi\| = \|\lambda_g \xi \lambda_g^* - \xi\| = \|\lambda_g \xi - \xi \lambda_g\| < \tilde{\delta}$. Since Γ has (T), $\exists \xi_0 \in (H)_1$ s.t. $\pi_g \xi_0 = \xi_0$ $\forall g \in \Gamma$ and $\|\xi - \xi_0\| < \varepsilon$. Now $\forall g \in \Gamma$, $\pi_g \xi_0 = \lambda_g \xi_0 \lambda_g^* = \xi_0 \iff \lambda_g \xi_0 = \xi_0 \lambda_g$. So ξ_0 is $L\Gamma$-central. Hence $L\Gamma$ has (T) by ② $\Rightarrow$ ① in the preceding lemma.